7 MATH3733

7.1 Information

Information is crucial for the market. Hence, a mathematical financial theory uses it under the name **filtration**; other names: **information**, **sigma-field**, **sigma-algebra**).

Definition 1 *Let market consist of one stock which is modelled by a random process* $X_t,\ t = 0, 1, 2, \ldots$. **A filtration or information to time** t *is* **the set** $\mathcal{F}_t^X$ (*this is a notation*) **of all events which correspond to the values of the process** X **until this time**, $(X_0, X_1, \ldots, X_t)$*; in the other words,* $\mathcal{F}_t^X$ *consists of* **all available information on the market till time** t.

The filtration includes the knowledge of any **event** connected with the values $X_0, X_1, \ldots, X_t$, that is, if we know information till time t, we must know whether any such event was realized or not.

7.2 CRR model: two-step market

This is a discrete time model with time $t = 0, \delta, 2\delta$ (of course, the *"tick"* $\delta > 0$). On our market there is a bank account B_t and a stock S_t. The interest rate is $r \geq 0$, i.e. $B_\delta = B_0 \exp(r\delta)$. The stock has only one *node* at time 0 with value S_0, two nodes at time δ, with values $S_\delta^1 > S_\delta^2$ (previously S_δ^+ and S_δ^-), and three nodes at time 2δ, $S_{2\delta}^1 > S_{2\delta}^2 > S_{2\delta}^3$.

There are probabilities p_0 & $1 - p_0$ to move from S_0 to S_δ^1 or to S_δ^2 correspondingly, and also probabilities $p_\delta(S_\delta^1)$, $1 - p_\delta(S_\delta^1)$, $p_\delta(S_\delta^2)$, and $1 - p_\delta(S_\delta^2)$ to move from S_δ to $S_{2\delta}$. From each node at time δ the price S can move only to two possible nodes, and it may move to $S_{2\delta}^2$ from any of the two nodes at time δ.

For any possible value S_δ, we will still use slightly extended notations $S_\delta^\pm$ to denote two possible values for the stock price at $t = 2\delta$.

To avoid arbitrage possibilities, we assume conditions (similar to those for one-step market): for any $t < 2$,

$$\exp(-r\delta)S_t^- < S_t < \exp(-r\delta)S_t^+.$$

7.3 Markov process (two steps)

Definition 2 *We call process* $(X_t,\ t = 0, \delta, 2\delta)$ **Markov** *(or* **markovian***) if for any available* t*, the conditional probability of the event* $\{X_t \leq x\}$ **given all the past** *of the process* X *equals to the (again conditional) probability of this event* **given only the last value of the process***, that is, given* $X_{t-\delta}$.

7.4 CRR model: two-step analysis

We need a method to value the call option for $t = 0$ and $t = \delta$. The idea is

1. firstly to **use the one-step model** to value the option at $t = \delta$,
2. and secondly, after we did it and know the values of C at each node for time $t = \delta$, **use again the same one-step model** to find the price at $t = 0$. Let us write down formulas (and show how we should use filtrations).

Indeed, we can find C_0 once we know the values $C_\delta(S_\delta^1)$ and $C_\delta(S_\delta^2)$. In turn, we can find the latter since we know $C_{2\delta}$ at any node for time 2δ (i.e. at expiry).

Definition 3 **Self-financed portfolio** *is such a portfolio* $P_t = \phi_t S_t + \psi_t B_t, t = k\delta, k = 0, 1, \ldots,$ *that on each step, the owner just re-distributes the existing amount of money:*

$$\forall t, \qquad \phi_{t-\delta} S_t + \psi_{t-\delta} B_t = \phi_t S_t + \psi_t B_t,$$
$$\text{or, equivalently,} \qquad P_t - P_{t-\delta} = \phi_{t-\delta}(S_t - S_{t-\delta}) + \psi_{t-\delta}(B_t - B_{t-\delta}).$$

Theorem 1 *If* $T = 2\delta$, *then the European call price can be computed via the formula,*

$$C_0 = e^{-2\delta} \tilde{E} C_{2\delta}; \qquad \tilde{E} \text{ here means “implicit expectation”.}$$

7.5 Random vectors and conditional distributions

Definition 4 *1. Let* $X_1, X_2, \ldots, X_n$ *be* n *random variables. The* n*-tuple* $(X_1, X_2, \ldots, X_n)$ *is called* n**-dimensional random vector**. *Its* **joint distribution function** *is* $F(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, \ldots, X_n \le x_n)$.

2. This random vector is called **continuous** *or* **continuously distributed** *if*

$$\partial^n F(x_1, \ldots, x_n) / \partial x_1 \ldots \partial x_n = p(x_1, \ldots, x_n),$$

or

$$\underbrace{\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n}}_{n} p(u_1, \ldots, u_n)\, du_1 \ldots du_n = F(x_1, \ldots, x_n).$$

Exercise 1 *Show that all marginal densities (also called one-dimensional densities) are equal to* $(n-1)$*-fold integrals,*

$$p_{X_1}(x_1) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} p(x_1, x_2, \ldots, x_n)\, dx_2 \ldots dx_n, \quad \ldots \quad \text{etc.},$$

$$p_{X_n}(x_n) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1} p(x_1, \ldots, x_{n-1}, x_n)\, dx_1 \ldots dx_{n-1}.$$