7^* Lecture: Central Limit Theorem – Another Proof

1. **Theorem** Let $X_1, X_2, \ldots$ are IIDRV's with $EX_1 = 0$, $var(X_1) = \sigma^2 \in (0, \infty)$, $S_n = \sum_{k=1}^n X_k$. Then,

$$S_n/\sqrt{n\sigma^2} \Longrightarrow \mathcal{N}(0,1) \tag{1}$$

2. **Proof (not compulsory)**: Hint-1. It suffices to show convergence

$$Eg(S_n/\sqrt{n\sigma^2}) \to Eg(Z) \quad (Z \sim \mathcal{N}(0,1)) \tag{2}$$

for $g \in C_u^2$ – twice continuously differentiable and bounded with two derivatives, such that the second derivative is uniformly continuous. Also, it suffices to assume $\sigma^2 = 1$.

3. Hint-2. Let $Z_1, Z_2, \ldots$ – IIDRV's $\sim \mathcal{N}(0,1)$, also independent from X_i's. Then $\sum_{k=1}^n Z_k/\sqrt{n} \sim \mathcal{N}(0,1)$, and it suffices to show

$$Eg(n^{-1/2}S_n) - Eg(n^{-1/2}\sum_{k=1}^n Z_k) \to 0, \; n \to \infty. \tag{3}$$

4. Hint-3. To prove (3), we use Taylor's expansion up to the second order:

$$\begin{aligned}
&Eg(n^{-1/2}\textstyle\sum_{k=1}^n X_k) - Eg(n^{-1/2}\sum_{k=1}^n Z_k) \qquad (4)\\
&= \{Eg(n^{-1/2}(X_1 + \ldots + X_n)) - Eg(n^{-1/2}(X_1 + \ldots + X_{n-1} + Z_n))\}\\
&+\{Eg(n^{-1/2}(X_1 + \ldots + X_{n-1} + Z_n)) - Eg(n^{-1/2}(X_1 + \ldots + X_{n-1} + Z_n))\}\\
&+ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+\{Eg(n^{-1/2}(X_1 + Z_2 \ldots + Z_n)) - Eg(n^{-1/2}(Z_1 + \ldots + Z_{n-1} + Z_n))\}.
\end{aligned}$$

All n terms here are considered similarly, and we restrict ourselves to the first one. Due to Taylor's expansion (twice apply the First Theorem of the Calculus for composite functions, $g(x) - g(y) = \int_0^1 g'(\alpha x + (1-\alpha)y)(x-y)\,d\alpha$) at $n^{-1/2}S_{n-1}$:

$$g(n^{-1/2}(X_1 + \ldots + X_n)) - g(n^{-1/2}(X_1 + \ldots + X_{n-1} + Z_n))$$

$$= g(n^{-1/2}S_{n-1}) + g'(n^{-1/2}S_{n-1})n^{-1/2}X_n$$

$$
\begin{aligned}
&+\int_0^1\int_0^1 g''(n^{-1/2}S_{n-1}+\alpha\beta n^{-1/2}X_n)\alpha n^{-1}X_n^2 d\alpha d\beta\\
&-g(n^{-1/2}S_{n-1})-g'(n^{-1/2}S_{n-1})n^{-1/2}Z_n\\
&-\int_0^1\int_0^1 g''(n^{-1/2}S_{n-1}+\alpha\beta n^{-1/2}Z_n)\alpha n^{-1}X_n^2 d\alpha d\beta\\
&= g(n^{-1/2}S_{n-1})+g'(n^{-1/2}S_{n-1})n^{-1/2}X_n+(1/2)g''(n^{-1/2}S_{n-1})n^{-1}X_n^2\\
&+\int_0^1\int_0^1 (g''(n^{-1/2}S_{n-1}+\alpha\beta n^{-1/2}X_n)-g''(n^{-1/2}S_{n-1}))\alpha n^{-1}X_n^2 d\alpha d\beta\\
&-g(n^{-1/2}S_{n-1})-g'(n^{-1/2}S_{n-1})n^{-1/2}Z_n-(1/2)g''(n^{-1/2}S_{n-1})n^{-1}Z_n^2\\
&-\int_0^1\int_0^1 (g''(n^{-1/2}S_{n-1}+\alpha\beta n^{-1/2}Z_n)-g''(n^{-1/2}S_{n-1}))\alpha n^{-1}X_n^2 d\alpha d\beta.
\end{aligned}
$$

Take expectation using $EX_n = EZ_n = 0$ и $EX_n^2 = 1 = EZ_n^2$. All integrals are cancelled out except two:

$$
\begin{aligned}
&Eg(n^{-1/2}(X_1+\ldots+X_n)) - Eg(n^{-1/2}(X_1+\ldots+X_{n-1}+Z_n))\\
&= E\int_0^1\int_0^1 (g''(n^{-1/2}S_{n-1}+\alpha\beta n^{-1/2}X_n)-g''(n^{-1/2}S_{n-1}))\alpha n^{-1}X_n^2 d\alpha d\beta\\
&-E\int_0^1\int_0^1 (g''(n^{-1/2}S_{n-1}+\alpha\beta n^{-1/2}Z_n)-g''(n^{-1/2}S_{n-1}))\alpha n^{-1}Z_n^2 d\alpha d\beta.
\end{aligned}
$$

5. Let us show that both integrals are of the order $o(1/n)$. Denote $\rho(u) = \sup_x \sup_{|v|\le u}(g''(x+v)-g''(x))$ – the modulus of continuity for g''. The function ρ is increasing, continuous, bounded, and $\rho(0)=0$.

   It can be shown that $E\rho(n^{-1/2}X_n)X_n^2 \to 0$, $n\to\infty$ [follows from the Theorem "on bounded convergence under the integral" (Lebesgue); if *additionally* assume $E|X_1|^3<\infty$ and take $g\in C_b^3$, then $\rho(u)\le C|u|$, so that $E\rho(n^{-1/2}X_n)X_n^2 \le n^{-1/2}E|X_1|^3 \to 0,\ n\to\infty$]. Thus,

$$
\begin{aligned}
&|Eg(n^{-1/2}(X_1+\ldots+X_n)) - Eg(n^{-1/2}(X_1+\ldots+X_{n-1}+Z_n))|\\
&\le E\int_0^1\int_0^1 \rho(\alpha\beta n^{-1/2}X_n)\alpha n^{-1}X_n^2 d\alpha d\beta\\
&+E\int_0^1\int_0^1 \rho(\alpha\beta n^{-1/2}Z_n)\alpha n^{-1}Z_n^2 d\alpha d\beta = n^{-1}o(1).
\end{aligned}
$$

   Since all other terms are estimated similarly, we finally get,

$$
|Eg(n^{-1/2}(X_1+\ldots+X_n)) - Eg(n^{-1/2}(Z_1+\ldots+Z_n))| = o(1),\ n\to\infty,
$$

   which shows (3).