18 MATH3733

18.1 Solving PDE via SDE: constant coefficients – 1

Consider a Partial Differential Equation (PDE) with constant coefficients,

$$u_t + \frac{1}{2}\sigma^2 u_{xx} + bu_x = 0, \quad u(T,x) = g(x) \tag{1}$$

("heat equation with a drift"). Consider a solution to SDE

$$dX_t = \sigma dW_t + bdt, \ t \ge 0, \quad \text{with initial data } X_0 = x. \tag{2}$$

Then,

$$X_t = x + \sigma W_t + bt \sim \mathcal{N}(x + bt, \sigma^2 t). \tag{3}$$

Suppose $u(t,x)$ is a solution to (1). Apply the Itô formula to $u(t,X_t)$:

$$du(t,X_t) = [u_t(t,X_t) + \frac{1}{2}\sigma^2 u_{xx}(t,X_t) + bu_x(t,X_t)]\,dt + \sigma u_x(t,X_t)\,dW_t = \sigma u_x(t,X_t)\,dW_t.$$

Hence,

$$u(T,X_T) - u(0,x) = \int_0^t \sigma u_x(s,X_s)\,dW_s,$$

and therefore (notation E_x means **expectation for the process starting from** x)

$$u(0,x) = E_x\,u(T,X_T) = E_x\,g(X_T); \quad \text{likewise, } u(t,x) = E_x\,g(X_{T-t}). \tag{4}$$

18.2 Solving PDE via SDE: constant coefficients – 2

Consider a PDE with constant coefficients and a "potential" $(-ru)$,

$$u_t + \frac{1}{2}\sigma^2 u_{xx} + bu_x - ru = 0, \quad u(T,x) = g(x). \tag{5}$$

Consider a solution to SDE (2). Suppose $u(t,x)$ is a solution to (5). Apply the Itô formula to $\exp(-rt)u(t,X_t)$:

$$d\exp(-rt)u(t,X_t) = \exp(-rt)[u_t(t,X_t) + \frac{1}{2}\sigma^2 u_{xx}(t,X_t) + bu_x(t,X_t) - ru(t,X_t)]\,dt$$

$$+ \exp(-rt)\sigma u_x(t,X_t)\,dW_t = \exp(-rt)\sigma u_x(t,X_t)\,dW_t.$$

Hence,

$$\exp(-rT)u(T,X_T) - u(0,x) = \int_0^t \exp(-rs)\sigma u_x(s,X_s)\,dW_s,$$

and so

$$u(0,x) = \exp(-rT)E_x\,g(X_T); \quad \text{likewise, } u(t,x) = \exp(-r(T-t))E_x\,g(X_{T-t}). \tag{6}$$

18.3 Solving PDE via SDE: linear coefficients – 1

Consider a PDE with coefficients $\sigma^2 x^2/2$ and $b\,x$,

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + bxu_x = 0, \quad u(T,x) = g(x). \tag{7}$$

Consider a solution to linear SDE

$$dX_t = \sigma X_t dW_t + bX_t dt, \quad X_0 = x. \tag{8}$$

Then,

$$X_t = x\exp(bt - \sigma^2 t/2 + \sigma W_t). \tag{9}$$

Suppose $u(t,x)$ is a solution to (1). Apply the Itô formula to $u(t,X_t)$:

$$du(t,X_t) = [u_t(t,X_t) + \frac{1}{2}\sigma^2 X_t^2 u_{xx}(t,X_t) + bX_t u_x(t,X_t)]\,dt + \sigma X_t u_x(t,X_t)\,dW_t$$

$$= \sigma X_t u_x(t,X_t)\,dW_t.$$

Hence, $u(T,X_T) - u(0,x) = \int_0^t \sigma X_s u_x(s,X_s)\,dW_s$, and therefore

$$u(0,x) = E_x\, g(X_T); \quad \text{likewise, } u(t,x) = E_x\, g(X_{T-t}). \tag{10}$$

18.4 Solving PDE via SDE: linear coefficients – 2

Consider a PDE with coefficients $\sigma^2 x^2/2$ and $b\,x$, and a "potential" $(-ru)$,

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + bxu_x - ru = 0, \quad u(T,x) = g(x). \tag{11}$$

Consider a solution to SDE (8). Suppose $u(t,x)$ is a solution to (11). Apply the Itô formula to $\exp(-rt)u(t,X_t)$:

$$d\exp(-rt)u(t,X_t) = \exp(-rt)[u_t(t,X_t) + \frac{1}{2}\sigma^2 X_t^2 u_{xx}(t,X_t) + bX_t u_x(t,X_t) - ru(t,X_t]\,dt$$

$$+\sigma X_t u_x(t,X_t)\,dW_t = \exp(-rt)\sigma X_t u_x(t,X_t)\,dW_t.$$

Hence, $\exp(-rT)u(T,X_T) - u(0,x) = \int_0^t \exp(-rs)\sigma X_s u_x(s,X_s)\,dW_s$, and therefore

$$u(0,x) = \exp(-rT)\,E_x\, g(X_T); \quad \text{likewise, } u(t,x) = \exp(-r(T-t))E_x\, g(X_{T-t}). \tag{12}$$

Note. One can now check directly corresponding equations using the distributions of X_t in each case: in (4) and (6) it is a normal distribution, and in (10) and (12) a log-normal one.