MATH3733 EXERCISES 2

All questions of Exercises 2 and 3 will constitute parts of the calculus in the proofs of the Black-Scholes formula and the Cameron-Martin-Girsanov theorem.

1 Assuming that

$$\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \exp(-x^2/2)\, dx = 1,$$

show the following equality:

$$\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \exp(x - 1/2) \exp(-x^2/2)\, dx = 1.$$

2 Prove $(T > 0)$

(a)

$$\int_{-\infty}^{\infty} \frac{1}{(2\pi T)^{1/2}} \exp(-x^2/(2T))\, dx = 1,$$

(b)

$$\int_{-\infty}^{\infty} \frac{1}{(2\pi T)^{1/2}} \exp(x - T/2) \exp(-x^2/(2T))\, dx = 1,$$

(c) and (with any μ)

$$\int_{-\infty}^{\infty} \exp(x) \frac{1}{(2\pi\sigma^2)^{1/2}} \exp(-(x-\mu)^2/(2\sigma^2))\, dx = \exp(\mu + \sigma^2/2).$$

3 Prove the assertions $(T > 0)$

$$\int_{a}^{\infty} a\, \frac{1}{(2\pi T)^{1/2}} \exp(-x^2/(2T))\, dx = a\,\Phi\left(\frac{-a}{\sqrt{T}}\right),$$

where $\Phi(z) = \int_{-\infty}^{z} \frac{1}{(2\pi)^{1/2}} \exp(-x^2/2)\, dx$ is a cumulative distribution function of the standard normal random value (= Laplace function).

4 Prove (always $T > 0$)

$$\int_{a}^{\infty} b \exp(x - T/2)\, \frac{1}{(2\pi T)^{1/2}} \exp(-x^2/(2T))\, dx = b\,\Phi\left(\frac{-a+T}{\sqrt{T}}\right).$$

5* $X_k \sim \mathcal{N}(a, \sigma^2),\ 1 \le k \le n$ – IID. Show that

$$\sum_{k=1}^{n} X_k \sim \mathcal{N}(na, n\sigma^2).$$