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School of Mathematics

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MATH271501

Statistical Methods

Time Allowed: 2 hours

Answer all questions in Section A and all questions from Section B.

Each question in Section A carries 2 marks, and each question in Section B carries 20 marks.

Questions A1 to A10 require you to write down a single letter answer.
Questions A11 to A20 require you to write down a short explanation or draw a sketch.

Your answers to Section A questions and Section B questions may be written in the same answer book.
SECTION A

Answer ALL questions in Section A

Questions A1 to A10 require you to write down a single letter answer.

A1. If a random variable $X$ has an exponential$(\lambda)$ distribution, then $E[X^r] = r!/\lambda^r$ for $r = 1, 2, 3, \ldots$. If $E[X] = \mu$, what is the value of $\mu^3 = E[(X - \mu)^3]$?

A: 0, B: $-2/\lambda^3$, C: $-1/\lambda^3$, D: $1/\lambda^3$, E: $2/\lambda^3$.

A2. In question A1 above, suppose $X_1, X_2, \ldots, X_n$ are independent exponential$(\lambda)$ random variables and $\bar{X}$ is their mean. What is the variance of $\bar{X}$?

A: $2n/\lambda^2$, B: $n\lambda^2$, C: $n/\lambda^2$, D: $1/n\lambda^2$, E: $\lambda^2/n$.

A3. In question A1 above, suppose that a random sample $x = (x_1, \ldots, x_n)$ is available and has sample mean $\bar{x}$. What is the method of moments estimate $\tilde{\lambda}$ of $\lambda$?

A: $1/\lambda$, B: $n\bar{x}$, C: $1/\bar{x}$, D: $\bar{x}$, E: $1/n \log(x_1x_2 \ldots x_n)$.

A4. In question A3 above, the method of moments estimator $\tilde{\lambda}$ of $\lambda$ satisfies $E[\tilde{\lambda}] = n\lambda/n - 1$. What does the bias of $\tilde{\lambda}$ equal?

A: 0, B: $\lambda/n - 1$, C: $n\lambda/n - 1$, D: $2/n - 1$, E: $n/n - 1$.

A5. If $Z \sim N(0, \sigma^2)$, what is the value of $E[Z^3]$?

A: 0, B: $\sigma^2$, C: $\sigma^4$, D: $3\sigma^4$, E: 1.
A6. Given constants $\epsilon > 0$ and $\eta > 0$, what would you have to show to demonstrate that a sequence of random variables $Y_1, Y_2, Y_3, \ldots$, converges in probability to zero?

A: $\exists N : P\{|Y_n| < \epsilon\} < \eta, \forall n > N$,  
B: $\exists N : P\{|Y_n| < \epsilon\} < \eta, \forall n < N$,  
C: $\exists N : P\{|Y_n| > \epsilon\} > \eta, \forall n > N$,  
D: $\exists N : P\{|Y_n| > \epsilon\} < \eta, \forall n > N$.

A7. If a random variable $X$ has zero mean and variance $\sigma^2 = 1$, which of the following statements is always true?

A: $P\{|X| \geq k\} \leq \frac{1}{k^2}$,  
B: $P\{|X| \geq k\} \geq \frac{1}{k^2}$,  
C: $P\{|X| \leq k\} \leq \frac{1}{k^2}$,  
D: $P\{|X| \leq k\} \geq \frac{1}{k^2}$.

A8. Suppose that $(X,Y)$ have a bivariate normal distribution with mean $\mu$ and variance matrix $\Sigma$ where

$$\mu = \left(\begin{array}{c} 1 \\ 2 \end{array}\right), \quad \Sigma = \left(\begin{array}{cc} 1 & 0.4 \\ 0.4 & 4 \end{array}\right).$$

What is the value of $\text{corr}(X,Y)$?

A: 0.0,  B: 0.1,  C: 0.2,  D: 0.4,  E: 0.5.

A9. In question A8 above, if $c$ is a constant, what is the value of $\text{Var}[Y - cX]$?

A: $4 - 0.8c + c^2$,  B: $4 - 0.4c + c^2$,  C: $2 - 0.4c + c^2$,  D: $2 - 0.8c + c^2$,  E: $2 - 0.2c + c^2$.

A10. In question A8 above, what value of $c$ makes $X$ and $Y - cX$ uncorrelated?

A: 0.0,  B: 0.1,  C: 0.2,  D: 0.4,  E: 0.5.
Questions A11 to A20 require you to write down a short answer or draw a sketch.

A11. Suppose that random variables \((X, Y)\) have joint probability density function given by \(f_{XY}(x, y) = 3x\) for \(0 < y < x < 1\) (and \(f_{XY}(x, y) = 0\) outside this region). Sketch the \((x, y)\)-region where \(f_{XY}(x, y)\) is non-zero.

A12. In question A11 above, show that the marginal probability density function of \(Y\) is \(f_Y(y) = \frac{3}{2}(1 - y^2)\) for \(0 < y < 1\).

A13. In question A12 above, obtain the cumulative distribution function \(F_Y(y)\) of \(Y\) for \(0 < y < 1\).

A14. A random sample \(x = (x_1, x_2, \ldots, x_n)\) is taken from a distribution with probability density function \(f_X(x; \lambda) = \lambda^2 xe^{-\lambda x}\) for \(x > 0\) where \(\lambda\) is an unknown parameter. Obtain the log-likelihood function \(\log L(\lambda; x)\).

A15. In question A14 above, show that the maximum likelihood estimator of \(\lambda\) is \(\hat{\lambda} = \frac{2}{\bar{x}}\) where \(\bar{x}\) is the sample mean of \((x_1, x_2, \ldots, x_n)\).

A16. In question A14 above, show that the asymptotic variance of \(\hat{\lambda}\) is \(\frac{\lambda^2}{2n}\).

A17. A Cauchy random variable \(X\) has probability density function \(f_X(x) = \frac{1}{\pi(1 + x^2)}\) for \(-\infty < x < \infty\) and characteristic function \(\phi_X(t) = E[e^{itX}] = e^{-|t|}\). If \(U = a + bX\), what is the characteristic function of \(U\)?

A18. Suppose that \(X_1, X_2, \ldots, X_n\) are mutually independent Cauchy random variables, each with characteristic function \(\phi_X(t) = e^{-|t|}\), as defined in question A17 above. If 

\[
\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n),
\]

show that \(\bar{X}_n\) has characteristic function \(e^{-|t|}\).

A19. What are the implications of the result in question A18 above for the central limit theorem?

A20. Imagine estimating a parameter \(\theta\) using a sample of data \(x\). If you are given a posterior density function \(\pi(\theta|x)\), briefly explain how to construct a Bayesian interval estimate for \(\theta\).
SECTION B

Answer all questions from Section B.

B1.

The random variable $X$ has a beta distribution with parameters $a > 0$ and $b > 0$ and probability density function

$$f_X(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \quad 0 < x < 1.$$  

Here $B(a,b)$ is the beta function which satisfies

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

and where $\Gamma(z)$ is the gamma function satisfying $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1) = 1$.

(a) Show that $E[X] = a/(a + b)$.

(b) Consider the transformation

$$U = \frac{X}{1-X}.$$  

Obtain the probability density function of $U$. What is the range of $U$?

(c) Consider now the case $a = 1$. Obtain $E[U]$ and suggest a method of moments estimator for the parameter $b$ based upon a random sample of $n$ values $u_1, u_2, \ldots, u_n$.

(d) Consider again the case $a = 1$, and suppose you now have available a random sample of $n$ values $x_1, x_2, \ldots, x_n$ taken from the distribution of $X$. Using your answer in part (a) above, what would be your method of moments estimator for the parameter $b$ in this case?

(e) Without making any detailed calculations, outline how you would choose between your competing estimators for the parameter $b$ in parts (c) and (d) above.
B2.

(a) A random variable $X$ has a standard normal distribution with probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad -\infty < x < +\infty.$$ 

Show that $U = X^2$ has probability density function $f_U(u)$ which satisfies

$$f_U(u) = \frac{1}{\sqrt{2\pi \sqrt{u}}} e^{-\frac{1}{2}u}.$$ 

Is this a 1-1 transformation? What is the range of $U$?

(b) Now consider the random variable $Y$ having a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ and with probability density function

$$f_Y(y) = \frac{\lambda^\alpha y^{\alpha-1}e^{-\lambda y}}{\Gamma(\alpha)}, \quad y > 0,$$

and where $\Gamma(\alpha)$ is the gamma function which satisfies $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(i) Explain why you know that

$$\int_0^\infty y^{\alpha-1}e^{-\lambda y} dy = \frac{\Gamma(\alpha)}{\lambda^\alpha}.$$ 

(ii) Prove that $Y$ has moment generating function $m_Y(t) = \text{E}[e^{tY}]$ satisfying

$$m_Y(t) = \left(\frac{\lambda}{\lambda-t}\right)^\alpha, \quad t < \lambda.$$ 

Why is the requirement $t < \lambda$ imposed?

(c) By comparing the probability density function of $U$ in part (a) above with that of a gamma distribution, or otherwise, deduce that the moment generating function of $U$ is $m_U(t) = (1 - 2t)^{-\frac{1}{2}}$.

Obtain the mean and variance of $U$.

(d) If $X_1$ and $X_2$ are independent standard normal random variables, obtain the moment generating function of

$$V = X_1^2 + X_2^2.$$
B3.

(a) A random variable $Y$ is said to have an inverse gamma distribution with parameters $k > 0$ and $\lambda > 0$ if it has probability density function

$$f_Y(y) = \frac{\lambda^k y^{-k-1} e^{-\lambda/y}}{\Gamma(k)}, \quad y > 0.$$  

Here $\Gamma(k)$ is the usual gamma function and satisfies $\Gamma(k) = (k - 1)\Gamma(k - 1)$.

Show that the mean of this inverse gamma distribution is $\frac{\lambda}{k-1}$ for $k > 1$.

(b) Let $x = (x_1, x_2, \ldots, x_n)$ be a random sample of $n$ values from a normal distribution with zero mean and variance $\theta$, and having common probability density function

$$f_X(x; \theta) = \frac{1}{\sqrt{(2\pi\theta)}} \exp\left(-\frac{x^2}{2\theta}\right), \quad -\infty < x < \infty.$$  

(i) Write down the likelihood function for $x$.

(ii) As a prior distribution for $\theta$ an inverse gamma distribution with parameters $k$ and $\lambda$ is chosen. Show that the posterior density function $\pi(\theta|x)$ satisfies

$$\pi(\theta|x) \propto \theta^{-k-1 - \frac{1}{2}n} \exp\left\{-\frac{1}{\theta}\left(\lambda + \frac{1}{2} \sum_{i=1}^{n} x_i^2\right)\right\}, \quad \theta > 0.$$  

What is the posterior distribution of $\theta$? What is the mean of this posterior distribution?

(iii) If you use the mean of the posterior density function to estimate $\theta$, what happens to your estimate for large $n$? Comment briefly on your result.

(iv) Consider again this case with $n$ large in question (b)(iii) above. Without doing any explicit calculations, explain what distribution you expect your estimator $\hat{\theta}$ for $\theta$ to have. Give a reason for your answer. What is the mean of this distribution of $\hat{\theta}$?  

(It is not necessary to specify the variance of this distribution.)

(v) Without doing any calculations, suggest one other way to estimate $\theta$ given the posterior density function.