Answer all questions in Section A and all questions in Section B.

Each question in Section A carries 2 marks, and each question in Section B carries 20 marks.

Questions A1 to A10 require you to write down a single letter answer.
Questions A11 to A20 require you to write down a short answer or draw a sketch.

Your answers to Section A questions and Section B questions may be written in the same answer book.
SECTION A

Answer ALL questions in Section A
Questions A1 to A10 require you to write down a single letter answer.

A1. Suppose that $X$ has probability density function

$$f_X(x) = 6x(1-x), \quad 0 < x < 1.$$ 

What is the value of the cumulative distribution function $F_X(x)$ at $x = 0.1$?

A: 0.012, B: 0.028, C: 0.124, D: 0.352, E: 0.540.

A2. For the random variable $X$ defined in question A1 above, what is the value of $E[X^2]$?

A: 0.300, B: 0.400, C: 0.500, D: 0.667, E: 0.800.

A3. For the random variable $X$ defined in question A1 above, if $U = X^2$, what is the probability density function $f_U(u)$ of $U$?

A: $3\sqrt{u}(1-\sqrt{u})$, B: $12u(1-\sqrt{u})$, C: $6\sqrt{u}(1-\sqrt{u})$, D: $3(1-\sqrt{u})$, E: $6u(1-\sqrt{u})$.

A4. The table below shows the joint probability function for two discrete random variables $X$ and $Y$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.1</td>
<td>0.1</td>
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<tr>
<td>1</td>
<td>0.2</td>
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<tr>
<td>2</td>
<td>0.2</td>
<td>0.0</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>

What is the value of $P\{X \leq 1\}$?

A: 0.4, B: 0.5, C: 0.6, D: 0.7, E: 0.8.
A5. In question A4 above, what is the value of $E[XY]$?
   A: 0.4, B: 0.5, C: 0.6, D: 0.7, E: 0.8.

A6. In question A4 above, what is the value of the conditional probability $P\{X = 2 | Y = 0\}$?
   A: 0.4, B: 0.5, C: 0.6, D: 0.7, E: 0.8.

A7. Suppose that $(X, Y)$ have a bivariate normal distribution with mean $\mu$ and variance matrix $\Sigma$ where
   \[
   \mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 0.8 \\ 0.8 & 9 \end{pmatrix}.
   \]
   What is the value of $\text{Var}[X + Y]$?
   A: 13.0, B: 13.8, C: 14.6, D: 17.8, E: 22.6.

A8. A random variable $X$ has probability density function $f_X(x) = \theta x^{\theta - 1}$ for $0 < x < 1$ where $\theta$ is an unknown parameter satisfying $\theta > 0$. A random sample $x = (x_1, \ldots, x_n)$ is available from this distribution and has sample mean $\bar{x}$. What is the maximum likelihood estimate $\hat{\theta}$ of $\theta$?
   A: $\bar{x} + \bar{x}$, B: $-n \log(x_1 x_2 \ldots x_n)$, C: $\log(x_1 x_2 \ldots x_n) / n$, D: $\bar{x}$, E: $\bar{x} / (1 - \bar{x})$.

A9. In question A8 above, $E[X] = \frac{\theta}{1 + \theta}$. What is the method of moments estimate $\tilde{\theta}$ of $\theta$?
   A: $\frac{\bar{x}}{1 + \bar{x}}$, B: $-n \log(x_1 x_2 \ldots x_n)$, C: $\log(x_1 x_2 \ldots x_n) / n$, D: $\bar{x}$, E: $\frac{\bar{x}}{1 - \bar{x}}$.

A10. Which of the following statements are true about method of moments estimators in general?
   (i) They give unbiased estimators.
   (ii) They are asymptotically normally distributed.
   (iii) They have minimum variance.
   A: All of these, B: (i) and (ii), C: (i) and (iii), D: (ii) only, E: None of these.
Questions A11 to A20 require you to write down a short answer or draw a sketch.

A11. Explain briefly what you understand by the phrase “a Poisson process with rate $\lambda$.”

A12. Explain briefly the difference between an estimate and an estimator.

A13. Suppose that $X$ has a gamma$(k, \lambda)$ distribution for $\lambda > 0$ and integer $k > 0$, and having probability density function

$$f_X(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}, \quad x > 0.$$  

Show that $E[X] = k/\lambda$.

A14. Suppose that $X_1, X_2, \ldots, X_n$ are independent normal $N(\mu, \sigma^2)$ random variables. It is known that $U = \sum_{i=1}^{n} (X_i - \bar{X})^2$ satisfies $U/\sigma^2 \sim \chi^2_{n-1}$ where $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Also recall that a $\chi^2_k$ random variable has mean $k$ and variance $2k$. It is intended to estimate the variance $\sigma^2$ using the statistic

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

(with divisor $n$). Show that $E[\hat{\sigma}^2] = \frac{(n-1)\sigma^2}{n}$ and obtain the variance of $\hat{\sigma}^2$.

A15. In question A14 above, obtain the bias and mean square error of $\hat{\sigma}^2$.

A16. Suppose $X$ and $Y$ are independent random variables with probability density functions $f_X(x) = e^{-x}$ for $x > 0$ and $f_Y(y) = ye^{-y}$ for $y > 0$ respectively. Write down the joint probability density function of $(X, Y)$.

A17. In question A16 above, consider the mapping $u = x - y$ and $v = y$ where $x > 0$ and $y > 0$. Show on a sketch in the $(x, y)$-plane the line corresponding to $u = +1$ and the line corresponding to $u = -1$.

A18. In question A16 above, for the mapping $U = X - Y$ and $V = Y$ show that the joint probability density function $f_{UV}(u, v)$ of $(U, V)$ satisfies

$$f_{UV}(u,v) = ve^{-u-2v}.$$  

A19. For the joint probability density function in question A18 above, obtain the marginal probability density function $f_U(u)$ in the case $u > 0$.

A20. “A sequence of random variables $Y_1, Y_2, Y_3, \ldots$ converges in probability to 0 so that $p \lim_{n \to \infty} Y_n = 0$.”  
Define carefully what is meant by this statement.
SECTION B

Answer all questions from Section B.

B1.

Suppose the random variable $X$ has an exponential distribution with parameter $\theta$ and probability density function $f_X(x) = \theta e^{-\theta x}$, $x > 0$.

(a) Obtain the moment generating function $m_X(t) = E[e^{tX}]$ of $X$. For what values of $t$ is this defined?

(b) By expanding $m_X(t)$ as a power series in $t$, or otherwise, deduce that $E[X^r] = r!/\theta^r$ for $r = 1, 2, 3, \ldots$.

(c) Suppose now that $X_1, \ldots, X_n$ are mutually independent and identically distributed random variables each having an exponential distribution with parameter $\theta$. Let $S_n = X_1 + \cdots + X_n$.

Write down, together with a brief statement of what results you have used, the moment generating function of $S_n$.

(d) Suppose that $Y = \frac{\theta S_n}{\sqrt{n}} - \sqrt{n}$.

(i) Obtain the the values of $E[S_n]$ and $\text{Var}[S_n]$ and so deduce the values of $E[Y]$ and $\text{Var}[Y]$.

(ii) If $Y$ has moment generating function $m_Y(t)$, show that

$$m_Y(t) = e^{-t\sqrt{n}} \left(1 - \frac{t}{\sqrt{n}}\right)^{-n}.$$ 

(iii) Obtain $\log m_Y(t)$ and deduce what happens as $n \to \infty$. What do you conclude about the distribution of $Y$ for large $n$?

You may use the result that if $U \sim N(\mu, \sigma^2)$, then it has moment generating function $m_U(t) = \exp(\mu t + \frac{1}{2} \sigma^2 t^2)$. Also you should recall that

$$\log(1 - u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \cdots$$

for $|u| < 1$ and

$$(1 - u)^{-1} = 1 + u + u^2 + u^3 + \cdots.$$
B2.

Suppose $x_1, x_2, \ldots, x_n$ forms a random sample from a Rayleigh distribution with common probability density function

$$f_X(x) = \frac{x}{\sigma^2} \exp \left( \frac{-x^2}{2\sigma^2} \right), \quad x > 0.$$ 

(a) Show that $E[X^2] = 2\sigma^2$.

(b) Obtain the maximum likelihood estimate $\hat{\sigma}^2$ for $\sigma^2$.

(c) Verify that $E[\hat{\sigma}^2] = \sigma^2$ and obtain $\text{Var}[\hat{\sigma}^2]$ for large $n$.

(d) Consider testing the null hypothesis $H_0$: $\sigma^2 = 1$ against the alternative hypothesis $H_1$: $\sigma^2 \neq 1$.

If $L_0$ denotes the maximised likelihood under $H_0$ and $L_1$ denotes the maximised likelihood under $H_0 \cup H_1$, obtain the form of the likelihood ratio test $\lambda = L_0/L_1$. What distribution does $-2\log \lambda$ have for large $n$?
B3.
Suppose a random sample $x = (x_1, x_2, \ldots, x_n)$ is taken from a normal $N(\theta, 1)$ distribution. It is desired to estimate the mean $\theta$. A normal distribution with zero mean and variance $1/t^2$ is used as a prior distribution for $\theta$.

(a) Show that the posterior density $\pi(\theta|x)$ for $\theta$ satisfies

$$
\pi(\theta|x) \propto \exp\left\{-\frac{1}{2} \left(\theta^2(n + t^2) - 2n\bar{x}\theta\right)\right\}
$$

where

$$
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.
$$

(b) By writing this posterior density in the form

$$
\pi(\theta|x) \propto \exp\left(-\frac{(\theta - m)^2}{2v^2}\right),
$$

or otherwise, deduce that $v^2 = 1/(n + t^2)$ and obtain a similar expression for $m$. What is the posterior distribution of $\theta$?

(c) Using your posterior distribution for $\theta$, what is your estimate for $\theta$? By recalling that $X_i \sim N(\theta, 1)$, obtain the mean and variance of your estimator for $\theta$.

(d) Discuss briefly what happens to your estimate in part (c) above:

(i) if $n$ is large,  
(ii) if $t$ is large,  
(iii) if $t$ is small.

Discuss briefly why someone might choose the case:  
(1) large $t$,  
(2) small $t$.

(e) Obtain the 95% Bayesian credibility interval for $\theta$.

You may use the result that if a random variable $Y \sim N(\mu, \sigma^2)$, then $Y$ has probability density function

$$
f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y - \mu)^2}{2\sigma^2}\right)
$$

for $-\infty < y < \infty$.

END OF QUESTIONS