Introduction to Brownian Motion (for non-probabilists)

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1. **Definition**

Q: How can one define Brownian Motion?
A: There are many ways to do this. In this talk we define BMP(W) (P for Process, W for Wiener) as a *measure* on functions (trajectories) on the interval $[0,T]$. This measure P is Gaussian, on cylindrical sets it can be described as follows: for any n and any partitions $t_0 := 0 \le t_1 < t_2 < \ldots < t_n \le T$, and any a_k, $1 \le k \le n$,

$$P(\prod_1^n X_{t_k} \le a_k) = \int_{-\infty}^{a_1} \ldots \int_{-\infty}^{a_n} \prod_{k=1}^n \frac{1}{\sqrt{2\pi(t_k - t_{k-1})}} e^{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}} \prod_1^n dx_k$$

(here $x_0 = 0$). Of course, T here can be arbitrary, so this measure, in fact, can be defined on the whole half line $[0,\infty)$.
Q: What is the reason to consider this measure? In the other words, how can one guess that it is worth to define this measure?
A: Suppose we are going to solve an "approximate Laplace equation" on a lattice with a small constant "size" h in a bounded interval $O \subset R^1$ (a bounded domain $O \subset R^2$ would be more instructive to *draw*, and on the board the lecturer has used indeed R^2) with a boundary Γ:

$$\Delta^h u^h(x) := \frac{u^h(x+h) - 2u^h(x) + u^h(x-h)}{h^2} = 0, \ x \in O, \quad \& \ \ u^h|_\Gamma = \varphi.$$

Consider a simple Random Walk (RW) on the lattice, with jumps $\pm h$ at $t = kh^2$, $k = 1, 2, \ldots$ with equal probabilities $1/2$, starting from $x \in O$. This RW may be understood as a measure on trajectories on this lattice with (independent) jumps $\pm h$ at times kh^2 (i.e. only to neighbouring points on the lattice) with the condition, every trajectory of length n has a measure 2^{-n}. Then, dropping some technical details, the solution of the Laplace equation can be written in the form,

$$u^h(x) = E_x \varphi(X_\tau)$$

(in probability language, E means an integration sign; x in E_x means that every trajectory starts from x; and X_t is simply a coordinate of a trajectory: perhaps it would be right to use a notation X_t^h that is more precise, anyway we have in mind this dependence on h), where τ is defined an *exit time from* O, that is, $\tau := \inf(t = kh^2 : X_{kh^2} \notin O)$ (that is, not in O, i.e., in Γ). The fact that this u^h satisfies the equation inside O follows from the complete probability formula,

$$u^h(x) = \frac{1}{2}\left(u^h(x-h) + u^h(x+h)\right),$$

this equality being equivalent to the Laplace equation $\Delta^h u^h = 0$; and the boundary condition is "evident" (as outside O, "of course", $\tau = 0$)[1].

Denote the measure on the trajectories on this lattice by P^h. It can be proved that the measure P^h has a limit, P (the notation P^0 was used on the board), that is a measure on trajectories on $[0, T]$ for any $T > 0$, and eventually on $[0, \infty)$, and this limiting measure is exactly our BMP(W) measure.

Q: How do we know that the limiting measure exists and is Gaussian?

A: The existence can be proved by standard probability technique that concerns a rigorous construction of a measure for a process given finite-dimensional distributions. The Gaussian property for the limiting measure as we introduced it follows from the Central Limit Theorem: *e.g.*, by the CLT, assuming $1 = Nh^2$, and denoting the k-th jump by hZ_k, we get,

$$P^h_x (X_1 \le a) = P^h \left(x + h \sum_{k=1}^N Z_k\right)$$

$$= \quad P^h \left(x + N^{-1/2} \sum_{k=1}^N Z_k\right) \to \int_{-\infty}^{a-x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy, \quad \text{as } N \to \infty.$$

Q: Are there any other properties of BMP(W) that are included in the standard definition and that the lecturer has hidden from the audience?

A: Yes, the property (condition) $P(X_t \text{ is continuous}) = 1$ is usually included in the definition of the BMP(W). The lecturer did not like to mention this in the beginning because this is, in fact, a technical lemma, that discontinuous paths have measure zero. If anybody is interested, this property can be established using so called continuity theorems: the most popular is Kolmogorov's Theorem (it uses an assumption, $E|X_t - X_s|^\alpha \le C|t-s|^{d+\varepsilon}$, where $\alpha, \varepsilon > 0$, and $d = dim$), and there are extensions (Loeve's Theorem, et al.), all of them can be found in advanced textbooks on probability or stochastic processes.

Q: We have heard that CLT is a basic theorem of Statistics, but here you use it to explain how stochastic processes (BMP(W)) relate to approximation methods in PDEs and eventually to Laplace operator, could you comment on it? Is there any relation of Laplace operator to Statistics?

A: Yes, CLT is an indispensable tool in Statistics, roughly, it justifies under mild conditions why *normal approximations* can be legally used in a majority of statistical models. Probably a full role of Laplace operator/equation in Statistics is yet to be discovered: e.g., it could be a good PhD or postdoc project in either of our three Departments. The Pure or Applied participants of this seminar may consider this as an invitation for a collaboration (still, we would need at least one PhD student for this).

2. **Laplace and Poisson equations**

Q: Now once we have a definition of the BMP(W) and some explanatory comments, could you say more precisely how it can be used in studies of the Laplace (or even Poisson?) equation?

A: Yes, however, it would be better to start with the simplest heat equation instead,

$$u_t(t, x) - \frac{1}{2}\Delta_x u(t, x) = 0, \quad u(0, x) = g(x).$$

This is a Cauchy problem "in the whole space", and we do not present all precise assumptions here. A (unique – under appropriate assumptions) solution u can be

[1] One may guess, of course, that this "evident" may be nontrivial or even wrong in some complex cases; however, as "usually" this is true, we do not discuss the details here.

written as an integral,

$$u(t,x) = \int_{-\infty}^{\infty} g(y) \frac{1}{\sqrt{2\pi t}}\, e^{-(y-x)^2/2t}\, dy = \int_{-\infty}^{\infty} g(x+y) \frac{1}{\sqrt{2\pi t}}\, e^{-y^2/2t}\, dy.$$

It can be also re-written in the form,

$$u(t,x) = E_x g(X_t),$$

where X_t, $t \geq 0$ is our Brownian Motion (BMP(W)) starting at x. This reminds us a similar representation for the solution of the approximate Laplace equation (second formula on page 1), does it not? There is some difference, however, the formula on page one is in some sense more difficult because it involves a *stopping time*, τ, while the last formula only uses non-random times. If we wished to consider a Cauchy problem in a bounded domain O, it would be necessary to introduce stopping time here as well, however, we just want to show the simplest connections between the heat equation and BMP(W) here.
Q: Please, we are intrigued and would like to see something similar to a solution of the approximate Laplace equation now.
A: We are moving towards it. Firstly imagine that we *integrate* our solution $u(t,x)$ with respect to t:

$$v(x) := \int_0^{\infty} u(t,x)\, dt.$$

What would we get then? Leaving a technical question about convergence of this integral, we would, of course, guess, that the function v should satisfy the Laplace equation,

$$\frac{1}{2}\Delta v(x) = ?$$

Indeed, what shall we use in the right hand side here? A natural attempt would be to try

$$? = \int_0^{\infty} u_t(t,x)\, dt = u(\infty, x) - g(x).$$

E.g., if due to any reason the limit $u(\infty, x)$ exists and equals zero (yes, it may be natural and it happen in some cases), then we would get,

$$\frac{1}{2}\Delta v(x) = -g(x).$$

This our guess makes some sense, and can be, indeed, justified *in some cases*: we will not go into further details here, just notice that the problem of convergence may be not elementary at all, and the order of integration (with respect to dt and dP) may be important; however it is not in all cases true that one must establish $\int_0^{\infty} E|f(X_t)|\, dt < \infty$ in order to justify the intuitive calculus.
Q: You are telling us some quizzes. In your last equation, although it looks like Poisson's equation (you promised us Laplace's one, actually), there is no boundary conditions, and the equation is in the whole space? Can you be a bit more precise please? What about equations in a bounded domain?
A: This is a two hour talk, nothing less and nothing more. I simply wanted to present an idea that it is easier to interpret a parabolic equation via BMP(W) rather than an elliptic one; however, as a next step, once we integrate a solution of a parabolic equation, evidently, time should disappear, hence, we might hope to solve some elliptic equations in this way, too. But instead of a justification of the latter formula,[2] I shall tell you now about a BMP(W) representation of the Laplace

[2]This equation may be important in some asymptotical results related to a so called diffusion approximation.

(and Poisson) equation, indeed, in the bounded domain O:

$$\frac{1}{2}\Delta v(x) = -g(x),\ x \in O, \quad \text{and} \quad u|_\Gamma = \varphi.$$

For simplicity of presentation, let us drop g here, i.e., assume $g \equiv 0$.[3] Then the analogue of the formula on page one is as follow,

$$u(x) = E_x \varphi(X_\tau),$$

where $(X_t,\, t \geq 0)$ is a path of our Brownian Motion (i.e., in the measure theory language simply X_t is a variable of integration), *and* τ is a *stopping time*: $\tau = \inf(t : X_t \in \Gamma)$. The formula with g reads, $u(x) = E_x[\int_0^\tau g(X_s)\, ds + \varphi(X_\tau)]$.
Q: You are using many times measure language. But how to understand "time" and, moreover, "stopping time" in this language?
A: This is the point. Time can be easily considered simply as an index of a coordinate, or variable of integration. The notion of stopping time is a bit more delicate. K. L. Chung (2002)[4] explains that the notion of time and especially stopping time is what separates probability or stochastic processes from integration theory. The latter notion can be interpreted in integration language as follows. For illustration purposes we will use "infinitesimal difference" language like Feynman's path integral: e.g., we could write down the solution to heat equation above symbolically as

$$u(t,x) = \int \dots \int \varphi(x + \sum_k y_k) \prod_k \frac{1}{\sqrt{2\pi\delta}}\, e^{-\frac{y_k^2}{2\delta}}\, dy_k\,.$$

Here δ means an infinitesimal increment in time; the sum and product upper limit "should be" t/δ. Then, the formula for $u(x)$ – solution to Laplace equation – can be written down also symbolically as

$$u(x) = \sum_{N=0}^{\infty} \int \dots \int \varphi(x + \sum_{k=1}^{N} y_k)\, 1(x + \sum_{k=1}^{j} y_k \in O, \forall j < N,$$
$$\&\ \ x + \sum_{k=1}^{N} y_k \notin O) \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi\delta}}\, e^{-\frac{y_k^2}{2\delta}}\, dy_k\,.$$

Q: Returning to integration with respect to time, is there any other probabilistic approach that allows to get an elliptic equation from a parabolic one? E.g., instead of integrating with respect to t, could we use a limit as $t \to \infty$?
A: Yes, although not for BMP(W) in Euclidean spaces: there is an area related to stabilization of marginal distributions for processes that correspond to elliptic operators of the second order, with keywords "mixing" or "weak dependence". In the PDE language, mixing corresponds (but probably is not identical) to a stabilization of solutions to a homogeneous parabolic equation. BMP(W) itself does possess mixing properties only on compact manifolds, but this is not our topic today.

[3]Hence, Poisson will be some other time. *A propos*, about Poisson and statistics: there is a famous *Fisher* or *KPP* equation which is a *nonlinear* parabolic equation where running waves can arise. This topic can be also studied by PDE methods and by using BMP(W), however this is not our topic today, but just to mention some nice connections between poisson and fish. (KPP means Kolmogorov-Petrovskii-Piskunov.)

[4]This book may be recommended as an excellent first reading about BMP(W) *for non-probabilists*; I tried to avoid a frequent use of this book in my talk, that was not easy given practically the same aim, so that you could enjoy this text so to say from scratch.

3. **Generator**

Q: You have told us something about Brownian Motion (BMP(W)) and its relations to the Laplace and even Poisson equations. Is there anything else about its relation to the Laplace operator itself that we should know? In geometry, we would eventually like to know whether BMP(W) relates somehow to *harmonic maps*.

A: Yes, and now we will learn the notion of *generator* of the Brownian motion; this will lead us to the relation to harmonic maps (or at least harmonic functions). The words "$\Delta/2$ is a generator of BMP(W)" are understood in the following sense:

1 BMP(W) is a Markov (and strong Markov) process. Roughly, this means that this process "starts as a new process at any point"; we will postpone a discussion of this notion as well as an exact definition for a while: till further seminars? Just notice that our Random Walk defined above has this property, hence, it may be not surprising to have the same feature in the limit. (We do not pretend to be rigorous!)

2 For any smooth function $f : R \to R$,

$$\lim_{t\to 0} \frac{E_x f(X_t) - f(x)}{t} = \frac{\Delta}{2} f(x).$$

(The rigorous definition uses stopping times here. However, we prefer this easier version, that is called, strictly speaking, quasi-generator, and is sufficient for most of purposes.)

Naturally, to verify the definition, one should prove the latter limiting relation. It can be done rigorously, e.g., using the representation via the Gaussian density. Hence, *if we consider a harmonic function*, – that is, a function satisfying $\Delta f(x) \equiv 0$ in the whole space, we shall get,

$$E_x f(X_t) - f(x) = 0, \quad \forall x, t.$$

There is a similar although a bit more involved interpretation for functions that are harmonic in some bounded domain, too. Hence, BMP(W) relates directly to harmonic functions. E.g., if a harmonic function is given on the boundary of some bounded domain, one can reconstruct the values of this function inside the domain using the formula on page 4 which represents a solution to the Laplace equation via BMP(W).

Q: You use always expectations/integations, like $Ef(X_t)$. We are just curious, can one comment on $f(X_t)$ *without* the expectation?

A: This is a very good question, as speakers often say about a question which they do not know how to comment. Yes, we know something interesting about $f(X_t)$, too. To formulate the statement, we have to know the notion of *Itô's stochastic integral* $\int_0^t \ldots dX_s$ (recall that we denote our BMP(W) by X_t; more popular notations here are dW_t or dB_t). I drop all details and only show you the result (for a harmonic function[5]):

$$f(X_t) = f(x) + \int_0^t f'(X_s)\, dX_s$$

(or ∇f in the case $dim > 1$). There are several exciting points here. First: $\int \ldots dX_s$ is *not* a Lebesgue integral, because X_s has no derivative (or there is a Sobolev derivative as a linear functional, not a usual function). Second: this integral can be defined rigorously in "L_2-sense", and it is a *martingale* as a function of t; in

[5] for non-harmonic functions, $f(X_t) = f(x) + \int_0^t f'(X_s)\, dX_s + \int_0^t \Delta f(X_s)/2\, ds$.

particular, after taking an expectation, this term $\int_0^t f'(X_s)\,dX_s$ disappears. Third: in fact, this Itô's integral leads to similar representations for solutions of much more general elliptic and parabolic equations, and to processes more general that BMP(W), called *diffusions*. Roughly speaking, a diffusion X_t with a generator $L = \sum a_{ij}(x)\partial_{x_i x_j} + \sum b(x)\partial_{x_i}$ corresponds to the second order elliptic equations

$$Lu = 0, \quad \text{or, more generally,} \quad Lu = -g.$$

These processes, however, can be constructed via the BMP(W) as *solutions of stochastic differential equations*,

$$dX_t = \sqrt{2a}(X_t)\,dW_t + b(X_t)\,dt, \; t \geq 0, \quad X_0 = x,$$

where W_t is one of the two more usual notations for BMP(W), while X_t now is a new process, – our more general diffusion, – that is to be found. In the case $dim > 1$, the function $\sqrt{2a}$ (where a is a matrix $d \times d$) may be considered as a function square root applied to this matrix, via the Cauchy integral formula. Naturally, any differential equality like the latter one is understood in the integral sense, and this is why we need the notion of stochastic integral. It is reasonable to stop this lecture here.

References

[1] Chung, K. L. (2002) Brown, Green and Probability & Brownian Motion on the Line, World Scientific, NJ et al.