20 MATH3733

20.1 American options: definition

Definition 1 *An American option is a contract similar to European option with a corresponding payoff (call $(S_t - K)_+$, put $(K - S_t)_+$, general option $g_t(S_t) \geq 0$), but which may be exercised any time to expiry.*

20.2 Binomial discrete time model

- Let $\delta > 0$ be time step, $T = n\delta$ expiry, $V_T = g_T(S_T)$ payoff of the option at expiry. Then at any node x at expiry,
$$V_{n\delta}(x) = g_{n\delta}(x). \tag{1}$$
Consider time before expiry, $(n-1)\delta$. Either the option is exercised with payoff $g_{(n-1)\delta}(x)$, or it is not; in the latter case its value at the node x is $\exp(-r\delta)\tilde{V}_{(n-1)\delta}(x)$; here $\tilde{V}_{(n-1)\delta}(x)$ denotes $(\tilde{p}_x g(x^+) + (1-\tilde{p}_x)g(x^-))$ where $\tilde{p}_x$ is a corresponding implied probability at this node. So it is reasonable to exercise the option at time $(n-1)\delta$ in the node x iff $g(x) > \exp(-r\delta)\tilde{V}_{(n-1)\delta}(x)$. Hence, the value of the option in the node x at time $(n-1)\delta$ must be equal to
$$V_{(n-1)\delta}(x) = \max\left(g(x), \exp(-r\delta)\tilde{V}_{(n-1)\delta}(x))\right). \tag{2}$$

- This can be repeated by induction backward in time from T to zero: at any time $k\delta$ between 0 and $(n-1)\delta$,
$$V_{k\delta}(x) = \max\left(g(x), \exp(-r\delta)\tilde{V}_{(k+1)\delta}(x))\right), \tag{3}$$
provided $\tilde{V}_{(k+1)\delta}(x)$ has already been calculated.

- There is no explicit formula for the price V_0, like for European options, except for call and some very special options. However, the general result may be presented in the form
$$V_0 = \max_{\tau} \tilde{E}\left(e^{-r\tau} g_\tau(S_\tau)\right), \quad V_t = \max_{\tau \geq t} \tilde{E}\left(e^{-r(\tau-t)} g_\tau(S_\tau)|\mathcal{F}_t\right), \tag{4}$$
where τ is any **stopping time**, i.e. a random time which satisfies the property: for any t, the set $(\tau > t) \in \mathcal{F}_t$, that is, one must decide whether to stop at t or proceed further only using information to time t. The max in the first formula above is taken over all possible stopping times taking values $0, \delta, 2\delta, \ldots, n\delta = T$.

- We will check it only for $n = 1$. Indeed, if $t = T = \delta$, then by definition the value V_t from (4) is a correct price of the option. We only need to check that (4) gives a correct price for $t = 0$. At $t = 0$ we must decide whether to stop or wait till expiry, or, in other words, we choose between $g(x)$ and $\exp(-r\delta)(\tilde{p}_x g(x^+) + (1-\tilde{p}_x)g(x^-))$, i.e. take the maximum of these two values. This coincides with equation (2), so (4) indeed provides a fair option price.

- Once (suppose) we have found the price, the next question is when to stop, that is, how to choose τ in order to optimize the profit, actually to achieve the maximum? Notice that the correct price may not be achieved if the owner does not take care, that is, if he does not choose the optimal stopping rule. The optimal rule which follows from the algorithm above is that one must stop when for the first time $g_t(S_t) \geq \exp(-r\delta)\tilde{V}_t(S_t)$, or

$$\tau = \min(0 \leq t \leq n\delta : \ g_t(S_t) \geq e^{-r\delta}\tilde{V}_t(S_t)).$$

- An amazing fact is that for American call the price is the same as for European option (that is, given by the Black and Scholes formula) with the same parameters. Correspondingly, the optimal stopping rule for American call is wait until expiry, that is, not exercise the right to sell the option earlier. (Notice that there are modifications of the standard American call, e.g., with "discounted payoff"; the latter does not apply to all of them, in general).

20.3 Continuous time model

- It may be shown that the price $V = V(t, S)$ solves the following PDE (called PDE with a free boundary condition):

$$\begin{cases} \left[V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV\right] = 0, & \text{for } (t,x): \ V(t,x) > g_t(x), \\ \left[V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV\right] \leq 0, & \text{for } (t,x): \ V(t,x) \leq g_t(x). \end{cases} \tag{5}$$

  In other words, there are two regions in $[0, T] \times R$: one in which one should continue waiting ($V(t,x) > g_t(x)$), and the other where one should immediately stop ($V(t,x) \leq g_t(x)$). At expiry, $V(T,x) = g_T(x)$.

- Correspondingly, the optimal stopping rule is wait until $V(t, S_t) \leq g_t(S_t)$ for the first time, then stop.

- How intuitively derive the PDE: in the region where one should continue waiting, the no-arbitrage approach with replicating self-financing portfolio gives the same equations as for European options,

$$\left[Y_t + \frac{1}{2}\sigma^2 S^2 Y_{SS} + rSY_S - rY\right] = 0.$$

- It may be shown (using the stochastic calculus) that under certain assumptions (4) provides the solution to equation (5).

- How to solve equation (5)? Numerically: choose a small δ and adjust the binomial model (i.e. prices and probabilities of up and down jumps); then use the algorithm described in (1) - (3).

- Again, for American call the price coincides with that of European call (with all parameters equal), and the optimal stopping rule is wait until expiry.

- Next reading: [Shiryaev, Chapter VI].