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MATH271501

Statistical Methods

Time Allowed: 2 hours

Answer all questions in Section A and no more than three questions from Section B.

Each question in Section A carries 2 marks, and each question in Section B carries 20 marks.

Questions A1 to A10 require you to write down a single letter answer.
Questions A11 to A20 require you to write down a short explanation or draw a sketch.

Your answers to Section A questions and Section B questions may be written in the same answer book.
SECTION A

Answer ALL questions in Section A

Questions A1 to A10 require you to write down a single letter answer.

A1. A discrete random variable $X$ only takes values 0 or $\theta$ and satisfies

$$
P\{X = x\} = \begin{cases} 
\frac{1}{2} & \text{if } X = 0, \\
\frac{1}{2} & \text{if } X = \theta.
\end{cases}
$$

If $\mu$ denotes the mean of $X$, what does the third moment $\mu_3 = \mathbb{E}[(X - \mu)^3]$ equal?

A: 0,  
B: $\frac{1}{8}\theta^3$,  
C: $\frac{1}{4}\theta^3$,  
D: $\frac{1}{2}\theta^3$.

A2. A continuous random variable $X$ has probability density function given by $f_X(x) = \frac{1}{\theta}$ for $0 < x < \theta$.

A random sample $x_1, x_2, \ldots, x_n$ of $n$ observations is taken from this distribution and has sample mean $\bar{x}$ and sample variance $s^2$. What does the method of moments estimator $\tilde{\theta}$ for $\theta$ equal?

A: $\bar{x}$,  
B: $2\bar{x}$,  
C: $s^2$,  
D: $\max_i x_i$.

A3. The method of moments estimator $\tilde{\theta}$ in question A2 above is an unbiased estimator of $\theta$. If $v^2$ is the variance of this estimator, what does the mean square error of the estimator equal?

A: $v^2$,  
B: $\theta + v^2$,  
C: $2\theta + v^2$,  
D: $\theta^2 + v^2$.

A4. In question A3 above, which of the properties below is certainly true about the estimator $\tilde{\theta}$?

A: $\tilde{\theta} = 0$,  
B: $\tilde{\theta} = \theta$,  
C: $\mathbb{E}[\tilde{\theta}] = \theta$,  
D: $\tilde{\theta} \sim \mathcal{N}(\theta, v^2)$ $\forall n$.

A5. A sequence of random variables $Y_1, Y_2, \ldots$ converges in probability to a constant $C$. What does this mean?

A: For given $\epsilon > 0$ and $\eta > 0$ and $\forall n_0$, $\mathbb{P}\{|Y_n - C| > \epsilon\} < \eta$ $\forall n > n_0$.
B: For given $\epsilon > 0$ and $\eta > 0$, $\exists n_0 : \mathbb{P}\{|Y_n - C| > \epsilon\} < \eta$ $\forall n > 0$.
C: For given $\epsilon > 0$ and $\eta > 0$, $\exists n_0 : \mathbb{P}\{|Y_n - C| > \epsilon\} < \eta$ $\forall n > n_0$.
D: For given $\epsilon > 0$ and $\eta > 0$, $\exists n_0 : \mathbb{P}\{|Y_n - C| > \epsilon\} > \eta$ $\forall n > n_0$. 

TURN OVER...
A6. Consider a Bayesian estimation procedure for a parameter $\theta$ based upon a sample $x$. Suppose the likelihood function is $p(x|\theta)$, the prior density function is $\pi(\theta)$ and the posterior density function $\pi(\theta|x)$. Which of the following statements are always true?

(i) $\pi(\theta) \propto p(x|\theta)$.
(ii) $\pi(\theta|x) \propto p(x|\theta)\pi(\theta)$.
(iii) $\pi(\theta|x) < \pi(\theta)$ $\forall \theta$.

A: (i) and (iii) only, B: (ii) and (iii) only, C: (i) and (ii) only, D: (ii) only.

A7. In question A6 above, it is required to construct a Bayesian 95% credibility interval $(\theta_1, \theta_2)$ for $\theta$. Which of the following properties are true for the interval?

(i) $\pi(\theta_1) = \pi(\theta_2)$.
(ii) $\int_{\theta_1}^{\theta_2} \pi(\theta|x)d\theta = 0.95$.
(iii) $\pi(\theta|x) \geq \pi(\theta_1|x)$ for all $\theta \in (\theta_1, \theta_2)$.

A: (i) and (iii) only, B: (ii) and (iii) only, C: (i) and (ii) only, D: (ii) only.

A8. Suppose that $X = (X_1, X_2)$ has a bivariate normal distribution so that $X \sim N(\mu, \Sigma)$ where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \frac{1}{2}\sigma_2^2 \\ \frac{1}{2}\sigma_2^2 & \sigma_2^2 \end{pmatrix}.$$

What is the variance of $U = X_2 - 2X_1$?

A: $\sigma^2$, B: $2\sigma^2$, C: $3\sigma^2$, D: $4\sigma^2$.

A9. In question A8 above, what does $\text{cov}(X_1 + X_2, U)$ equal?

A: $-\frac{3}{2}\sigma^2$, B: $-\frac{1}{2}\sigma^2$, C: 0, D: $+\frac{1}{2}\sigma^2$, E: $+\frac{3}{2}\sigma^2$.

A10. In question A8 above, which of the following statements are true?

(i) $X_2$ and $U$ are independent.
(ii) $X_2$ and $U$ are uncorrelated.
(iii) $X_2$ and $U$ have a bivariate normal distribution.

A: (i) and (iii) only, B: (i), (ii) and (iii), C: (i) and (ii) only, D: (ii) only.
Questions A11 to A20 require you to write down a short answer or draw a sketch.

A11. A random variable $X$ has an exponential distribution with parameter $\theta > 0$ and probability density function $f_X(x) = \theta e^{-\theta x}$ for $x > 0$. Show that the cumulative distribution function $F_X(x)$ of $X$ is $F_X(x) = 1 - e^{-\theta x}$ for $x > 0$.

A12. For the random variable $X$ in question A11 above, show that the mean of $X$ is $E[X] = 1/\theta$.

A13. Suppose that $X$ and $Y$ are independent exponential random variables, each with parameter $\theta$, as in question A11 above. Write down the joint probability density function $f_{XY}(x, y)$ for $(X, Y)$. What is the range of $X$ and $Y$?

A14. If $u = x + y$ and $v = y$, explain why the region $x > 0$ and $y > 0$ in the $x$-$y$ plane maps to the region $u > 0$ and $0 < v < u$ in the $u$-$v$ plane. Sketch this region in the $u$-$v$ plane.

A15. If $X$ and $Y$ are independent exponential random variables, each with parameter $\theta$, and if $U = X + Y$ and $V = Y$, show that $(U, V)$ have joint probability density function $f_{UV}(u, v) = \theta^2 e^{-\theta u}$.

A16. For $(U, V)$ having joint probability density function $f_{UV}(u, v) = \theta^2 e^{-\theta u}$ as in question A15 above, obtain the marginal probability density function $f_U(u)$ of $U$. What is the range of $u$ where $f_U(u)$ is non-zero?

A17. A random sample $x_1, x_2, \ldots, x_n$ of size $n$ is taken from an exponential distribution with parameter $\theta$ as defined in question A11 above. Write down the likelihood and the log-likelihood functions for $\theta$.

A18. Obtain the maximum likelihood estimate for $\theta$ based on the random sample in question A17 above.

A19. For a non-negative continuous random variable $X$, having probability density function $f_X(x)$, prove the Markov inequality

$$P\{X \geq a\} \leq \frac{E[X]}{a},$$

where $a > 0$ is a constant.

(Hint: Consider the definition of $E[X]$ and split the range of integration at a suitable point.)

A20. Verify directly that if $X$ has an exponential distribution with parameter $\theta$, then it does indeed satisfy the Markov inequality in question A19 above. (You can use the fact that $e^c > c$ for all values $c > 0$.)
SECTION B

Answer no more than THREE questions from Section B.

B1.

(a) A random variable $W$ has a gamma $(\alpha, \lambda)$ distribution with probability density function

$$f_W(w) = \frac{\lambda^\alpha w^{\alpha-1}e^{-\lambda w}}{\Gamma(\alpha)}, \quad w > 0,$$

where $\Gamma(\alpha)$ is the usual gamma function satisfying $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Obtain the mean of $W$.

(b) Suppose $X \sim N(0, 1)$ and independently $Y \sim \chi^2_k$ with probability density functions respectively

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}, \quad -\infty < x < +\infty,$$

and

$$f_Y(y) = \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}} y^{\frac{k}{2}-1}e^{-\frac{1}{2}y}}{\Gamma\left(\frac{1}{2}k\right)}, \quad y > 0.$$

(i) Using the transformation $U = \frac{X}{\sqrt{Y/k}}$ and $V = Y$ show that the joint probability density function of $(U, V)$ satisfies

$$f_{UV}(u, v) = \frac{\left(\frac{1}{2}\right)^{\frac{k}{2}} u^{\frac{k}{2}(k-1)}e^{-\frac{1}{2}v(1+u^2/k)}}{\sqrt{2\pi} \sqrt{k} \Gamma\left(\frac{1}{2}k\right)}.$$

What are the ranges of $U$ and $V$?

(ii) By making the substitution $c = \frac{1}{2} \left(1 + u^2/k\right)$, or otherwise, show that the probability density function $f_U(u)$ of $U$ satisfies

$$f_U(u) = \frac{\Gamma\left(\frac{1}{2}(k+1)\right)}{\Gamma\left(\frac{1}{2}k\right) \sqrt{k\pi}} \cdot \frac{1}{(1 + u^2/k)^{\frac{1}{2}(k+1)}}.$$

(iii) Consider now the case $k = 1$. What is the probability density function of $U$ in this case? What is the name of this distribution? Briefly describe the key properties of this distribution.
Suppose a random variable \( X \) has a geometric distribution with probability function 
\[
P\{X = x\} = \theta^x(1 - \theta), \quad x = 0, 1, 2, 3, \ldots
\]
where \( 0 < \theta < 1 \).

(a) (i) Show that \( X \) has moment generating function \( m_X(t) = \mathbb{E}[e^{tX}] \) given by
\[
m_X(t) = \frac{1 - \theta}{1 - \theta e^t}.
\]
(You may note that \( \sum_{x=0}^{\infty} u^x = \frac{1}{1 - u} \) for \( |u| < 1 \).)

(ii) What is the range of \( t \) for which this moment generating function is defined?

(iii) Obtain the mean of \( X \). Show that the variance of \( X \) is \( \text{Var}[X] = \theta(1 - \theta) \).

(b) Suppose that \( X_1, X_2, \ldots, X_n \) are independent geometric random variables as defined above. Let
\[
S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad \bar{X}_n = \frac{S_n}{n}
\]

(i) If \( n \to \infty \) and \( \theta \to 0 \) such that \( n\theta = \lambda \) is a constant, show that the moment generating function of \( S_n \) tends towards \( \exp(\lambda(e^t - 1)) \) which is the moment generating function of a Poisson(\( \lambda \)) random variable. What do you conclude about the distribution of \( S_n \) for large \( n \) in this case?
(You may use the result that \( \log(1 - v) = -v - \frac{1}{2}v^2 - \frac{1}{3}v^3 - \cdots \) if \( |v| < 1 \).)

(ii) Now suppose that \( \theta \) is fixed and \( n \to \infty \). Explain why \( Z_n = \bar{X}_n - \mu \sigma/\sqrt{n} \) is asymptotically a normal distribution, where \( \mu = \mathbb{E}[X] \) and \( \sigma^2 = \text{Var}[X] \). (There is no need to give a detailed explicit proof but do outline the key steps in your proof.)

B3.

(a) Outline the key properties of maximum likelihood estimators.

(b) Suppose \( X_1, X_2, \ldots, X_n \) are independent \( N(\mu, \sigma^2) \) random variables with common probability density function
\[
f_X(x) = \frac{1}{\sqrt{2\pi} \sigma} \cdot \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right), \quad -\infty < x < +\infty.
\]

(i) Obtain the maximum likelihood estimates \( \hat{\mu} \) and \( \hat{\sigma} \) for \( \mu \) and \( \sigma \).

(ii) Show that for large \( n \), \( \text{Var}[\hat{\mu}] = \sigma^2/n \), \( \text{Var}[\hat{\sigma}] = \sigma^2/(2n) \) and \( \text{cov}(\hat{\mu}, \hat{\sigma}) = 0 \).

(iii) The normal density curve has points of inflexion at \( \theta = \mu + \sigma \) and \( \phi = \mu - \sigma \). What are the maximum likelihood estimates \( \hat{\theta} \) and \( \hat{\phi} \) for \( \theta \) and \( \phi \)? Obtain \( \text{Var}[\hat{\theta}] \), \( \text{Var}[\hat{\phi}] \) and \( \text{corr}(\hat{\theta}, \hat{\phi}) \).
B4.

(a) A random variable $X$ has a beta $(a, b)$ distribution with probability density function

$$f_X(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a, b)}, \quad 0 < x < 1,$$

where $a > 0$ and $b > 0$, and where $B(a, b)$ is a beta function satisfying

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

where $\Gamma(\alpha)$ is a gamma function satisfying $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ and $\Gamma(1) = 1$.

(i) Explain briefly why the beta distribution is a useful distribution to model proportions.

(ii) Obtain the mean of $X$.

(iii) Show that if $a > 1$ and $b > 1$, then the mode of $X$ is

$$x = \frac{a-1}{a+b-2}$$

(b) The negative binomial distribution is a frequently used model in ecological capture-recapture studies and satisfies

$$P\{N = n\} = \binom{n-1}{m-1} \theta^m (1-\theta)^{n-m}, \quad n = m, m+1, m+2, \ldots,$$

where $m$ is a fixed positive integer and $N$ represents the number of “trials” to obtain $m$ “successes” where $\theta$ is the probability of a “success”.

As a prior density for $\theta$ a beta $(a, b)$ distribution is used.

(i) For given $m$ and $n$, obtain the posterior distribution for $\theta$.

(ii) Obtain two different point estimators for $\theta$ by using the posterior distribution you found in question (b-i) above.

(iii) What does the posterior mode equal if the prior density for $\theta$ is a uniform $(0, 1)$ distribution? Why does this give an intuitively sensible estimator for $\theta$ in this case?

(iv) How does your answer to question (b)(iii) change if you are given $k$ independent observations $n_1, n_2, \ldots, n_k$?

END OF QUESTIONS