

CLT

For *IID* random variables with $EX_k = 0$ and $\text{var}(X_k) = 1$, the following weak convergence holds true,

$$n^{-1/2} \sum_1^n X_k \implies \mathcal{N}(0, 1), \quad n \rightarrow \infty. \quad (1)$$

Under additional assumptions, the rate of convergence is available. Let us show how this can be applied for the approximate solution of the heat equation ($0 \leq t \leq 1$)

$$\left(u_t + \frac{1}{2} u_{xx} \right) (t, x) = 0, \quad u(1, x) = g(x). \quad (2)$$

Indeed, to solve the equation (2) approximately, one can use, e.g., the method

$$\frac{u\left(\frac{k-1}{n}, \frac{m}{\sqrt{n}}\right) - u\left(\frac{k}{n}, \frac{m}{\sqrt{n}}\right)}{1/n} \quad (3)$$

$$= \frac{u\left(\frac{k}{n}, \frac{m+1}{\sqrt{n}}\right) - 2u\left(\frac{k}{n}, \frac{m}{\sqrt{n}}\right) + u\left(\frac{k}{n}, \frac{m-1}{\sqrt{n}}\right)}{2/n}, \quad (4)$$

or, equivalently,

$$u\left(\frac{k-1}{n}, \frac{m}{\sqrt{n}}\right) = \frac{u\left(\frac{k}{n}, \frac{m+1}{\sqrt{n}}\right) + u\left(\frac{k}{n}, \frac{m-1}{\sqrt{n}}\right)}{2}. \quad (5)$$

Let $W_{k/n}^n = x + n^{-1/2} \sum_1^n X_k$ with $X_k = \pm 1$ with probability half each, and $x = i/\sqrt{n}$. By induction, we conclude that

$$u\left(\frac{n-k}{n}, x\right) = E_x g(W_{k/n}^n). \quad (6)$$

Hence, as $n \rightarrow \infty$, for (at least) any $g \in C_b$,

$$E_x g(W_{n/n}^n) \rightarrow E g(x + W_1) = \int \frac{g(x+y)}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy. \quad (7)$$

which means exactly *convergence of the method* to the exact solution. That is, the CLT for the Bernoulli case is **equivalent** to the convergence of this simplest approximation method on appropriate functions g .

Naturally, any bound of the convergence (6) means the same bound of the convergence of the approximate solution to the PDE (2).