# MATH3733 – Stochastic Financial Modelling

## "beta– version" 30.11.2008

- **Semester 1; Year 2008/2009**
- **Lecturer:** Prof. Alexander Veretennikov, e-mail: A.Veretennikov@leeds.ac.uk, office 10.18d; home-page: http://www.maths.leeds.ac.uk/~veretenn/
- **Programmes of Study:** BSc and MMath Mathematics; Mathematical Studies; Mathematics with Finance; Joint Honours (Arts); Joint Honours (Science).
- **Pre-requisites:** MATH2750, or equivalent. This means that the notion of **Markov processes** will be exploited.
- **Objectives:** To develop a general methodology based on stochastic analysis for the pricing of European call and put options on the market of options. By the end of this module, students should be able to:
  (a) describe certain main instruments available in financial markets – underlying assets, options;
  (b) use filtrations and martingales to model any evolving state of knowledge in a *fair market*;
  (c) use appropriate stochastic methods to evaluate return rates on risky assets;
  (d) value options using the Black-Scholes theorem.

**Form of assessment:** One 3 hour examination at end of semester (80%). Coursework (20%) – practicals.

## Literature:


1. M.Baxter, A.Rennie, Financial Calculus, Cambridge University Press, Cambridge, 1996. [BR]
2. S.N.Neftci, An Introduction to the Mathematics of Financial Derivatives, Academic Press, San Diego et al., 1996. [N]
3. P.Wilmott, S.Howison, J.Dewynne, The Mathematics of Financial Derivatives, Cambridge University Press, Cambridge, 1995. [WHD]
4. A.N.Shiriaev, Essentials of stochastic finance : facts, models, theory. Singapore River Edge, N.J. : World Scientific, 1999. [Sh]
5. Some additional references have been provided in the handouts.

# PLAN OF THE COURSE

You are going to study **two major parts:** discrete time and continuous time theory of option pricing. The discrete time part is easier; however, the main goal of the course is to learn continuous time counterpart, in particular, the **Black-Scholes formula.**

This is a mathematics, thus, it is rigorous. Hence, the first advice is: **learn exact definitions** of all notions used in the theory. Another advise is to learn definitions of all basic notions you already know from previous years: it will be **not** sufficient to know them intuitively for this course. In the very beginning there will be a test which will help you realize what is needed.

To help you follow the whole course, here is its (approximate) plan below.

- European options, introduction. The main question is: how to value a European option? You will learn the notion of arbitrage, one of the main tools of the theory. Lectures 1 – 4.

- Reminder of basic probability notions, and some new theoretical material: lectures 5 – 6. You will learn new terms such as conditional expectation, martingale and discounted martingale.

- Simplest "binomial" model, option pricing for this model: lectures 6 – 9. You will also learn that option price is a (discounted) martingale after a strange transformation: change of probabilities; new probabilities are called implied ones, a very important notion. How better acquire the concept of change of probabilities: read the first two pages in the book [BR] "The parable of the bookmaker" which explains that the latter values the bets not following real chances (probabilities) of each horse to win, but rather following the bets themselves, – this is the change of probabilities in practice.

- Central Limit Theorem (which you started studying in your year one); application of this theorem to the first appearance of the Black-Scholes formula, the goal of the course – lectures 10 – 13.

- Elements of stochastic calculus: this is a new material and it will appear difficult. I will try to explain everything at the intuitive level. However, there is no way to learn this part other than real hard study, like ancient Greeks told that "There is no royal way to mathematics": **you** must learn it. New notions you will learn: Brownian motion, stochastic integral, Itô's formula, stochastic differential equation, Cameron - Martin - Girsanov transformation of measure. Lectures 14 – 20.

- Continuous time model of stock price; Black-Scholes formula via transformation of measure. Lectures 21 – 24.

  Another way of establishing Black – Scholes formula is based on he original idea of its authors, on solving a certain parabolic PDE, called Black–Scholes partial differential equation. The part related to this way will be given as *advanced reading*.

- Two lectures for revision, lectures 25 – 26. Other slots - for example classes.
- Probability background will be given as a third part. Some part of it is a revision, and some is a new material. A must from that part is, in particular (but not limited to), conditional distributions and expectations, CLT, characteristic functions, Gaussian densities, definitions of Markov process, martingale, discounted martingale.

## 0.1 Introduction

The modern theory of stochastic financial mathematics was created during last 25-30 years, though started in the very beginning of 20th century by L. Bachelier, see http://cepa.newschool.edu/het/profiles/bachelier.htm. Two of the founders of the modern theory received a Nobel prize in economics in 1997: Robert S. Merton (http://almaz.com/nobel/economics/1997a.html) and Myron S. Scholes (http://almaz.com/nobel/economics/1997b.html). One more Nobel prize winner related to stochastic finance is Paul A. Samuelson (http://www.almaz.com/nobel/economics/1970a.html). The theory is indeed used in financial world.

**Knowledge required for a person who wishes to become a financial analyst:**
1. economics background;
2. stochastic integrals, Itô's formula, martingales, Girsanov's transformation;
3. approximation methods of solving partial differential equations (PDEs) and stochastic differential equations (SDEs);
4. computer modelling.

# Part I – discrete time binomial model

# 1 Elementary economics background, some notions

## 1.1 Markets

- **Stock markets.** *Stock* is a printed paper issued by a company to attract investments, money. Usually with dividends. [NYSE (New York Stock Exchange), NASDAQ (National Association of Securities Dealers Automatic Quotation), EUSDAQ, London Stock Exchange, et al.]

- **Bond markets.** "*Bond*: a printed paper given by a government, city or business company saying that money has been received and and will be paid back, usually with interest," which is fixed.

- **Currency markets or foreign exchange markets.**

- **Commodity markets** (oil, gold, copper, wheat, electricity, etc).

- **Futures and options markets.** "*Futures*: goods and stocks bought at prices agreed upon at the time of the purchase but paid for and delivered afterwards." "*Option*: the right to buy or sell something at a certain price within a certain period of time." [CBOE (Chicago Board Option Exchange).]

## 1.2 Why study option models and option pricing?

Because there is a huge financial industry which requires a corresponding mathematics. Options is just a simplest example.

## 1.3 What is option

A *European call option* is a *contract* with the following conditions ([WHD, section 1.2]):

- At a prescribed time in the future, known as the expiry date or expiration date or simply expiry $T$, the holder of the option *may*

- purchase a prescribed asset, known as an *underlying asset* for a

- prescribed amount, known as the *exercise price* or *strike price*, $K$.

- The call option can be purchased for a price $C_t$ called the *premium* at time $t$, $0 \le t < T$. The price varies with $t$.

Emphasize, it is a *right* and not an *obligation* for the holder of this contract. The other party of the contract who is known as the *writer*, does have a potential obligation: he **must** sell the asset if the holder chooses to buy it.

The problem of option pricing is the problem how determine the right premium.

A *European put option* is similar, but gives the owner the right to *sell* an asset at a specified price ar expiration.

In contrast to European options, *American options* can be exercised *any time* between the writing and the expiration of the contract. Option of any type can be traded *all over the world,* independently of their titles, as well Asian, Russian, etc. Some brief introduction to American option pricing will be given in the last lecture.

We always suppose that there is *no commissions or fees, nor dividends*, though all this in principle may be considered in more advanced theories. Also may be considered several underlying assets. On the markets which we study any product which exists there *may be bought and sold* in any quantity, even non-integer. "Short–selling" is available in our market models: this means that a dealer is allowed to borrow not money, but, for example, an asset, and then he may sell it, with the obligation to return at prescribed time the same asset.

# 2 More about options

## 2.1 European Put Option: example

Suppose today is April 2001 and the price of one share of stock XYZ is $47. A corresponding line in a newspaper would read

XYZ July-2001 50 put 8

**How to understand this?**

1. XYZ is a name of a company.
2. July 2001 is an *expiry date*; remember in the UK the expiry date is always *18.00 of the third Wednesday of a month.*
3. $ 50 is an *exercise price* or *strike price* of this *put* option. That is, at expiry the owner has a right to *sell* a share of stock for $50. If the real market price will be *less*, the owner will exercise his right, i.e. he will *buy a share of stock for the market price and immediately sell it to the writer of the put option contract for the strike price*; hence, he gets a profit. If the market price at expiry is greater than $50 the owner of this put would lose money invested in this option.
4. Put means put option.
5. $8 is the price of this option today (we assumed it is April 2001). This is the trading price or *premium price* of the option. Notice it is much less than the strike price.

**What may happen in July** if I buy the European option put and wait till July?

(a) One share of stock costs \$45 (for example). Then I will exercise my right to sell a share of stock for the exercise price, \$50. I buy a stock *shortly* for \$45 and immediately sell it for \$50. My gain or payoff is \$50 - \$45 = \$5 (ignoring for the moment initial payment \$8).

(b) The stock costs \$54. Then it is not reasonable to exercise my right. The gain or payoff is zero.

Notice that the **profit or loss (P&L)** might be negative even if the profit is positive: in our example in case (a) the profit is equal to \$5 but the correspondent P&L is - \$3. However, if the stock at expiry costs less than \$42, the P&L would be positive. Also notice that the definition above of the P&L notion is correct only if the interest rate $r = 0$; otherwise we are to compare the two values which should be transformed to the same date, as a money value depends on time. We will not operate with this notion, it is mentioned just for orientation.

## 2.2 Why buy options, not bonds, or commodities?

1. **First reason - speculation.** There are chances to win a profit spending a small amount of money (it is called *leverage* or *gearing*), much less than if buying or selling underlying stocks: an option price is much less than that of a share of stock.

   (a) **For the European call option.**

      i. **For the owner** (called **long position**). A possible **loss** of the owner of a call option is bounded by its premium which is always much less than the price of the corresponding underlying asset – share of stock. On the other hand, a possible profit is only slightly less than a possible profit for the share of stock.

      ii. **For the writer** (called **short position**). A possible loss is unbounded, but still it is less than that for the share of stock. Also, there is a possibility that the price of a share of stock at expiry will be less than the strike price (=exercise price); in this case the writer keeps his premium.

   (b) **For the European put option.**

      i. **For the owner** (**long position**). A loss is bounded by the premium – which is much less than the stock price, – while there is a chance to gain a profit if the strike price at expiry is greater than the stock price.

      ii. **For the writer** (**short position**). A loss = strike price minus stock price (provided this difference > 0) minus premium: this is less (- premium) than if selling a share of stock, and there is a chance to keep the premium.

2. **Second reason - hedging.** This is a very essential concept: options are used as some sort of **insurance** by dealers on the market. If a dealer is

   (a) **buying** some asset (**long position**) and wishing to reduce possible losses because of **decreasing** of its price, he might buy corresponding put option; this will

      i. reduce possible profit, but

      ii. give a cheap way to eliminate possible losses;

   (b) if he is **selling** some asset (**short position**), he might also wish to reduce his possible loss because of a possible **increase** of the price for this stock; then he might buy corresponding call option which again

      i. reduces his profit in the case if the stock price goes down, but

      ii. eliminates any risk for the case of the increase of the stock price.

# 3 Notations. First Theorem – put and call parity

## 3.1 European Call option: mathematical notations

**Assume that the standard interest rate** $r = 0$**.** Denote the price of our underlying stock and our call option at time $t$ by $S_t$ and $C_t$ (from Call) correspondingly. The price $S_t$ is given to us by the market, and it is natural to assume that it is **random** (cf. www.bloomberg.com). How can the market determine a correct price for the option, $C_t$? Assume that the price $C_t$ depends on values $S_t$, $K$ and $T-t$, which can be represented in the form

$$C_t = F(S_t, K, T-t).$$

At expiry, $T$, the price $C_T$ is known given the price $S_T$: namely, $C_T$ can be either $S_T - K$ or zero. Indeed, if the option is expiring **out-of-money**, that is,

$$S_T < K,$$

then the option will have no value. The underlying asset can be purchased in the market for $S_T$, so no option holder will exercise the right to buy the underlying asset for $K$. Thus,

$$S_T < K \Longrightarrow C_T = 0.$$

But, if the option expires **in-the-money**, that is,

$$S_T > K,$$

the option will have some value. One should clearly exercise the option. One can buy the underlying security for $K$ and sell it for a higher price $S_T$. The net profit will be $S_T - K$. In the other words,

$$S_T > K \Longrightarrow C_T = S_T - K.$$

A shorthand notation for both possibilities (equality included in either of them):

$$C_T = \max[S_T - K, 0] = (S_T - K)_+.$$

Sometimes they also use an expression **at-the-money** in the case $S_T = K$. All terms (*in-, out-, at-the-money*) are also used **for current option prices.** It turns out that the price, or **payoff** before expiration is **represented by a smooth curve which tends to** $(S_T - K)_+$ **as** $t \to T$.

Now, we look at this function (as function of the variable $S_T$ given $K$) taking into account the we paid $C_0$ at time 0: our profit is equal to

$$(S_T - K)_+ - C_0.$$

(**Profit = payoff − cost of the option,** the cost being equal to $C_0$.) The value $(S_T - K)_+$ is called **intrinsic value** of the European call option. If one buys the option at the moment $t < T$, he/she pais $C_t$, and at expiry gets a profit

$$(S_T - K)_+ - C_t.$$

## 3.2 Put and call parity

It turns out that the values of **call** and **put** options with the same parameters – i.e. the same exercise price, same expiry and same date of writing the contract (current time) – are closely connected, so that if we know the price of the call option (and all other parameters) then we can calculate the price of the put option by a simple formula.
**Assume that the interest rate $r = 0$.** In this case the formula reads

$$C_t - P_t = S_t - K \qquad (\forall t \le T).$$

Thus,

$$P_t = C_t - S_t + K.$$

If $K = 50$, $S = 47$, $C = 5$, we can calculate $P = 50 - 47 + 5 = 8$. So, we see that in our example (compare to the previous lecture with a similar example concerning a call option) the price \$8 corresponds to the price of a call option \$5 and zero interest rate and all other parameters: current price \$47, expiry July 2001, strike price \$50.

**The case $r > 0$.**
It turns out that a correct formula in this case is

$$C_t - P_t = S_t - \exp(-r(T-t))K. \tag{1}$$

This formula is called **a put and call parity**. There will be a rigorous proof in the next lecture.

### 3.3 Example

Assume $r = 6\%$ (per year). Remember that in our considerations the interest rate is compounded continuously. So, $T - t = 3/12 = 1/4 = 0.25$ (year) and *the put and call parity* has a form

$$C - P = S - \exp(-0.06 \times 0.25)K = S - \exp(-0.015)K = S - 0.98511194\,K.$$

Thus, we can calculate,

$$P = C - S + 0.98511194\,K = 5 - 47 + 0.98511194 \times 50 = -42 + 49.255597 = 7.255597.$$

Hence, if the interest rate is taken into account, the price \$8 (remind lecture 2) is not correct. Indeed, it provides an **arbitrage opportunity** which is not permitted on the "fair market". Roughly speaking it means **"free lunch"**, i.e. some profit for nothing. We will require that *on our market models* this is not possible. Such markets are called **fair**, – i.e. where free lunch is not possible. In this module we will consider only fair markets.

# 4 Basic principle: no arbitrage

### 4.1 Example of arbitrage

Suppose the interest rate is 6% while the quoted price of the put option is indeed \$8. What can a trader do? He *sells a put option for the quoted price \$8, immediately buys a call option for \$5 and sells a stock for \$47 (maybe shortly)*. Currently, his P&L is equal to \$8 - \$5 + \$47 = \$50, but remember he has borrowed a stock. *Then he invests \$50 to a bank with the interest rate* $r = 6\%$.

Then he waits until July. What happens in July? His bank account will be $\$\exp(r(T-t)) \times 50 = \$1.0151131 \times 50 = \$50.755653$. He also is the owner of a call option, he ows a stock and he wrote a put option with the same expiry date in July. Consider the plot of his P&L due to call - stock - put, or in the other words, *long call + short stock + short put*. The loss is constant and equal to \$50. So, **the trader wins \$0.755653 per share in July prices without any risk**. This is what we call arbitrage and this is forbidden on the market. All this may happen only because the put price was not valued correctly.

### 4.2 Proof of put and call parity: Arbitrage reasoning

Let us explain the formula for put & call parity using the arbitrage arguments. Whatever the put and call options prices (with the same expiry dates and the same strike prices) are, if we are buying a put now (at time $t$), suppose we decide also to sell a call and buy a share of stock. It costs $P_t - C_t + S_t$ which turns out to be positive, - remember that S is much greater than C and P. To finance this operation, we borrow the amount $P_t - C_t + S_t$ in the bank. Since both options are **European**, we wait till July (expiry date) keeping the share of stock as well: this is our **portfolio = the set of all financial securities** which we have at the moment. Let us look at the P&L plot for this portfolio **at expiry.** Our P&L is positive and *constant*, it is equal to K:

$$P_T - C_T + S_T = K.$$

We can check it using the definitions of payoff functions:

$$P_T - C_T + S_T = (K - S_T)_+ - (S_T - K)_+ + S_T \equiv K.$$

Indeed, if $S_T > K$ then the value $(K - S_T)_+ - (S_T - K)_+ + S_T$ equals to

$$0 - (S_T - K) + S_T = K.$$

If $S_T \leq K$ then it equals to

$$(K - S_T) - 0 + S_T = K.$$

So, our profit is **deterministic, not random**. Hence, it must be exactly the same as if it were invested into a bank account with the interest rate $r$, that is,

$$K = \exp(r(T-t))(P_t - C_t + S_t).$$

Otherwise an **arbitrage opportunity** would arise. So, we come to the **put and call parity formula**:

$$P - C + S = \exp(-r(T-t))K.$$

*Exercise.* Explain why: consider two cases:

$$K > \exp(r(T-t))(P - C + S) \quad \text{and} \quad K < \exp(r(T-t))(P - C + S).$$

*Hint.* Use the same arguments as above (concerning \$0.7...).

Let $K > \exp(r(T-t))(P - C + S)$, or $\exp(-r(T-t))K > P - C + S$. Denote

$$\alpha = \exp(-r(T-t))K - (P - C + S).$$

We can borrow $\exp(-r(T-t))K - \alpha = P - C + S$ at time $t$ to buy this portfolio (that is, we buy put P, we sell call C and we buy a share of stock S). Then we wait till expiry and get a profit K (see the plot) and pay our debt to the bank $exp(r(T-t))(P-C+S)$ *which is less than our gain* $K$. We get a pure non-random riskless profit

$$K - \exp(r(T-t))(P - C + S) > 0.$$

This is a riskless free lunch, i.e. an arbitrage. **So inequality** $K > \exp(r(T-t))(P - C + S)$ **is impossible**, such prices cannot exist on the market.

Similarly the opposite case can be considered: this time we sell the same portfolio (shortly) and lend the money to a bank. At expiry our profit is again positive. So $K < \exp(r(T-t))(P - C + S)$ is also wrong, the prices are not realistic. It remains the only possibility

$$K = \exp(r(T-t))(P - C + S).$$

*Exercise.* In the example from [WHD, p.7-10], we know the values $S, C, P, K, T-t$ for certain options. Is it possible to find the interest rate $r$ looking at any of the options, assuming that calls are European (even though they are not)?

**We conclude that if we can value a call option we will value a put option as well, using the put-call parity. In remains to find the answer to the main question: how value a call option** ? To do this, we need to *specify our model.*

# 5 Cox–Ross–Rubinstein binomial model

## 5.1 Cox-Ross-Rubinstein model: One step analysis

This is a discrete time model with time $t = 0$ and $t = \delta > 0$. On our market there is a bank account $B_t$ (a riskless bond in [BR, p. 10]) and a stock. The interest rate is $r \geq 0$, i.e. $B_\delta = B_0 \exp(r\delta)$. The stock has only one *node* at time 0 with value $S_0$, and two nodes at time $\delta$, with values $S_\delta^+$ and $S_\delta^-$. Assume $S_\delta^+ \neq S_\delta^-$ (otherwise there is no randomness, it is not a stock); let

$$S_\delta^+ > S_\delta^-.$$

There are probabilities $p$ & $1-p$ to move from $S_0$ to $S_\delta^+$ or to $S_\delta^-$ correspondingly. We assume (to avoid arbitrage possibilities)

$$\exp(-r\delta)S_\delta^- < S_0 < \exp(-r\delta)S_\delta^+.$$

**Exercise 1** *Show why otherwise an arbitrage exists. Hint: if any of two inequalities is not true, then either bank account or the stock provides a strictly greater profit without a risk, hence, it cannot exist on the market.*

On this simple market we are to introduce a call option and **to value it**.

## 5.2 Portfolio, strategy; equivalent portfolio principle

We have an amount of money which we are to invest in our market. We can choose between bank account and stock. Our **portfolio** is determined by the formula

$$P_0 = \phi S_0 + \psi B_0$$

which means that at time $t = 0$ we buy $\phi$ shares of stock and put $\psi B_0$ to our bank account. *Any values of $\phi$ and $\psi$ are admissible, positive or negative, integer or not.*

The couple $(\phi, \psi)$ is called a **strategy** at time zero. Then, at time $t = \delta$ our portfolio will cost

$$P_\delta = \phi S_\delta + \psi B_\delta$$

which equals $\phi S_\delta^+ + \psi B_0 \exp(r\delta)$ with probability $p$ and $\phi S_\delta^- + \psi B_0 \exp(r\delta)$ with probability $1 - p$. Note that the *expected value* of this random variable is

$$E(P_\delta) = \left(\phi \left(S_\delta^+ p + S_\delta^-(1-p)\right) + \psi B_0 \exp(r\delta)\right).$$

Now, choosing a strategy, we are going to imitate the call option, namely, construct a portfolio which has the **same payoff** as our call option, it will **mimic** or **replicate** the call. Fix a strike price $K > 0$ for our call, – it must satisfy inequalities

$$S_- \leq K \leq S_+.$$

The expiry is $T = \delta$. The problem is to construct such a strategy $(\phi, \psi)$ that the payoff of this portfolio were $(S_\delta - K)_+$. If we find such $\phi, \psi$ it will be called a *synthetic derivative*, – synthetic call. If its payoff is **exactly** the same as for the call, this portfolio will be equivalent to the call; we say that it will **replicate** our call option. **Hence, their prices coincide**, $C_0 = P_0$.

How to find this strategy? The couple $(\phi, \psi)$ must satisfy two equations:

$$\begin{cases} \phi S_\delta^+ + \psi B_0 \exp(r\delta) & = S_\delta^+ - K, \\ \phi S_\delta^- + \psi B_0 \exp(r\delta) & = 0. \end{cases}$$

This is always possible (remember $S_\delta^+ > S_\delta^-$). So, we have constructed a *synthetic derivative* which is equivalent to our call option! Solution is

$$\phi = \frac{S_\delta^+ - K}{S_\delta^+ - S_\delta^-}, \quad \psi = B_0^{-1} e^{-r\delta} \left( (S_\delta^+ - K) - \frac{S_\delta^+ - K}{S_\delta^+ - S_\delta^-} S_\delta^+ \right). \tag{2}$$

**Exercise 2** *Show it.*

Hence, **the value of our call option at time** $0$ **is**

$$C_0 = P_0 = \phi S_0 + \psi B_0 \tag{3}$$

with the strategy $\phi, \psi$ from (2).

## 5.3 Arbitrary synthetic derivative

In fact, we can construct **any synthetic derivative**, i.e. find a strategy for a portfolio with an **arbitrary payoff.** Indeed, suppose we would like to get a payoff $f(S_\delta)$, that is, $f(S_\delta^+)$ at $S_\delta = S_\delta^+$ and $f(S_\delta^-)$ at $S_\delta = S_\delta^-$. Then we have to solve a system of equations

$$\begin{cases} \phi S_\delta^+ + \psi B_0 \exp(r\delta) & = f(S_\delta^+), \\ \phi S_\delta^- + \psi B_0 \exp(r\delta) & = f(S_\delta^-). \end{cases}$$

Its solution is

$$\phi = \frac{f(S_\delta^+) - f(S_\delta^-)}{S_\delta^+ - S_\delta^-}, \quad \psi = B_0^{-1}\, e^{-r\delta} \left( f(S_\delta^+) - \frac{f(S_\delta^+) - f(S_\delta^-)}{S_\delta^+ - S_\delta^-}\, S_\delta^+ \right). \tag{4}$$

**Exercise 3** *Show it.*

Hence, the price of this synthetic derivative at time 0 equals

$$V_0 = \phi S_0 + \psi B_0 \tag{5}$$

with $\phi, \psi$ from (4).

**Exercise 4** *Show how to construct a synthetic put option and find the price.*

# 6 Implied probabilities in one step model

## 6.1 One step CRR model: "Implied" probabilities

Remind the formulae from the previous lecture for **call**

$$C_0 = \phi S_0 + \psi B_0 = \frac{S_\delta^+ - K}{S_\delta^+ - S_\delta^-} S_0 + \, e^{-r\delta} \left( (S_\delta^+ - K) - \frac{S_\delta^+ - K}{S_\delta^+ - S_\delta^-}\, S_\delta^+ \right),$$

and for **any synthetic derivative** with payoff $f(S_\delta^\pm)$, ($\phi, \psi$ **are different now**)

$$V_0 = \phi S_0 + \psi B_0 = \frac{f(S_\delta^+) - f(S_\delta^-)}{S_\delta^+ - S_\delta^-} S_0 + e^{-r\delta} \left( f(S_\delta^+) - \frac{f(S_\delta^+) - f(S_\delta^-)}{S_\delta^+ - S_\delta^-}\, S_\delta^+ \right).$$

**Denote**[1]

$$\tilde{p} = \frac{S_0 \exp(r\delta) - S_\delta^-}{S_\delta^+ - S_\delta^-}.$$

Since $0 < \tilde{p} < 1$ (*why?*), we have for **call**,

$$C_0 = e^{-r\delta} \left( \tilde{p}(S_\delta^+ - K) + (1 - \tilde{p})0 \right). \tag{6}$$

For a **general synthetic derivative** the price $V_0$ reads likewise,

$$V_0 = e^{-r\delta} \left( \tilde{p} f(S_\delta^+) + (1 - \tilde{p}) f(S_\delta^-) \right) \tag{7}$$

**with the same** $\tilde{p}$. This is a **discounted mean value of the payoff w.r.t. to probabilities** $\tilde{p}$ and $1-\tilde{p}$. The latter in general are not equal to the actual probabilities $p$ and $1 - p$; they are calculated without any mention of them. New probabilities $\tilde{p}$ and $1 - \tilde{p}$ are called **implied ones, risk-neutral, risk-adjusted, martingale ones**, and the expectation w.r.t. them we denote by $\tilde{E}$ (you may find $Q$ in some books).

[1] Here implied probabilities arise as if "from nowhere". However, it is useful to consider them as a certain "change of measure" from the original ones, by considering the ratios $\tilde{p}/p$ and $\tilde{q}/q$ as a special "density". Its analogue in the second part will be a density in the CMG Theorem.

**Exercise 5** *Show formulas (6) and (7).*

**Theorem 1** *Values $S_t$, $C_t$ and $B_t$ (the latter is not random) satisfy assertions*

$$e^{-r\delta}\tilde{E}S_\delta = S_0, \quad e^{-r\delta}\tilde{E}C_\delta = C_0, \quad \textit{and} \quad e^{-r\delta}B_\delta = B_0.$$

*Proof.* We only prove the first equality, the second being similar and the third evident (from definition of $B_t$). We substitute the values for $\tilde{p}$ and $1-\tilde{p}$:

$$e^{-r\delta}\left(\tilde{p}S_\delta^+ + (1-\tilde{p})S_\delta^-\right) = e^{-r\delta}\left(\frac{S_0e^{r\delta} - S_1^-}{S_\delta^+ - S_\delta^-}S_\delta^+ + \frac{S_\delta^+ - S_0e^{r\delta}}{S_\delta^+ - S_\delta^-}S_\delta^-\right)$$

$$= e^{-r\delta}\frac{[S_\delta^+S_\delta^- - S_\delta^+S_\delta^- + S_0e^{r\delta}(S_\delta^+ - S_\delta^-)]}{S_\delta^+ - S_\delta^-} = S_0.$$

**Exercise 6** *Show the assertion for $C_0$. Hint: it is written above, find where.*

# 7 Two step CRR model

## 7.1 CRR model: two-step market

This is a discrete time model with time $t = 0, \delta, 2\delta$ (of course, the *"tick"* $\delta > 0$). On our market there is a bank account $B_t$ and a stock $S_t$. The interest rate is $r \geq 0$, i.e. $B_\delta = B_0 \exp(r\delta)$. The stock has only one *node* at time 0 with value $S_0$, two nodes at time $\delta$, with values $S_\delta^1 > S_\delta^2$ (previously $S_\delta^+$ and $S_\delta^-$), and three nodes at time $2\delta$, $S_{2\delta}^1 > S_{2\delta}^2 > S_{2\delta}^3$.

There are probabilities $p_0$ & $1-p_0$ to move from $S_0$ to $S_\delta^1$ or to $S_\delta^2$ correspondingly, and also probabilities $p_\delta(S_\delta^1)$, $1-p_\delta(S_\delta^1)$, $p_\delta(S_\delta^2)$, and $1-p_\delta(S_\delta^2)$ to move from $S_\delta$ to $S_{2\delta}$. From each node at time $\delta$ the price $S$ can move only to two possible nodes, and it may move to $S_{2\delta}^2$ from any of the two nodes at time $\delta$.

For any possible value $S_\delta$, we will still use slightly extended notations $S_\delta^\pm$ to denote two possible values for the stock price at $t = 2\delta$.

To avoid arbitrage possibilities, we assume conditions (similar to those for one-step market): for any $t < 2$,

$$\exp(-r\delta)S_t^- < S_t < \exp(-r\delta)S_t^+.$$

## 7.2 CRR model: two-step analysis; self–financed portfolios

We need a method to value the call option for $t = 0$ and $t = \delta$. The idea is

1. firstly to **use the one-step model** to value the option at $t = \delta$,
2. and secondly, after we did it and know the values of $C$ at each node for time $t = \delta$, **use again the same one-step model** to find the price at $t = 0$. Let us write down formulas (and show how we should use filtrations).

Indeed, we can find $C_0$ once we know the values $C_\delta(S_\delta^1)$ and $C_\delta(S_\delta^2)$. In turn, we can find the latter since we know $C_{2\delta}$ at any node for time $2\delta$ (i.e. at expiry).

**Definition 1** **Self-financed portfolio** *is such a portfolio* $P_t = \phi_t S_t + \psi_t B_t, t = k\delta, k = 0, 1, \ldots,$ *that on each step, the owner just re-distributes the existing amount of money:*

$$\forall t, \qquad \phi_{t-\delta} S_t + \psi_{t-\delta} B_t = \phi_t S_t + \psi_t B_t,$$
$$\text{or, equivalently,} \qquad P_t - P_{t-\delta} = \phi_{t-\delta}(S_t - S_{t-\delta}) + \psi_{t-\delta}(B_t - B_{t-\delta}).$$

**Theorem 2** *If* $T = 2\delta$, *then the European call price can be computed via the formula,*

$$C_0 = e^{-2\delta} \tilde{E} C_{2\delta}; \qquad \tilde{E} \text{ here means "implicit expectation".}$$

# 8 General CRR model; martingale pricing theorem

## 8.1 General CRR (= binomial) model

This is a continuation of two-steps CRR model: a discrete time model with time $t = 0, \delta, 2\delta, \ldots$ with a bank account $B_t$ and a stock $S_t$. The interest rate is $r \geq 0$, i.e. $B_\delta = B_0 \exp(r\delta)$. The stock has only one *node* at time 0 with value $S_0$, two nodes at time $\delta$, with values $S_\delta^1 > S_\delta^2$, three nodes at time $2\delta$, $S_{2\delta}^1 > S_{2\delta}^2 > S_{2\delta}^3$, and so on.

There are probabilities $p_0$ & $1 - p_0$ to move from $S_0$ to $S_\delta^1$ or to $S_\delta^2$ correspondingly, probabilities $p_\delta(S_\delta^1)$, $1 - p_\delta(S_\delta^1)$, $p_\delta(S_\delta^2)$, and $1 - p_\delta(S_\delta^2)$ to move from $S_\delta$ to $S_{2\delta}$, etc. From each node at time $t$ the price $S$ can move only to two possible nodes at time $t + \delta$.

For any possible value $S_t$, we will keep notations $S_t^\pm$ to denote two possible values for the stock price at $t + \delta$ (in [BR]): $S_{now}, S_{up/down}, f_{now}, f_{up/down}$ are used). To avoid arbitrage possibilities, we assume for any $t < T$,

$$\exp(-r\delta) S_t^- < S_t < \exp(-r\delta) S_t^+.$$

The method how to value the call option for any time $t$:

1. firstly to **use the one-step model** to value the option at $t = T - \delta$;
2. secondly, after we did it and know the values of $C$ at each node for time $t = T - \delta$, use again the same one-step model to find the price at $T - 2\delta$;
3. and so on, by induction, which leads finally to the value $\boxed{C_0 = e^{-rT} \tilde{E} C_T.}$

# 9 Pricing theorem for any time $t \in [0, T]$

## 9.1 Formula for call in general CRR model

The theorem 1 from lecture 6 for **any** step may be presented as

$$\exp(-r\delta)\tilde{E}(C_{t+\delta}|\mathcal{F}_t) = C_t,$$

that is, for any step the process $C_t$ is a discounted martingale with respect to implied probabilities. (The same holds true with $\mathcal{F}_t^C$ instead of $\mathcal{F}_t$; this form was used in the handouts.) Using induction and the remark on conditional expectations above, we may conclude that for any $t$ and any $k > 0$,

$$\exp(-rk\delta)\tilde{E}(C_{t+k\delta}|\mathcal{F}_t^C) = C_t.$$

In particular, with $t = 0$ and $k\delta = T$ this gives us the value of call option in general CRR model:

$$\exp(-rT)\tilde{E}(C_T|\mathcal{F}_0^C) \equiv \exp(-rT)\tilde{E}(C_T) = C_0. \tag{8}$$

# 10 Drift and volatility; limit procedure

## 10.1 Drift and volatility in the binomial model

We consider a binomial (CRR) model with several steps and probabilities up and down 1/2 & 1/2; *assume* that at any node $S$ the values $S^{\pm}$ are defined as

$$S^+ = S\exp(\mu\delta + \sigma\sqrt{\delta}), \quad S^- = S\exp(\mu\delta - \sigma\sqrt{\delta}).$$

The mean value of the of $\log S$ increment for one step given $S_0 = S$ reads:

$$E(\log S_\delta - \log S_0 | S_0 = S) = \frac{1}{2}(\mu\delta + \sigma\sqrt{\delta}) + \frac{1}{2}(\mu\delta - \sigma\sqrt{\delta}) = \mu\delta.$$

Similarly (by induction) we may show that for any $t = k\delta$,

$$E(\log S_t - \log S_0 | S_0 = S) = \mu t.$$

For the variance,

$$E((\log S_\delta - \log S_0 - \mu\delta)^2 | S_0 = S) = \frac{1}{2}(\mu\delta + \sigma\sqrt{\delta} - \mu\delta)^2 + \frac{1}{2}(\mu\delta - \sigma\sqrt{\delta} - \mu\delta)^2 = \sigma^2\delta,$$

and similarly (using properties of the variance for independent variables) for any $t = k\delta$,

$$E((\log S_t - \log S_0 - \mu t)^2 | S_0 = S) = \sigma^2 t.$$

**Definition 2** *In this case, the drift of the price $S_t$ equals $\mu$, and volatility is $\sigma^2$.*

## 10.2 The limit for the binomial model

In this model, we are going to fix expiry $T$ and consider time step $\delta = T/n$ which tends to zero. The question is, what happens then to the call price $C_0$ (lecture 9, formula (1)):

$$C_0 = \exp(-rT)\tilde{E}(C_T) \equiv \exp(-rT)\tilde{E}(S_T - K)_+. \tag{9}$$

Notice that implied probabilities are permanent for any step; hence, the increments of the log-price **remain independent under implied probabilities** (like under original ones), as for any Bernoulli trials. $S_\delta$ can be represented in the form

$$S_\delta = S_0 \exp(\mu\delta + \sigma\sqrt{\delta}X_1),$$

where $X_1 = \pm 1$ with probabilities $p$ and $q$; similarly, for $T = n\delta$ we may represent

$$S_T = S_{n\delta} = S_0 \exp(n\mu\delta + \sigma\sqrt{\delta}\sum_{k=1}^{n} X_k) = S_0 \exp(\mu T + \sigma n^{-1/2} T^{1/2} \sum_{k=1}^{n} X_k), \tag{10}$$

where all $X_k = \pm 1$ with probabilities $p$ and $q$ and are independent (that is, this is a standard Bernoulli trials scheme). We are going to use the Central Limit Theorem, for which there is a prompt because $\sqrt{\delta}\sum_{k=1}^{n} X_k = \sqrt{T}\left(n^{-1/2}\sum_{k=1}^{n} X_k\right)$ (see lecture 9), **but under implied probabilities**. Hence, we recalculate the mean value and variance for $S_n = \sum_{k=1}^{n} X_k$ (under original probabilities $EX_k = 0$ and $var(X_k) = 1$). We will use approximate expressions as $\delta \to 0$, with $\{a \approx b\} \iff \{a - b = o(\sqrt{\delta})\}$:

$$\tilde{p} = \frac{\exp(r\delta) - \exp(\mu\delta - \sigma\sqrt{\delta})}{\exp(\mu\delta)(\exp(\sigma\sqrt{\delta}) - \exp(-\sigma\sqrt{\delta})} \approx \frac{1}{2} + \sqrt{\delta}\frac{r - \mu - \sigma^2/2}{2\sigma}.$$

Now we are able to find *implied* mean value and *implied* variance of $X_k$:

$$a_n = \tilde{E}X_1 = 1\tilde{p} - 1(1 - \tilde{p}) \approx \sqrt{\delta}\frac{r - \mu - \sigma^2/2}{\sigma} = \sqrt{T}\,\frac{r - \mu - \sigma^2/2}{\sigma\sqrt{n}},$$

$$\tilde{E}X_1^2 = 1, \quad \text{hence,} \quad \sigma_n^2 := \widetilde{var}(X_1) \approx 1 - \left(\sqrt{\delta}\frac{r - \mu - \sigma^2/2}{\sigma}\right)^2 \approx 1.$$

Since $\tilde{E}|X_1|^3 = 1$, **we can apply theorem 2 from lecture 9**:

$$\frac{\sum_{k=1}^{n} X_k - na_n}{\sigma_n\sqrt{n}} \overset{\tilde{P}}{\Longrightarrow} \eta \sim \mathcal{N}(0,1).$$

So, formula (10) may be represented as

$$S_T = S_0 \exp(\mu T + \sigma\sqrt{\delta}\sum_{k=1}^{n}(X_k - a_n) + n\sigma\sqrt{\delta}a_n)$$

$$= S_0 \exp(\mu T + \sigma\sigma_n\sqrt{T}\frac{\sum_{k=1}^{n}(X_k - a_n)}{\sigma_n\sqrt{n}} + n\sigma\left(\sqrt{T/n}\right)\left(\sqrt{T}\frac{(r - \mu - \sigma^2/2)}{(\sigma\sqrt{n})}\right)$$

$$= S_0 \exp(\mu T + \sigma\sqrt{\delta}\sum_{k=1}^{n}(X_k - a_n) + T(r - \mu - \sigma^2/2))$$

$$\overset{\tilde{P}}{\Longrightarrow} S_0 \exp((r - \sigma^2/2)T + \sigma\sqrt{T}\eta), \quad \eta \sim \mathcal{N}(0,1) \tag{11}$$

(i.e. weak convergence[2] under *implied* probabilities). Hence, as $n \to \infty$,

$$\boxed{C_0 = \exp(-rT)\tilde{E}(S_0 \exp((r-\sigma^2/2)T + \sigma\sqrt{T}\eta) - K)_+.}$$

We have used the second definition of weak convergence (slightly extended, however correct). This is a preliminary version of our goal – **Black-Scholes formula[3].**

**Remark.** *A rigorous justification of the latter convergence includes a bound* $\overline{\lim}_{n\to\infty}\tilde{E}\exp(2n^{-1/2}\sum_{k=1}^n X_k) < \infty$*. This follows from the expression for* $\tilde{p}$*,* inde- pendence *of all* $X_k$*'s under* $\tilde{P}$*, and the limit* $\lim_{n\to\infty}(1+c/n+o(1/n))^n = \exp(c) < \infty$*. This rigor is not required in this module, so that this remark is regarded as advanced reading.*

## Computing $\tilde{p}$

This is a more detailed calculus for computing the expressions for implied probabilities $\tilde{p}$, $\tilde{q}$, and related quantities. It uses only elementary material from the calculus, concerning the limit

$$\lim_{x\to 0}\frac{\exp(cx)-1}{x} = c,$$

which we assume to be known.

INSERT THE REST!

[2]Why there is $\tilde{P}$ above the sign $\Longrightarrow$? Because $\Longrightarrow$ denotes weak convergence related to some specific measure (revise the definition), and in our case we have two different measures, $P$ and $\tilde{P}$. So, this notation simply indicates which measure is used in the definition of weak convergence. Often the latter is clear, and, hence, this indication is not necessary.

[3]The final version will use Gaussian integration and eventually Laplace function in order to present more explicitly the same expression.