19 MATH3733

19.1 Black-Scholes' equation

We assume that the stock price process is given by the formula

$$S_t = S_0 \exp(\mu t + \sigma W_t), \quad t \geq 0.$$

Assume that the (European) call option price is a function of time and S_t (and not its values in the past):

$$C_t = Y(t, S_t)$$

where $Y(t, S)$ is a function with two continuous derivatives in S and one in t.

Theorem 1 (Black-Scholes) *The function Y satisfies an equation*

$$(BS) \qquad \frac{\partial Y}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2} + rS\frac{\partial Y}{\partial S} - rY = 0,$$

with a terminal condition

$$Y(T, S) = (S - K)_+.$$

This equation possesses a (unique) explicit solution $(0 \leq t \leq T)$

$$Y(t, S) = S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-), \tag{1}$$

where

$$d_\pm = d_\pm(t) = \frac{\log(S/K) + (r \pm \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}.$$

Notice that the equation does not depend on μ. So, the solution does not depend on it either. This effect is important in practice: the volatility σ may be determined much more easily and precisely than the trend μ. So, even if two traders evaluate the trend differently, they may agree with the value of a call since they use the same estimated volatility.

Proof of Theorem:

1. Straightforward calculations show that the function given by the formula (1) does satisfy (BS) and the terminal condition (the latter in the sense that $\lim_{t\to T} Y(t, S) = (S - K)_+$), which is recommended to check. Though, in the end of the proof we will get to the formula independently.

2. Consider a portfolio

   $$X_t = \phi_t S_t + \psi_t B_t$$

   with some *policy* (ϕ_t, ψ_t), $0 \leq t \leq T$. Similarly to CRR model, we are looking for such a policy that

   $$Y(t, S_t) = X_t, \quad \forall\ 0 \leq t \leq T.$$

   We consider only self-financing policies, that is, by definition,

   $$dX_t = \phi_t dS_t + \psi_t dB_t,$$

   compare to the notion of self-financing policies in discrete time. The sense is that only changes of prices S_t and B_t make changes in the price of the portfolio.

3. Since $Y(t, S_t) \equiv X_t$, the stochastic differentials should also be equal:
$$dY(t, S_t) \equiv dX_t.$$
Remind that
$$dS_t = \sigma S_t dW_t + \mu' S_t\, dt, \quad \mu' = \mu + \sigma^2/2, \quad dB_t = rB_t\, dt.$$
By Itô's formula,
$$dX_t = \psi_t r B_t\, dt + \phi_t \left(\sigma S_t dW_t + \mu' S_t\, dt\right)$$
and
$$dY(t, S_t) = \left(\frac{\partial Y}{\partial t} + \mu' S \frac{\partial Y}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2}\right) dt + \frac{\partial Y}{\partial S}\, \sigma\, S_t\, dW_t.$$

4. So we get two equations:
$$\phi_t \sigma S_t = \sigma S_t \frac{\partial Y}{\partial S},$$
and
$$\phi_t \mu' S_t + \psi_t r B_t = \frac{\partial Y}{\partial t} + \mu' S \frac{\partial Y}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2}.$$

5. From the first one,
$$\phi_t = \frac{\partial Y}{\partial S} \quad \text{(this value is called \textbf{Delta} of a call option, important for hedging).}$$

6. We substitute this to the second equation and get
$$r\psi_t B_t = \frac{\partial Y}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2}.$$

7. Since
$$\psi_t B_t = X_t - \phi_t S_t = Y(t, S_t) - \phi_t S_t = Y(t, S_t) - S_t \frac{\partial Y}{\partial S},$$
we finally obtain
$$rY = \frac{\partial Y}{\partial t} + rS \frac{\partial Y}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2}.$$

8. From results of lecture 16 (how to solve PDEs via SDEs), it follows that solution of equation (BS) must have a form
$$Y(0, S) = \exp(-rT) E(S_0 \exp((r - \sigma^2/2)T + \sigma\sqrt{T}\eta) - K)_+, \quad \eta \sim \mathcal{N}(0, 1) \quad (2)$$
(compare with last formula from lecture 10).

9. Calculus leading from (2) to (1) was fulfilled in **Exercises 3 and 4**, and will be repeated in the final handout.

Exercise 1 *Show that the portfolio with $\phi_t = \partial Y(t, S_t)/\partial S$ and $\psi_t = (Y(t, S_t) - S_t \partial Y(t, S_t)/\partial S)/B_t$ is indeed self-financing. Hint: show that $X_t = Y(t, S_t)$ and find $dY(t, S_t)$ taking into account the Black-Scholes equation.*