6 MATH3733

6.1 One step CRR model: "Implied" probabilities

Remind the formulae from the previous lecture for **call**

$$C_0 = \phi S_0 + \psi B_0 = \frac{S_\delta^+ - K}{S_\delta^+ - S_\delta^-} S_0 + e^{-r\delta} \left((S_\delta^+ - K) - \frac{S_\delta^+ - K}{S_\delta^+ - S_\delta^-} S_\delta^+ \right),$$

and for **any synthetic derivative** with payoff $f(S_\delta^\pm)$, (ϕ, ψ **are different now**)

$$V_0 = \phi S_0 + \psi B_0 = \frac{f(S_\delta^+) - f(S_\delta^-)}{S_\delta^+ - S_\delta^-} S_0 + e^{-r\delta} \left(f(S_\delta^+) - \frac{f(S_\delta^+) - f(S_\delta^-)}{S_\delta^+ - S_\delta^-} S_\delta^+ \right).$$

Denote

$$\tilde{p} = \frac{S_0 \exp(r\delta) - S_\delta^-}{S_\delta^+ - S_\delta^-}.$$

Since $0 < \tilde{p} < 1$ (*why?*), we have for **call**,

$$C_0 = e^{-r\delta} \left(\tilde{p}(S_\delta^+ - K) + (1 - \tilde{p})0 \right). \tag{1}$$

For a **general synthetic derivative** the price V_0 reads likewise,

$$V_0 = e^{-r\delta} \left(\tilde{p} f(S_\delta^+) + (1 - \tilde{p}) f(S_\delta^-) \right) \tag{2}$$

with the same $\tilde{p}$. This is a **discounted mean value of the payoff w.r.t. to probabilities** $\tilde{p}$ and $1-\tilde{p}$. The latter in general are not equal to the actual probabilities p and $1-p$; they are calculated without any mention of them. New probabilities $\tilde{p}$ and $1-\tilde{p}$ are called **implied ones, risk-neutral, risk-adjusted, martingale ones**, and the expectation w.r.t. them we denote by $\tilde{E}$ (you may find Q in some books).

Exercise 1 *Show formulas (1) and (2).*

Theorem 1 *Values S_t, C_t and B_t (the latter is not random) satisfy assertions*

$$e^{-r\delta} \tilde{E} S_\delta = S_0, \quad e^{-r\delta} \tilde{E} C_\delta = C_0, \quad \textit{and} \quad e^{-r\delta} B_\delta = B_0.$$

Proof. We only prove the first equality, the second being similar and the third evident (from definition of B_t). We substitute the values for $\tilde{p}$ and $1-\tilde{p}$:

$$e^{-r\delta} \left(\tilde{p} S_\delta^+ + (1-\tilde{p}) S_\delta^- \right) = e^{-r\delta} \left(\frac{S_0 e^{r\delta} - S_1^-}{S_\delta^+ - S_\delta^-} S_\delta^+ + \frac{S_\delta^+ - S_0 e^{r\delta}}{S_\delta^+ - S_\delta^-} S_\delta^- \right)$$

$$= e^{-r\delta} \frac{[S_\delta^+ S_\delta^- - S_\delta^+ S_\delta^- + S_0 e^{r\delta}(S_\delta^+ - S_\delta^-)]}{S_\delta^+ - S_\delta^-} = S_0.$$

Exercise 2 *Show the assertion for C_0. Hint: it is written above, find where.*

6.2 Probability, random variables (revision)

Definition 1 **Probability space** *is a triple* $(\Omega; F; P)$ *where* Ω *is any nonempty set which consists of points called* **outcomes** *or* **elementary events***;* F *is the set of* **all events** *which are subsets of* Ω *and for which* **probability** P *is defined.*

Example 1 *If we toss a coin once, we can describe this with the help of a probability space* $(\Omega; F; P)$ *with* $\Omega = \{0; 1\}$ *which represents two possible outcomes. An* **event** *is* **any subset of** Ω*, – remember that there is an empty set which is a subset of any other set, – and* **probability** *is* **a function** *on all events defined by the formula* $P(A) = \#A/\#\Omega$ *where* $\#A$ *means the number of points in* A*.*

Example 2 *If we roll a die once, we can describe this with the help of a* **probability space** $(\Omega; F; P)$ *with* $\Omega = \{1, 2, 3, 4, 5, 6\}$*, which represents six possible* **outcomes***. In this example* **an event** *is again* **any subset of** Ω *and* **probability** *is* **a function** *on all events defined by the formula* $P(A) = \#A/\#\Omega$*.*

Example 3 *If we throw a point on the interval* $[0; 1]$ *"randomly", we can describe this with the help of a* **probability space** $(\Omega; F; P)$ *with* $\Omega = [0; 1]$ *which represents a continuum of all possible* **outcomes***. In this example* **an event** *is* **any collection of sub intervals** *of* $[0; 1]$*, and* **probability** *is a* **function** *on the set of all events defined by the formula* $P(A) = |A|/|\Omega|$ *where* $|A|$ *means the full length of* A *and* $|\Omega| = 1$*.*

Probability P must satisfy certain properties or axioms:

1. $P(A \bigcup B) = P(A) + P(B)$ if $A \bigcap B \equiv AB = \emptyset$;
2. $P(\Omega) = 1$.

Definition 2 1. *Events* A *and* B *are called* **independent** *if*

$$P(AB) = P(A)P(B);$$

2. *Events* $A_1, A_2, \dots, A_n$ *are called* **independent** *if for any subset of* <u>*different*</u> *indices* $i_1, i_2, \dots, i_k,\ k \geq 1$,

$$P(A_{i_1}, \dots, A_{i_k}) = \prod_{j=1}^{k} P(A_{i_j});$$

3. *If* $P(B) > 0$ *then conditional probability* $P(A|B)$ *is defined by the formula*

$$P(A|B) = P(AB)/P(B) \qquad (\text{if } P(B) = 0, \text{ then } P(A|B) \text{ is not defined}).$$

Exercise 3 *Show the* <u>*multiplication rule*</u> *for probabilities:*

$$P(AB) = P(B)P(A|B).$$

Definition 3 *1.* **Random variable** X *is a* **function on a probability space** $(\Omega; F; P)$ *with values in* R *such that for any* $x \in R$, *the set* $\{X \leq x\}$ *is an* **event**, *that is,* $\{X \leq x\} \in F$ *and hence the probability* $P(X \leq x)$ *is defined.*

2. Then the function

$$F(x) = P(X \leq x)$$

is called a **distribution function** *or* **cumulative distribution function** *of* X. *Function* $p(x)$ *is called a* **density** *or* **distributional density** *of* X *if* $F'(x) = p(x)$, *or for any* x,

$$F(x) = \int_{-\infty}^{x} p(y)\, dy.$$

Definition 4 *1. Let* X, Y *be two random variables. The couple* (X, Y) *is called* **random vector**. *Its* **joint distribution function** *is* $F(x; y) = P(X \leq x,\ Y \leq y)$.

2. A random vector (X, Y) *is called* **continuous** *or* **continuously distributed** *if for any* x, y,

$$\partial^2 F(x, y)/\partial x \partial y = p(x; y),$$

or

$$\int_{-\infty}^{x} \int_{-\infty}^{y} p(u, v)\, dudv = F(x, y).$$

Exercise 4 *Show that marginal densities (also called one-dimensional densities) of both components are expressed as*

$$p_X(x) = \int_{-\infty}^{\infty} p(x, y)\, dy, \quad p_Y(y) = \int_{-\infty}^{\infty} p(x, y)\, dx$$

3. In continuous case, **conditional density** *of r.v.* X **given** Y *is function*

$$p_{X|Y}(x|Y) := \left. \frac{p(x, y)}{p_Y(y)} \right|_{y=Y} = \frac{p(x, Y)}{p_Y(Y)}.$$

4. In discrete case, **conditional distribution** *of r.v.* X **given** Y *is function*

$$P(X = x_k|Y)\,|_{Y=y_m} := \frac{P(X = x_k, Y = y_m)}{P(Y = y_m)}, \quad m = 1, \ldots, n.$$

Otherwise it may be expressed by

$$P(X = x_k|Y) := \sum_{m=1}^{n} \frac{P(X = x_k, Y = y_m)}{P(Y = y_m)} 1(Y = y_m).$$

6.3 Expectations, martingales

Definition 5 *1. The expectation or mean value of X is*

$$EX = \int_{-\infty}^{\infty} x p_X(x)\,dx \ \ \textit{(continuous case)}, \quad EX = \sum_k x_k P(X = x_k) \ \ \textit{(discrete case)}.$$

2. **Conditional expectation** **of X given Y** *is*

$$E(X|Y) = \int_{-\infty}^{\infty} x p_{X|Y}(x|Y)\,dx \qquad \textit{(continuous case)}$$

(that is, integration with respect to the conditional density), and

$$E(X|Y)|_{Y=y_m} = \sum_k x_k P(X = x_k | Y = y_m), \quad m = 1, \ldots, n \quad \textit{(discrete case)},$$

$$\textit{or equivalently, } E(X|Y) = \sum_k x_k P(X = x_k | Y = y_m) 1(Y = y_m), \quad m = 1, \ldots, n.$$

Lemma 1 *1. In either case, for any function f such that expressions make sense,*

$$Ef(X) = \int_{-\infty}^{\infty} f(x) p_X(x)\,dx \ \textit{(continuous)}, \ Ef(X) = \sum_k f(x_k) P(X = x_k) \ \ \textit{(discrete)}.$$

2. In either case,

$$E(E(X|Y)) = EX. \tag{3}$$

Proof of (3) for continuous case.

$$E(E(X|Y)) = E\left(\int_{-\infty}^{\infty} x p_{X|Y}(x|Y)\,dx\right) = \int \left[\int x \frac{p(x,y)}{\int p(x',y)\,dx'}\,dx\right] p_Y(y)\,dy$$

$$= \int \left[\frac{\int x p(x,y)\,dx}{\int p(x',y)\,dx'}\right] p_Y(y)\,dy = \int x \left[\int p(x,y)\,dy\right] dx = \int x p_X(x)\,dx = E(X).$$

Exercise 5 *Consider discrete case.*

Definition 6 *A* **random process** *(X_t) is a set of random variables while t runs over $(0, 1, 2, \ldots)$ or $[0, \infty)$.*

Definition 7 *A* **Markov** *(will be recalled on the next lecture) random process (X_t) is called* **a martingale** *if for any $t < t'$, $E|X_t| < \infty$ and*

$$E(X_{t'} - X_t | X_t) = 0, \qquad \textit{or equivalently,} \qquad E(X_{t'} | X_t) = X_t.$$

Lemma 2 *If (X_t) is a martingale, then*

$$E(X_t) = const \qquad \textit{(i.e. the mean value does not depend on time).}$$

Proof. By previous lemma, $E(X_t - X_0) = E(E(X_t - X_0 | X_0)) = 0$, hence, $E(X_t) = E(X_0)$, $\forall t$.