MATH3733

Additional handout

Solving PDE via SDE: linear coefficients – 3

Consider a PDE with coefficients $\sigma^2 x^2/2$ and $b\,x$, and a "potential" $(-ru)$, *and a right hand side* (-1),

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + bxu_x - ru = -1, \quad u(T,x) = g(x). \tag{1}$$

Consider a solution to SDE

$$dX_t = \sigma X_t dW_t + bX_t dt, \qquad X_0 = x.$$

Suppose $u(t,x)$ is a solution to (1). Apply the Itô formula to $\exp(-rt)u(t,X_t)$:

$$d\exp(-rt)u(t,X_t) = \exp(-rt)[u_t(t,X_t) + \frac{1}{2}\sigma^2 X_t^2 u_{xx}(t,X_t) + bX_t u_x(t,X_t) - ru(t,X_t]\,dt$$

$$+\sigma X_t u_x(t,X_t)\,dW_t = \exp(-rt)[-1]\,dt + \exp(-rt)\sigma X_t u_x(t,X_t)\,dW_t.$$

Hence, in the integral form,

$$\exp(-rT)u(T,X_T) - u(0,x) = -\int_0^T \exp(-rs)\,ds + \int_0^T \exp(-rs)\sigma X_s u_x(s,X_s)\,dW_s,$$

and, therefore, after taking expectation,

$$\exp(-rT)\,E_x\,g(X_T) - u(0,x) = +\frac{1}{r}\,\exp(-rs)\Big|_0^T,$$

or, equivalently,

$$\boxed{u(0,x) = \exp(-rT)\,E_x\,g(X_T) + \tfrac{1}{r}\left[1 - \exp(-rT)\right].}$$

Itô's formula applied to complex valued functions

Let $(W_t,\, t \geq 0)$ be a Wiener process, and $(f(x),\, x \in R^1)$ be some complex valued function, that is,

$$f(x) = A(x) + i\, B(x).$$

Let us assume that functions A and B have two continuous derivatives. Then, we can apply Itô's formula to $A(W_t)$ and to $B(W_t)$:

$$dA(W_t) = A'(W_t)\, dW_t + \frac{1}{2} A''(W_t)\, dt,$$

and

$$dB(W_t) = B'(W_t)\, dW_t + \frac{1}{2} B''(W_t)\, dt.$$

In the integral form, equivalently, we have,

$$A(W_t) - A(0) = \int_0^t A'(W_s)\, dW_s + \int_0^t \frac{1}{2} A''(W_s)\, ds,$$

and

$$B(W_t) - B(0) = \int_0^t B'(W_s)\, dW_s + \int_0^t \frac{1}{2} B''(W_s)\, ds.$$

Hence, multiplying the latter line by $i = \sqrt{-1}$ and taking a sum, we get the following version of Itô's formula:

$$A(W_t) + i\, B(W_t) - A(0) - i\, B(0) = \int_0^t \left(A'(W_s) + iB'(W_s)\right)\, dW_s + \int_0^t \frac{1}{2} \left(A''(W_s) + iB''(W_s)\right)\, ds.$$

Equivalently,

$$\boxed{f(W_t) - f(0) = \int_0^t f'(W_s)\, dW_s + \int_0^t \tfrac{1}{2} f''(W_s)\, ds.}$$