4 MATH3733

4.1 Example of arbitrage

Suppose the interest rate is 6% while the quoted price of the put option is indeed \$8. What can a trader do? He *sells a put option for the quoted price \$8, immediately buys a call option for \$5 and sells a stock for \$47 (maybe shortly)*. Currently, his P&L is equal to \$8 - \$5 + \$47 = \$50, but remember he has borrowed a stock. *Then he invests \$50 to a bank with the interest rate* $r = 6\%$.

Then he waits until July. What happens in July? His bank account will be $\$\exp(r(T-t)) \times 50 = \$1.0151131 \times 50 = \$50.755653$. He also is the owner of a call option, he ows a stock and he wrote a put option with the same expiry date in July. Consider the plot of his P&L due to call - stock - put, or in the other words, *long call + short stock + short put*. The loss is constant and equal to \$50. So, **the trader wins \$0.755653 per share in July prices without any risk**. This is what we call arbitrage and this is forbidden on the market. All this may happen only because the put price was not valued correctly.

4.2 Proof of put and call parity: Arbitrage reasoning

Let us explain the formula for put & call parity using the arbitrage arguments. Whatever the put and call options prices (with the same expiry dates and the same strike prices) are, if we are buying a put now (at time t), suppose we decide also to sell a call and buy a share of stock. It costs $P_t - C_t + S_t$ which turns out to be positive, - remember that S is much greater than C and P. To finance this operation, we borrow the amount $P_t - C_t + S_t$ in the bank. Since both options are **European**, we wait till July (expiry date) keeping the share of stock as well: this is our **portfolio = the set of all financial securities** which we have at the moment. Let us look at the P&L plot for this portfolio **at expiry.** Our P&L is positive and *constant*, it is equal to K:

$$P_T - C_T + S_T = K.$$

We can check it using the definitions of payoff functions:

$$P_T - C_T + S_T = (K - S_T)_+ - (S_T - K)_+ + S_T \equiv K.$$

Indeed, if $S_T > K$ then the value $(K - S_T)_+ - (S_T - K)_+ + S_T$ equals to

$$0 - (S_T - K) + S_T = K.$$

If $S_T \leq K$ then it equals to

$$(K - S_T) - 0 + S_T = K.$$

So, our profit is **deterministic, not random**. Hence, it must be exactly the same as if it were invested into a bank account with the interest rate r, that is,

$$K = \exp(r(T-t))(P_t - C_t + S_t).$$

Otherwise an **arbitrage opportunity** would arise. So, we come to the **put and call parity formula**:

$$P - C + S = \exp(-r(T-t))K.$$

Exercise. Explain why: consider two cases:

$$K > \exp(r(T-t))(P - C + S) \quad \text{and} \quad K < \exp(r(T-t))(P - C + S).$$

Hint. Use the same arguments as above (concerning \$0.7...).

Let $K > \exp(r(T-t))(P - C + S)$, or $\exp(-r(T-t))K > P - C + S$. Denote

$$\alpha = \exp(-r(T-t))K - (P - C + S).$$

We can borrow $\exp(-r(T-t))K - \alpha = P - C + S$ at time t to buy this portfolio (that is, we buy put P, we sell call C and we buy a share of stock S). Then we wait till expiry and get a profit K (see the plot) and pay our debt to the bank $exp(r(T-t))(P-C+S)$ *which is less than our gain* K. We get a pure non-random <u>riskless</u> profit

$$K - \exp(r(T-t))(P - C + S) > 0.$$

This is a riskless free lunch, i.e. an arbitrage. **So inequality** $K > \exp(r(T-t))(P - C + S)$ **is impossible**, such prices cannot exist on the market.

Similarly the opposite case can be considered: this time we sell the same portfolio (shortly) and lend the money to a bank. At expiry our profit is again positive. So $K < \exp(r(T-t))(P - C + S)$ is also wrong, the prices are not realistic. It remains the only possibility

$$K = \exp(r(T-t))(P - C + S).$$

Exercise. In the example from [WHD, p.7-10], we know the values $S, C, P, K, T-t$ for certain options. Is it possible to find the interest rate r looking at any of the options, assuming that calls are European (even though they are not)?

We conclude that if we can value a call option we will value a put option as well, using the put-call parity. In remains to find the answer to the main question: <u>how value a call option</u> ?