CLT

For *IID* random variables with $EX_k = 0$ and $\texttt{var}(X_k) = 1$, the following weak convergence holds true,

$$n^{-1/2}\sum_{1}^{n} X_k \Longrightarrow \mathcal{N}(0,1), \quad n \to \infty. \tag{1}$$

Under additional assumptions, the rate of convergence is available. Let us show how this can be applied for the approximate solution of the heat equation ($0 \le t \le 1$)

$$\left(u_t + \frac{1}{2}u_{xx}\right)(t,x) = 0,\ u(1,x) = g(x). \tag{2}$$

Indeed, to solve the equation (2) approximately, one can use, e.g., the method

$$\frac{u\left(\frac{k-1}{n}, \frac{m}{\sqrt{n}}\right) - u\left(\frac{k}{n}, \frac{m}{\sqrt{n}}\right)}{1/n} \tag{3}$$

$$= \frac{u\left(\frac{k}{n}, \frac{m+1}{\sqrt{n}}\right) - 2u\left(\frac{k}{n}, \frac{m}{\sqrt{n}}\right) + u\left(\frac{k}{n}, \frac{m-1}{\sqrt{n}}\right)}{2/n}, \tag{4}$$

or, equivalently,

$$u\left(\frac{k-1}{n},\frac{m}{\sqrt{n}}\right)=\frac{u\left(\frac{k}{n},\frac{m+1}{\sqrt{n}}\right)+u\left(\frac{k}{n},\frac{m-1}{\sqrt{n}}\right)}{2}. \quad (5)$$

Let $W^n_{k/n} = x + n^{-1/2}\sum_1^n X_k$ with $X_k = \pm 1$ with probability half each, and $x = i/\sqrt{n}$. By induction, we conclude that

$$u\left(\frac{n-k}{n},x\right)=E_x g(W^n_{k/n}). \quad (6)$$

Hence, as $n \to \infty$, for (at least) any $g \in C_b$,

$$E_x g(W^n_{n/n}) \to Eg(x+W_1)=\int \frac{g(x+y)}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy. \quad (7)$$

which means exactly *convergence of the method* to the exact solution. That is, the CLT for the Bernoulli case is **equivalent** to the convergence of this simplest approximation method on appropriate functions g.

Naturally, any bound of the convergence (6) means the same bound of the convergence of the approximate solution to the PDE (2).