17 MATH3733

17.1 Black-Scholes' formula via Cameron-Martin-Girsanov

This is another way to find the value of call option which is better in the sense that you need not know how to solve partial differential equations; all you need is to know Gaussian integrals from Exercises 1 and 2. We assume again that the stock price is given by the equation

$$S_t = S_0 \exp(\mu t + \sigma W_t).$$

We are going to find a new probability measure $\tilde{P}$ under which $e^{-rt}S_t$, $0 \le t \le T$, is a martingale. For this aim we represent W_t in the form (c is a constant which will be chosen later)

$$W_t = \tilde{W}_t - ct, \quad t \ge 0,$$

which is equivalent to

$$\tilde{W}_t = W_t + ct, \quad t \ge 0.$$

Due to Cameron-Martin-Girsanov theorem, ($\tilde{W}_t$, $0 \le t \le T$) is a martingale under the measure $\tilde{P}$ defined by its density

$$\gamma_T = \exp(-cW_T - c^2 T/2).$$

Now, S_t may be represented using $\tilde{W}_t$ by the formula

$$S_t = S_0 \exp(\mu t + \sigma(\tilde{W}_t - ct)) = S_0 \exp((\mu - \sigma c)t + \sigma \tilde{W}_t),$$

so that

$$\exp(-rt)S_t = S_0 \exp((\mu - \sigma c - r)t + \sigma \tilde{W}_t). \tag{1}$$

17.2

Lemma 1 *Let $c = (\mu - r + \sigma^2/2)/\sigma$. Then $e^{-rt}S_t$, $0 \le t \le T$, is a martingale under $\tilde{P}$.*

Proof. By the Itô formula, the stochastic differential of $\exp(-rt)S_t$ via $\tilde{W}_t$ (see (1)) is equal to

$$d\,(\exp(-rt)S_t) = S_t[\mu - \sigma c - r + \sigma^2/2]\,dt + S_t \sigma\, d\tilde{W}_t.$$

The choice of c means that $\mu - \sigma c - r + \sigma^2/2 = 0$. So, $d\,(\exp(-rt)S_t) = S_t\sigma\, d\tilde{W}_t$. Therefore, $(\exp(-rt)S_t)$ is a martingale under $\tilde{P}$.

Assume that under the implied probability measure $\tilde{P}$ the price C_t of the European call option is a discounted martingale, that is, the process $\exp(-rt)C_t$ is a martingale. The justification of such assumption is that this is true for the CRR model with small steps $\delta > 0$; hence, it should be true in the limit.

17.3

Theorem 1 (Black-Scholes) *The function C_t is given by the formula $(0 \le t \le T)$*

$$[BS] \qquad Y(t,S) = S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-),$$

where

$$d_\pm = d_\pm(t) = \frac{\log(S/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Proof:

1. It suffices to show the formula for $t = 0$.

2. All we have to do is calculate the expectation (remember that $\mu - c\sigma = r - \sigma^2/2$)

$$C_0 = e^{-rT}\tilde{E}(S_T - K)_+ \tag{2}$$

$$= e^{-rT}\tilde{E}\left(S_0 e^{\mu T + \sigma(\tilde{W}_T - cT)} - K\right)_+$$

$$= e^{-rT}\tilde{E}\left(S_0 e^{\left(r-\sigma^2/2\right)T + \sigma\tilde{W}_T} - K\right)_+.$$

3. Since $\tilde{W}_T \sim \mathcal{N}(0,T)$ under measure $\tilde{P}$, the random variable $\tilde{W}_T$ has the same normal distribution as $\sqrt{T}\xi$ where $\xi \sim \mathcal{N}(0,1)$ (always under measure $\tilde{P}$). So we may rewrite the last expression in the form

$$C_0 = e^{-rT}\tilde{E}\left(S_0 e^{\left(r-\sigma^2/2\right)T + \sigma\sqrt{T}\xi} - K\right)_+$$

$$= e^{-rT}\int\left(S_0 e^{\left(r-\sigma^2/2\right)T + \sigma\sqrt{T}z} - K\right)_+ p_1(z)\,dz.$$

We integrate over the half-line

$$z:\ S_0 e^{\left(r-\sigma^2/2\right)T + \sigma\sqrt{T}z} \ge K$$

(for all other z the expression in the brackets is equal to zero), or equivalently,

$$z:\ e^{\left(r-\sigma^2/2\right)T + \sigma\sqrt{T}z} \ge K/S_0,$$

or equivalently,

$$z:\ \left(r - \sigma^2/2\right)T + \sigma\sqrt{T}z \ge \log(K/S_0),$$

or equivalently,

$$z:\ \sigma\sqrt{T}z \ge \log(K/S_0) - \left(r - \sigma^2/2\right)T,$$

or, at last,

$$z \ge \frac{\log(K/S_0) - (r - \sigma^2/2)\,T}{\sigma\sqrt{T}}.$$

4. So (we write S instead of S_0),

$$C_0 = e^{-rT} \int_{\frac{\log(\frac{K}{S}) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} \left(S e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z} - K \right) p_1(z)\, dz$$

$$= e^{-rT} S \int_{\frac{\log(\frac{K}{S}) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z} p_1(z)\, dz - e^{-rT} K \int_{\frac{\log(\frac{K}{S}) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} p_1(z)\, dz$$

$$= e^{-rT} S \int_{-\infty}^{-\frac{\log(\frac{K}{S}) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}} e^{\left(r - \frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}z} p_1(z)\, dz - e^{-rT} K \int_{-\infty}^{-\frac{\log(\frac{K}{S}) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}} p_1(z)\, dz$$

$$= e^{-rT} S \int_{-\infty}^{\frac{\log(\frac{S}{K}) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}} e^{rT}\, e^{\left(-\frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}z} p_1(z)\, dz - e^{-rT} K \int_{-\infty}^{\frac{\log(\frac{S}{K}) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}} p_1(z)\, dz$$

$$= S \int_{-\infty}^{\frac{\log(\frac{S}{K}) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}} e^{\left(-\frac{\sigma^2}{2}\right)T - \sigma\sqrt{T}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz - e^{-rT} K \Phi\left(\frac{\log(\frac{S}{K}) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right)$$

$$= S \int_{-\infty}^{\frac{\log(\frac{S}{K}) + \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-(z + \sigma\sqrt{T})^2/2}\, dz - e^{-rT} K \Phi\left(\frac{\log(S_0/K) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right)$$

$$= S \int_{-\infty}^{\frac{\log(\frac{S}{K}) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-(z')^2/2}\, dz' - e^{-rT} K \Phi\left(\frac{\log(\frac{S}{K}) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right)$$

$$= S\Phi\left(\frac{\log(\frac{S}{K}) + (r + \sigma^2/2)\, T}{\sigma\sqrt{T}} \right) - e^{-rT} K \Phi\left(\frac{\log(\frac{S}{K}) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right).$$