Hilbert-Schmidt Operators vs. Integrable Systems of Elliptic Calogero-Moser Type I. The Eigenfunction Identities

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Received: 4 December 2007 / Revised: 9 July 2008 / Accepted: 10 October 2008 Published online: 9 December 2008 – © Springer-Verlag 2008

Abstract: In this series of papers we study Hilbert-Schmidt integral operators acting on the Hilbert spaces associated with elliptic Calogero-Moser type Hamiltonians. As shown in this first part, the integral kernels are joint eigenfunctions of differences of the latter Hamiltonians. On the relativistic (difference operator) level the kernel is built from the elliptic gamma function, whereas the building block in the nonrelativistic (differential operator) limit is basically the Weierstrass sigma-function. For the $A_{N-1}$ case we consider all of the commuting Hamiltonians at once, the eigenfunction properties reducing to a sequence of elliptic identities. For the $BC_N$ case we only treat the defining Hamiltonians. The functional identities encoding the eigenfunction properties have a remarkable corollary in the relativistic $BC_1$ case: They imply that the sum over eight-fold products of the four Jacobi theta functions is invariant under the Weyl group of $E_8$.

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1. Introduction

The nonrelativistic elliptic Calogero-Moser Hamiltonian is given by the PDO (partial differential operator)
Here, the pair potential is the Weierstrass $\wp$-function; its periods $\pi/r$ and $i\alpha$ are chosen positive and purely imaginary, so that the PDO is formally self-adjoint. As is well known, this Hamiltonian defines a quantum integrable system: There exist $N$ commuting PDOs

$$H_1 = -i \sum_{j=1}^{N} \partial_x^2, \quad H_2 = H_{nr}, \quad H_k = \frac{(-i)^k}{k} \sum_{j=1}^{N} \partial_x^k + 1. \text{ o.}, \quad k = 3, \ldots, N,$$

where l. o. denotes a PDO of lower order in the partials with elliptic coefficients \cite{1–3}. To date, however, no joint eigenfunctions are known to exist for arbitrary $g, r, \alpha > 0$, save for $N = 2$; in that case, the Schrödinger equation for (1.1) amounts to the Lamé equation.

The $N$ commuting Hamiltonians of the relativistic Calogero-Moser system can be chosen to be the A$\Delta$Os \cite{337–343}, and they have versions for other root systems as well. The $BC_N$ version of the nonrelativistic Calogero-Moser system is defined by the Inozemtsev Hamiltonian \cite{6}

$$H_{nr}(g, \lambda; x) \equiv - \frac{1}{2} \sum_{j=1}^{N} \partial_x^2 + 2g(g-1) \sum_{1 \leq j < k \leq N} \wp(x_j - x_k; \pi/2r, i\alpha/2), \quad g \in \mathbb{R}, \quad r, \alpha > 0. \quad (1.1)$$

The existence of $N-1$ commuting PDOs of higher order follows from papers by Oshima and coworkers \cite{2,3}. For $N = 1$ the Schrödinger equation for (1.5) reduces to the Heun
equation, which has been extensively studied in the literature. For $N > 1$, however, not much is known about joint eigenfunctions, or even about eigenfunctions of the single PDO (1.5).

A ‘relativistic’ generalization of the elliptic $BC_N$ systems was proposed by van Diejen [7], and shown to be integrable by Komori and Hikami [8]. For this case already the defining Hamiltonian for $N = 1$ involves an elaborate definition, whereas the commuting Hamiltonians for $N > 1$ are not even known in explicit form (with one exception [7]). To ease the exposition, we specify the defining Hamiltonian later on, cf. Subsect. 3.1 for $N = 1$ and Subsect. 4.1 for $N > 1$. Just as in the nonrelativistic case, very little is known about eigenfunctions of the defining $BC_N$ Hamiltonian for $N > 1$ (or even for $N = 1$).

A principal aim of this series of papers is to develop a novel approach towards proving the existence of joint Hilbert space eigenfunctions of the above elliptic Hamiltonians, a question that thus far has been wide open. Put more precisely, our starting point is such that it yields right away an ONB (orthonormal base) for the relevant Hilbert spaces, and the main problem consists in proving the conjecture that the functions in this ONB are joint eigenfunctions for the Hamiltonians with real eigenvalues, hence enabling a reinterpretation as commuting self-adjoint Hilbert space operators.

Since the above Hamiltonians are all invariant under permutations, translations over the real elliptic period $\pi/r$, and, in the $BC_N$ case, sign changes of $x_1, \ldots, x_N$, the ‘smallest’ Hilbert space that can be associated with the PDOs and $A\Delta$Os is given by

$$\mathcal{H} \equiv L^2(F, dx),$$

where

$$F \equiv \{ x \in \mathbb{R}^N \mid -\pi/2r < x_N < \cdots < x_1 \leq \pi/2r \}, \quad (A_{N-1}),$$

$$F \equiv \{ x \in \mathbb{R}^N \mid 0 < x_N < \cdots < x_1 \leq \pi/2r \}, \quad (BC_N).$$

The aforementioned ONB of this natural quantum arena $\mathcal{H}$ for the elliptic Calogero-Moser Hamiltonians now arises from certain Hilbert-Schmidt integral operators acting on $\mathcal{H}$. The kernels $\Psi(x, y)$ of these operators will be detailed later on. For now, we only mention their pertinent features, so as to clarify the relation of the Hilbert-Schmidt (from now on HS) operators to the above Hamiltonians.

First, the functions $\Psi(x, y)$ depend on the four cases at issue, and are expressed in terms of the elliptic gamma function and its limits. Second, they are eigenfunctions of differences $H(x) - H(-y)$ of the $A_{N-1}$ and $BC_N$ PDOs and $A\Delta$Os. Third, for suitably restricted parameters they are manifestly square-integrable over $F \times F$, thus defining HS integral operators $I$ on $\mathcal{H}$.

For the two $A_{N-1}$ cases there exists an infinite-dimensional family of kernels $\Psi(x, y)$ with these features. In particular, they all satisfy

$$(H(x) - H(-y))\Psi(x, y) = 0, \quad (A_{N-1}),$$

and give rise to HS operators all of which commute and are normal. The above ONB is the ONB of joint eigenvectors of this commuting family of HS operators, whose existence is an immediate consequence of the spectral theorem. For the two $BC_N$ cases we only know one function $\Psi(p; x, y)$, where $p$ denotes the coupling parameters (4 in the nonrelativistic and 8 in the relativistic case). It satisfies

$$(H(p; x) - H(p'; y))\Psi(p; x, y) = \sigma(p)\Psi(p; x, y), \quad (BC_N),$$

where

$$\sigma(p) \equiv \frac{1}{2} \left( \frac{1 - \cos(p)}{\sin(p)} \right)^{N-1}.$$
with \( p' \) linearly related to \( p \). The HS operator \( \mathcal{I}(p) \) with kernel \( \Psi(p; x, y) \) is not normal for general \( p \). The conjectured ONB of eigenvectors for the defining Hamiltonian \( H(p; x) \) is in this case the ONB of eigenvectors of the positive trace class operator \( \mathcal{I}(p) \mathcal{I}(p)^* \). (In fact, we believe this is also an ONB of eigenvectors for the remaining \( N - 1 \) commuting \( BC_N \) operators.)

We would like to stress that, as they stand, the above eigenfunction formulae (1.9)–(1.10) have no direct bearing on the reinterpretation of the commuting Calogero-Moser Hamiltonians as operators on \( \mathcal{H} \). As already mentioned, however, their relevance arises from the expected ONB relation; in this scenario, (1.9) is encoding a Hilbert space commutativity relation

\[
\hat{H} \mathcal{I} - \mathcal{I} \hat{H} = 0, \tag{1.11}
\]

whereas (1.10) corresponds to

\[
\hat{H}(p) \mathcal{I}(p) - \mathcal{I}(p) \hat{H}(p') = \sigma(p) \mathbf{1}, \tag{1.12}
\]

the definition domains in \( \mathcal{H} \) of the Hamiltonians \( \hat{H} \) being determined by the ONB vectors associated with \( \mathcal{I} \). In our lectures at the 2004 RIMS Workshop on Elliptic Integrable Systems (an account of which has appeared in [9,10]), we have already announced the eigenfunction relations (1.9)–(1.10), and explained the above scenario in more detail, cf. [10]. Moreover, a survey of elliptic eigenfunction literature up to 2005 can be found in [9].

As mentioned above, in this series of papers we intend to work out the consequences of the novel HS perspective for the joint Hilbert space eigenvector problem. The present first part has an algebraic and function-theoretic character, inasmuch as it is only concerned with the eigenfunction properties of \( \Psi(x, y) \), and the associated functional identities of elliptic type. In the second part we study some introductory issues for the \( AN_{-1} \) case [11], and in the third and fourth part the nonrelativistic and relativistic \( BC_1 \) cases [12,13]. We also intend to elaborate on the \( N > 1 \) cases.

We proceed to sketch the organization and results of this paper in more detail. We consider the cases \( AN_{-1}, BC_1 \) and \( BC_N \) in Sects. 2, 3 and 4, resp., referring to [14] for the proof of the joint eigenfunction property of \( \Psi(x, y) \) in the \( AN_{-1} \) case. On the relativistic level it is convenient to switch to analytic difference operators \( A \) with meromorphic coefficients via a similarity transformation with a weight function that is positive on \( F \), as follows:

\[
A(p; x) = w(p; x)^{-1/2} H(p; x) w(p; x)^{1/2}. \tag{1.13}
\]

Here, \( p \) denotes case-dependent parameters, which we continue to suppress in the \( AN_{-1} \) case. The eigenfunction identities in the \( AN_{-1} \) case are then of the form

\[
(A(x) - A(-y))S(x, y) = 0, \quad S(x, y) = \frac{\Psi(x, y)}{[w(x)w(y)]^{1/2}}, \tag{1.14}
\]

whereas for the \( BC_N \) case they read

\[
(A(p; x) - A(p'; y))S(p; x, y) = \sigma(p)S(p; x, y), \quad S(p; x, y) = \frac{\Psi(p; x, y)}{[w(p; x)w(p'; y)]^{1/2}}. \tag{1.15}
\]
In the nonrelativistic $BC_1$ (Heun) and $BC_N$ (Inozemtsev) cases an analogous similarity transformation can also be made, leading to the same structure for $\Psi(x, y)$. More specifically, again the function $S$ has the simplest form, with $\Psi$ related to it via the above formulas involving a weight function, but now the transformed PDOs have a more complicated appearance than the defining ones.

We begin each section by detailing the relevant A$\Delta$Os (analytic difference operators). Next we specify the weight function. Since it is of the form $w(x) = 1/c(x)c(-x)$, (1.16)

we need only define $c(x)$, which may be viewed as a generalized Harish-Chandra $c$-function. Then we introduce the special eigenfunction $S$ and consider eventual generalizations. The nonrelativistic (differential operator) $BC_1$ and $BC_N$ cases are studied in Subsect. 3.2 and 4.2, resp.

It is a remarkable fact that the ‘eigenvalue’ $\sigma(p)$ actually vanishes for the nonrelativistic $BC_N$ case. (This amounts to the function on the rhs of (4.34) being identically zero, an identity that is far from obvious.) This is no longer true for the relativistic $BC_N$ case. Indeed, for the $BC_1$ case this can be seen from an explicit formula for $\sigma(p)$. On the other hand, in this special ($N = 1$) case the additive constant in the defining A$\Delta$O can be changed in such a way that the new $\sigma(p)$ does vanish.

As a corollary of the explicit $\sigma(p)$-evaluation, it follows that a certain function is invariant under the Weyl group of $E_8$, cf. Proposition 3.3. Using formulas specified on p. 224 of the lecture notes [15], this function can be expressed in terms of Jacobi theta functions. Doing so, it is proportional to

$$\Sigma_8(z) \equiv \sum_{l=1}^{4} \prod_{m=1}^{8} \theta_l(z_m), \quad z \in \mathbb{C}^8.$$ (1.17)

To our knowledge, the resulting $E_8$ invariance of $\Sigma_8(z)$ has not been observed before. (Of course, $D_8$ invariance is plain. It is the invariance under the $E_8$ reflection

$$z_m \rightarrow -z_m + \frac{1}{4} \sum_{n=1}^{8} z_n, \quad m = 1, \ldots, 8,$$ (1.18)

that is striking.)

The elliptic gamma function plays a pivotal role in this series of papers. It can be defined by the infinite product

$$G(r, a_+, a_-; z) \equiv \prod_{m,n=0}^{\infty} \frac{1 - \exp(-(2m+1)ra_+ - (2n+1)ra_- - 2irz)}{1 - \exp(-(2m+1)ra_+ - (2n+1)ra_- + 2irz)}.$$ (1.19)

Here and throughout this paper we choose the parameters $r, a_+, a_-$ positive. From (1.19) it readily follows that $G(r, a_+, a_-; z)$ solves the first order analytic difference equations (from now on A$\Delta$Es)

$$\frac{G(z + ia_8/2)}{G(z - ia_8/2)} = R_\delta(z), \quad \delta = +, -.$$ (1.20)
with right-hand-side functions

\[ R_\delta(z) \equiv R(r, a_\delta; z) \equiv \prod_{l=0}^{\infty} \left( 1 - \exp[-(2l + 1)r a_\delta + 2irz] \right) \times \left( 1 - \exp[-(2l + 1)r a_\delta - 2irz] \right). \]  

(1.21)

In turn, this product formula entails that \( R_\delta(z) \) solves the A\(\Delta\)E,

\[ \frac{R_\delta(z + i a_\delta/2)}{R_\delta(z - i a_\delta/2)} = -\exp(-2irz). \]  

(1.22)

Defining the functions

\[ s_\delta(z) \equiv s(r, a_\delta; z) \equiv -ie^{irz} R_\delta(z + i a_\delta/2)/p_\delta, \]  

(1.23)

where \( p_+ \) and \( p_- \) denote the positive constants

\[ p_\delta \equiv p(r, a_\delta) \equiv 2r \prod_{k=1}^{\infty} (1 - e^{-2kra_\delta})^2, \quad \delta = +, -, \]  

(1.24)

we also obtain

\[ \frac{s_\delta(z + i a_\delta/2)}{s_\delta(z - i a_\delta/2)} = -\exp(-2irz). \]  

(1.25)

The latter functions are related to the Weierstrass \( \sigma \)-function via

\[ s(r, a_\delta; z) = \exp(-\eta_\delta z^2r/\pi) \sigma(z; \pi/2r, i a_\delta/2). \]  

(1.26)

We have occasion to use a few properties of the elliptic gamma function, which we proceed to collect. (We suppress dependence on the parameters when no confusion can occur.) First, we have the relations

\[ G(-z) = 1/G(z), \quad \text{(reflection equation)}, \]  

(1.27)

\[ G(z + \pi/r) = G(z), \quad \text{(periodicity)}, \]  

(1.28)

\[ G(r, a_-, a_+; z) = G(r, a_+, a_-; z), \quad \text{(modular invariance)}, \]  

(1.29)

which follow by inspection from (1.19). Second, we need the limit

\[ \lim_{\alpha \to 0} G(r, a_+, a_-; z + i a_- \lambda) = \exp((\lambda - \kappa) \ln R_\lambda(z)), \quad \lambda, \kappa \in \mathbb{R}, \]  

(1.30)

cf. the last paragraph of Subsect. IIIB in [16]. Third, we use the \( G \)-duplication formula

\[ G(r, a_+, a_-; 2z) = \prod_{l, m = +, -} G(r, a_+, a_-; z - i (la_+ + ma_-)/4) \cdot G(r, a_+, a_-; z - i (la_+ + ma_-)/4 - \pi/2r), \]  

(1.31)

cf. (3.106) in [16]. Via (1.20) it implies the \( R \)-duplication formula

\[ R(2z) = \prod_{\delta = +, -} R(z - \delta i \alpha/4)R(z - \delta i \alpha/4 - \pi/2r), \quad R(z) \equiv R(r, \alpha; z). \]  

(1.32)
Fourth, for later purposes we list the representation

\[ G(r, a_+, a_-; z) = \exp \left( i \sum_{n=1}^{\infty} \frac{\sin 2nrz}{2n \sinh nr a_+ \sinh nr a_-} \right), \quad |\Im z| < (a_+ + a_-)/2. \]  

(1.33)

The corresponding representation of \( R \) reads

\[ R(r, \alpha; z) = \exp \left( - \sum_{n=1}^{\infty} \frac{\cos 2nrz}{n \sinh nr \alpha} \right), \quad |\Im z| < \alpha / 2. \]  

(1.34)

2. The Commuting \( A_{N-1} \) Hamiltonians

In Sect. 1 we defined the \( 4N \) commuting \( A_{N-1} \) AΔOs in terms of the \( \sigma \)-function, cf. (1.3)–(1.4). Due to the \( R \)-function being the right-hand side of the elliptic gamma function AΔEs (recall (1.20)), it is however more convenient to switch to \( R \) and omit multiplicative constants arising from this change, cf. (1.26) and (1.23). Furthermore, we substitute \( \alpha, \beta \rightarrow a_+, a_- \).

After these changes, the similarity-transformed AΔOs are given by

\[ A_{l,\delta}(x) = \sum_{I \subset \{1, \ldots, N\}} \prod_{m \in I} \frac{R_\delta(x_m - x_n - \mu + ia_\delta/2)}{R_\delta(x_m - x_n + ia_\delta/2)} \cdot \prod_{m \in I} \exp(-ia_- \delta \partial x_m), \quad \text{ (2.1)} \]

where \( l = 1, \ldots, N \) and \( \delta = +, - \); furthermore, we have

\[ A_{-l,\delta}(x) \equiv A_{l,\delta}(-x), \quad l = 1, \ldots, N, \quad \delta = +, -. \]  

(2.2)

The weight function (1.16) is defined via the \( c \)-function

\[ c(x) \equiv \prod_{1 \leq j < k \leq N} \frac{G(x_j - x_k - \mu + i(a_+ + a_-)/2)}{G(x_j - x_k + i(a_+ + a_-)/2)}. \]  

(2.3)

Defining the special function

\[ S(x, y) = \prod_{j,k=1}^{N} \frac{G(x_j - y_k - \mu/2)}{G(x_j - y_k + \mu/2)} \]  

we now have the eigenfunction identities

\[ (A_k,\delta(x) - A_k,\delta(-y))S(x, y) = 0, \quad k \in \{\pm 1, \ldots, \pm N\}, \quad \delta \in \{+, -\}. \]  

(2.5)

Taking their validity for granted (for a proof, see Sect. 2 in [14]), it easily follows that the one-parameter generalization

\[ S_\xi(x, y) = \prod_{j,k=1}^{N} \frac{G(x_j - y_k - \xi - \mu/2)}{G(x_j - y_k + \xi + \mu/2)}, \quad \xi \in \mathbb{C}. \]  

(2.6)
satisfies (2.5), too. Moreover, the identities (2.5) still hold when we multiply $S_\xi$ by
\[
\phi\left(\sum_{j=1}^{N} (x_j - y_j)\right), \quad \phi(z) \text{ meromorphic.} \tag{2.7}
\]
Thus we obtain an infinite-dimensional space of joint zero-eigenvalue eigenfunctions of $4N$ differences of $A\Delta O$s.

In order to obtain the nonrelativistic counterparts, we put
\[
\mu = ia - g, \quad g \in \mathbb{R}, \quad a_+ \to \alpha, \tag{2.8}
\]
and let $a_-$ tend to 0. Then (1.30) implies that (2.6) converges to
\[
S_{nr,\xi}(x, y) = \prod_{j,k=1}^{N} R(x_j - y_k + \xi)^{-g}, \quad R(x) \equiv R(r, \alpha; x), \tag{2.9}
\]
while the weight function has the limit
\[
w_{nr}(x) = \left(\prod_{1 \leq j < k \leq N} R(x_j - x_k + i\alpha/2)R(x_j - x_k - i\alpha/2)\right)^g. \tag{2.10}
\]
By (1.22)–(1.24), we can rewrite this as
\[
w_{nr}(x) = p(r, \alpha)^{N(N-1)g} \left(\prod_{1 \leq j < k \leq N} s(x_j - x_k)\right)^{2g}. \tag{2.11}
\]
From this representation it is clear that $w_{nr}(x)$ is positive on the set (1.7) (with the obvious choice of logarithm branch).

The nonrelativistic limit of the commuting $A\Delta O$s (2.1)–(2.2) and corresponding Hamiltonians (1.13) is studied in Subsect. 4.2 and 4.3 of [5]. For the differences of Hamiltonians we may work with the renormalized eigenfunction
\[
\Psi_{nr,\xi}(x, y) = \left(\prod_{1 \leq j < k \leq N} s(x_j - x_k)s(y_j - y_k)\right)^g \frac{1}{\prod_{j,k=1}^{N} R(x_j - y_k + \xi)^g}, \tag{2.12}
\]
cf. (2.9)–(2.11). In particular, we obtain (see Section 3 in [14])
\[
(H_{nr}(x) - H_{nr}(y))\Psi_{nr,\xi}(x, y) = 0, \tag{2.13}
\]
where $H_{nr}(x)$ is the defining Hamiltonian (1.1) of the nonrelativistic Calogero-Moser system. The identity (2.13) was derived first by Langmann in a quite different context [17].
3. The $BC_1$ Hamiltonians

3.1. The relativistic case. In this one-variable case there are two commuting $A+\Delta Os A_+$ and $A_-$, the first one of which may be viewed as a similarity transform of the defining Hamiltonian. It is given by

$$A_+(h; x) \equiv V(h; x) \exp(-ia_--dx) + V(h; -x) \exp(ia_--dx) + V_b(h; x), \quad (3.1)$$

where

$$V(h; x) \equiv \prod_{n=0}^{7} \frac{R_+(x - h_n - ia_-/2)}{R_+(2x + ia_+/2)R_+(2x - ia_- + ia_+/2)}, \quad (3.2)$$

$$V_b(h; x) \equiv \frac{\sum_{l=0}^{3} p_l(h)[E_t(\mu; x) - E_t(\mu; z_l)]}{2R_+(\mu - ia_+/2)R_+(\mu - ia_- - ia_+/2)}, \quad z_0 = z_2 = \pi/2 r, \quad z_1 = z_3 = 0. \quad (3.3)$$

To define $p_t$ and $E_t$, we first introduce half-periods

$$\omega_0 = 0, \quad \omega_1 = \pi/2 r, \quad \omega_2 = ia_+/2, \quad \omega_3 = -\omega_1 - \omega_2. \quad (3.4)$$

Then the product functions $p_t$ are given by

$$p_0(h) \equiv \prod_n R_+(h_n), \quad p_1(h) \equiv \prod_n R_+(h_n - \omega_1), \quad (3.5)$$

$$p_2(h) \equiv \exp(-2ra_+) \prod_n \exp(-irh_n)R_+(h_n - \omega_2), \quad (3.6)$$

$$p_3(h) \equiv \exp(-2ra_+) \prod_n \exp(irh_n)R_+(h_n - \omega_3), \quad (3.7)$$

and $E_t$ reads

$$E_t(\mu; x) \equiv \frac{R_+(x + \mu - ia - \omega_t)R_+(x + \mu + ia - \omega_t)}{R_+(x - ia - \omega_t)R_+(x + ia - \omega_t)}, \quad t = 0, \ldots, 3. \quad (3.8)$$

Here and from now on, we use a new parameter $a$ defined by

$$a \equiv (a_+ + a_-)/2. \quad (3.9)$$

It is not clear by inspection, but true that the function $V_b(h; x)$ does not depend on the parameter $\mu \in \mathbb{C}$, cf. Lemma 3.2 in [15]. In particular, this entails the $\mu = 0$ representation

$$V_b(h; x) = \frac{\rho}{2R_+(ia_- + ia_+/2)} \times \sum_{t=0}^{3} p_t(h)[[L_+(x - ia - \omega_t) - L_+(x + ia - \omega_t)] - [x \to z_t]], \quad (3.10)$$

where

$$\rho \equiv \lim_{z \to -ia_+/2} \frac{z + ia_+/2}{R_+(z)} = -i \exp(ra_+/2)/p_+, \quad (3.11)$$
(cf. (1.23)) and \(L_+(z)\) is the logarithmic derivative

\[
L_+(z) \equiv R'_+(z)/R_+(z).
\] (3.12)

The \(A \Delta O A_+\) depends on 8 parameters \(h_0, \ldots, h_7\), apart from the parameters \(r, a_+, a_-\), which we usually omit. It is related to the defining \(BC_1\) Hamiltonian via (1.13), where the weight function is given by (1.16) and

\[
c(h; x) \equiv \frac{1}{G(2x + ia)} \prod_{n=0}^7 G(x - h_n).
\] (3.13)

More precisely, the \(BC_1\) Hamiltonians introduced in [7,8] are related to \(H(h; x)\) via some complicated multiplicative and additive constants, cf. Appendix B in [15].

The \(A \Delta O A_- (h; x)\) is obtained from \(A_+(h; x)\) by interchanging the step size parameters \(a_+\) and \(a_-\) wherever they occur. Noting \(V_b(h; x)\) is elliptic with periods \(\pi/r\) and \(ia_+\) and using the \(A \Delta E\) (1.22), it is easy to check that \(A_+\) and \(A_-\) commute. Clearly, \(A_\pm\) are even, invariant under shifting \(x\) and the \(h_n\)'s by \(2\omega_1\), and permutation invariant:

\[
A_\delta(h; x) = A_\delta(h; -x) = A_\delta(h; x + 2\omega_1),
\] (3.14)

\[
A_\delta(h; x) = A_\delta(h + 2\omega_1e_n; x), \quad n = 0, \ldots, 7,
\] (3.15)

\[
A_\delta(h; x) = A_\delta(wh; x), \quad \forall w \in S_8.
\] (3.16)

A moment’s thought shows that we also have

\[
A_\delta(h + \omega_1\zeta; x) \equiv A_\delta(h; \omega_1 - x), \quad \zeta \equiv \sum_{n=0}^7 e_n.
\] (3.17)

We now introduce the special function

\[
S(x, y) \equiv \prod_{\delta_1, \delta_2 = +, -} G(\delta_1 x + \delta_2 y - ia + \phi),
\] (3.18)

and proceed to answer the question: Given \(A_+(h; x)\) with generic parameters \(a_+, a_- \in (0, \infty)\) and \(h \in \mathbb{C}^8\), for which \(\phi \in \mathbb{C}\) and \(h' \in \mathbb{C}^8\) is the quotient function

\[
Q(x, y) \equiv \frac{1}{S(x, y)}(A_+(h; x)S(x, y) - A_+(h'; y)S(x, y))
\] (3.19)

independent of \(x\) and \(y\)? Using the \(G\)-\(A \Delta E\)s (1.20), we calculate the two summands of \(Q\):

\[
\mathcal{L}(x, y) \equiv S(x, y)^{-1}A_+(h; x)S(x, y)
\]

\[
= V(h; x) \prod_{\delta = +, -} \frac{R_+(x - ia_-/2 - \delta y + ia - \phi)}{R_+(x - ia_-/2 + \delta y - ia + \phi)}
\]

\[
+ (x \to -x) + V_b(h; x),
\] (3.20)

\[
\mathcal{R}(x, y) \equiv S(x, y)^{-1}A_+(h'; y)S(x, y)
\]

\[
= V(h'; y) \prod_{\delta = +, -} \frac{R_+(y - ia_-/2 - \delta x + ia - \phi)}{R_+(y - ia_-/2 + \delta x - ia + \phi)}
\]

\[
+ (y \to -y) + V_b(h'; y).
\] (3.21)
Clearly, all of the terms occurring in (3.20) and (3.21) have at most simple poles in \( x \) for generic parameters.

Now for \( \mathcal{L} = \mathcal{R} \) to be \( x \)-independent, it is necessary that the \( \mathcal{L} \)- and \( \mathcal{R} \)-residues at the pole occurring for

\[
x = -y + ia_- - \phi
\]

(3.22) cancel. Thus we need equality of

\[
V(h; -y + ia_- - \phi) \frac{R_+(-2y + ia_- + ia_+ - 2\phi)R_+(ia_- + ia_+ - 2\phi)}{R_+(-2y - ia_- / 2)} ,
\]

(3.23) and

\[
V(h'; y) \frac{R_+(2y - ia_- + ia_+ / 2)R_+(ia_- + ia_+ / 2 - 2\phi)}{R_+(2y - 2ia_- - ia_+ / 2 + 2\phi)} .
\]

(3.24)

Substituting (3.2), we see this amounts to

\[
\prod_{n=0}^{7} \frac{R_+(y + h_n - ia_- / 2 + \phi)}{R_+(y - h'_n - ia_- / 2)} = 1.
\]

(3.25)

This is satisfied when we choose

\[
h'_n = -h_n - \phi, \quad n = 0, \ldots, 7, \quad (\text{mod } \pi / r)
\]

(3.26) and for generic \( h \) this is the only way to satisfy (3.25) (modulo permutations, cf. (3.16)).

From now on we require (3.26). It is straightforward to verify that this entails that the \( \mathcal{L} \)- and \( \mathcal{R} \)-residues at the poles

\[
x = y + ia_- - \phi, \quad x = \pm y - ia_- + \phi,
\]

(3.27) cancel, too.

Next, consider the \( x \)-dependence of \( \mathcal{R}(x, y) \). From (1.22) we see that \( \mathcal{R} \) is elliptic in \( x \) with periods \( \pi / r, ia_+ \). For \( \mathcal{L} = \mathcal{R} \) to be \( x \)-independent, it is therefore necessary that \( \mathcal{L} \) be elliptic in \( x \) as well. This holds true for \( V_b(h; x) \), but the first term on the rhs of (3.20) is quasi-periodic with multiplier

\[
\exp(2ir(\sum_n h_n + 4\phi)),
\]

(3.28)

under the lattice translation \( x \to x + ia_+ \). Hence we must require

\[
\sum_n h_n + 4\phi = 0. \quad (\text{mod } \pi / r).
\]

(3.29)

Doing so from now on, we observe that the second term on the rhs of (3.20) has multiplier (3.28) with \( r \to -r \) under \( x \to x + ia_+ \), so (3.29) ensures it is elliptic as well. As a consequence, the functions \( \mathcal{L} \) and \( \mathcal{R} \) are elliptic in \( x \), so that they have equal residues at all \( y \)-dependent \( x \)-poles. (Indeed, any such pole differs from the poles at (3.22) and (3.27) by elliptic lattice translations.)

To prove that \( Q \) is constant in \( x \), it remains to show that the residues of the terms in \( \mathcal{L} \) at the \( y \)-independent \( x \)-poles cancel. Consider first the poles at \( x = \omega_t, t = 0, 1, 2, 3 \). Since \( \mathcal{L} \) is not only elliptic, but also even in \( x \), the residues at these poles cancel. Hence
we are left with the poles at \( \pm x = ia_-/2 + \omega_t \). By evenness, it suffices to show that the residues for \( x = ia_-/2 + \omega_t \) cancel. For \( t = 0, 1 \), this is readily verified. Now take \( t = 2 \). Using (3.11) and (1.22), we obtain the total residue

\[
\rho \cdot \frac{p_2}{2R_+(ia_- + ia_+/2)} + \frac{e^{-2\pi a_+ \rho}}{2} \cdot \prod_{\delta = +, -} \frac{R_+(ia_+/2 - \delta y + ia - \phi)}{R_+(ia_+/2 - \delta y - ia + \phi)} \prod_{n} \frac{R_+(ia_+/2 - \delta y + ia - \phi)}{R_+(ia_+/2 - \delta y - ia + \phi)}. \tag{3.30}
\]

Using (1.22) once more, we see that this is proportional to

\[
\prod_n e^{-i\pi h_n} - e^{2ir(ia_- + ia_+)} e^{2ir(2\phi - 2ia)} = e^{-i\pi \sum_n h_n} - e^{4i\pi \phi}. \tag{3.31}
\]

Thus the residue vanishes provided we sharpen the restriction (3.29) to

\[
\sum_n h_n + 4\phi = 0, \quad \text{(mod $2\pi/r$).} \tag{3.32}
\]

(At this point we would like to add that we overlooked essentially the same sharpening in Lemma 3.2 of [15]: for the $z$-independence of (3.39) in [15] one must require (3.36) mod $2\pi/r$.)

With the stronger requirement (3.32) in effect from now on, it is not hard to see that the residue of \( L \) at \( x = ia_-/2 + \omega_3 \) vanishes, too. As a consequence, \( Q(x, y) \) is constant in \( x \). Repeating the analysis for the \( y \)-dependence, we clearly need to require

\[
\sum_n h'_n + 4\phi = 0, \quad \text{(mod $2\pi/r$)} \tag{3.33}
\]

to guarantee constancy in \( y \). But in fact (3.33) is a consequence of (3.32) and our standing requirement (3.26). Therefore, the constraints (3.26) and (3.32) imply that \( Q(x, y) \) is constant.

We now reformulate and slightly extend these findings. To this end we introduce the \( E_8 \) reflection

\[
J_R \equiv 1_8 - \frac{1}{4} \zeta \otimes \zeta, \quad \zeta_n = 1, \quad n = 0, \ldots, 7. \tag{3.34}
\]

(Together with the \( D_8 \) reflections, \( J_R \) generates the Weyl group of \( E_8 \).)

**Proposition 3.1.** For \( h \in \mathbb{C}^8 \) and \( x, y \in \mathbb{C} \), let

\[
S(h; x, y) \equiv \prod_{\delta \in \{+, -\}} G(\delta_1 x + \delta_2 y - ia - (\zeta, h)/4). \tag{3.35}
\]

Then there exist constants \( \sigma_\pm(h) \) such that

\[
(A_\delta(h; x) - A_\delta(-J_R h; y))S(h; x, y) = \sigma_\delta(h)S(h; x, y), \quad \delta = +, -, \tag{3.36}
\]

\[
(A_\delta(h; x) - A_\delta(-J_R h + \omega_1 \zeta; y))S(h; x, y; \omega_1 - y) = \sigma_\delta(h)S(h; x, \omega_1 - y), \quad \delta = +, -. \tag{3.37}
\]

Moreover, the constants satisfy the relations

\[
\sigma_\delta(h) = -\sigma_\delta(-J_R h), \tag{3.38}
\]

\[
\sigma_\delta(h) = \sigma_\delta(wh), \quad \forall w \in S_8, \tag{3.39}
\]

\[
\sigma_\delta(h) = \sigma_\delta(h + \omega_1 \zeta). \tag{3.40}
\]
Proof. We have already shown the eigenfunction relation (3.36) for $\delta = +$. Since $S$ is invariant under interchange of $a_+$ and $a_-$ (recall (1.29) and (3.9)), it also follows for $\delta = -$. Taking $y \to \omega_1 - y$ in (3.36) and using (3.17), we deduce (3.37).

Now the reflection property (3.38) results from the readily verified identities

$$S(h; x, y) = S(h; y, x) = S(-J_R h; x, y),$$

(3.41)

while permutation invariance is clear from (3.16) and

$$S(h; x, y) = S(w h; x, y), \quad \forall w \in S_8.$$

(3.42)

It remains to prove (3.40). To this end we observe that we have

$$S(h; x, y) = S(h + \omega_1 \xi; x, y) = S(h; \omega_1 - x, \omega_1 - y).$$

(3.43)

Thus, taking $x \to \omega_1 - x$ in (3.37) and using (3.17), we deduce (3.40). 

Using the $\mu = 0$ formula (3.10) for $V_b$, we proceed to evaluate the shift $\sigma_+(h)$ explicitly. Since it equals $Q(x, y)$ (3.19) and $Q$ does not depend on $x, y$, we may and will start from the representation

$$\sigma_+(h) = \mathcal{L}(x, x) - \mathcal{R}(x, x),$$

(3.44)

cf. (3.20)–(3.21). Setting

$$u \equiv ia_-/2,$$

(3.45)

we now study the function

$$E(u) \equiv R_+(2u + \omega_2)\sigma_+(h)$$

$$= \mathcal{M}(h, u; x) + \mathcal{M}(h, u; -x)$$

$$- \frac{\rho}{2} \sum_t p_t(h) ([L_+(u - x + \omega_2 + \omega_t) + L_+(u + x + \omega_2 - \omega_t)] - [x \to z_t])$$

$$- (h \to h'), \quad \phi = - \sum_n h_n/4, \quad h_n' = -h_n - \phi,$$

(3.46)

where we have introduced

$$\mathcal{M}(h, u; x) \equiv \frac{R_+(2u + \omega_2) \prod_n R_+(u - x + h_n)}{R_+(2x + \omega_2) R_+(2u - x - \omega_2)} \frac{R_+(\omega_2 - \phi) R_+(2x + \omega_2 - \phi)}{R_+(2u + \omega_2 - \phi) R_+(2u - 2x + \omega_2 - \phi)}.$$

(3.47)

To start, we observe that by virtue of (1.22) $\mathcal{M}$ is elliptic in $u$ with periods $\pi/r$ and $ia_+$. Moreover, combining (1.22) and (3.12) we deduce

$$L_+(z + ia_+) = L_+(z) - 2ir,$$

(3.48)

so that the function on the third line of (3.46) is elliptic as well. Therefore $E(u)$ is an elliptic function of $u$. We continue to study the residues at the poles of its summands, which are simple for generic $x$ and $h$.

First, consider the pole at $u = x + \phi/2 + \omega_t$. Since $\phi' = \phi$, its residue sum is proportional to

$$\prod_n R_+(\phi/2 + \omega_t + h_n) - \prod_n R_+(\phi/2 + \omega_t + h_n').$$

(3.49)
Recalling \( h'_n = -h_n - \phi \), it is plain that it vanishes for \( t = 0 \) and \( t = 1 \). Using (1.22), its vanishing for \( t = 2 \) and \( t = 3 \) follows too. Likewise, the residues at \(-x + \phi/2 + \omega_t\) cancel.

Second, consider the pole at \( u = x + \omega_t \). For \( t = 0 \) and \( t = 1 \) we get residue sums whose vanishing is immediate. For \( t = 2 \) we can use (1.22) to write the residue sum as

\[
\rho \frac{R_+(2x + 3\omega_2) \prod_n R_n(h_n + \omega_2)}{2 R_+(2x + \omega_2)(-1)} \frac{R_+(\omega_2 - \phi) R_+(2x + \omega_2 - \phi)}{R_+(2x + 3\omega_2 - \phi) R_+(3\omega_2 - \phi)} - \frac{\rho}{2} p_z(h)
\]

\[\rightarrow (h \to h'). \quad (3.50)\]

Using (1.22) a few more times, it follows that this vanishes. Likewise, the residues for \( t = 3 \) cancel. By evenness in \( x \), the residue sums at \( u = -x + \omega_t \) vanish as well.

Third, consider the residue sum at the pole \( u = \phi/2 + \omega_t \). It is proportional to

\[
\frac{\prod_n R_n(\phi/2 + \omega_t - x + h_n)}{R_+(2x + \omega_2) R_n(\phi + 2\omega_t - 2x - \omega_2) R_+(2\omega_t - 2x + \omega_2) + (x \to -x)}
\]

\[\rightarrow (h \to h'). \quad (3.51)\]

This can be rewritten as

\[
\frac{(\prod_n R_n(\phi/2 + \omega_t - x + h_n) - \prod_n R_n(\phi/2 - \omega_t + x + h_n))}{R_+(2x + \omega_2) R_+(2x - \omega_2 - 2\omega_t)} - \frac{R_+(2x + \omega_2 - \phi)}{R_+(2x + \omega_2 - \phi - 2\omega_t)}
\]

\[+ (x \to -x). \quad (3.52)\]

For \( t = 0 \) and \( t = 1 \) it is evident that this vanishes. Vanishing for \( t = 2 \) and \( t = 3 \) can be verified by using (1.22).

The upshot is that the only poles of the elliptic function \( E(u) \) can arise from the \([x \to z_t]\) terms in (3.46). Now thus far we have not used our specific \( z_t \)-choice, which is detailed in (3.3). In particular, if we replace this choice by

\[\tilde{z}_t \equiv \omega_t, \quad (3.53)\]

then the corresponding function \( \tilde{E}(u) \) is also elliptic, and it has vanishing residue sums at the poles \( u = \pm x + \phi/2 + \omega_t, \pm x + \omega_t, \phi/2 + \omega_t \) of its summands. The only remaining pole in the period rectangle occurs at \( u = 0 \), so its residue must vanish and \( \tilde{E}(u) \) must be equal to a constant \( \tilde{C} \). Since this residue equals

\[\rho \sum_{t=0}^{3} (p_t(h) - p_t(h')), \quad (3.54)\]

we deduce the identity

\[\sum_{t=0}^{3} p_t(h) = \sum_{t=0}^{3} p_t(h'). \quad (3.55)\]

In turn, the identity (3.55) implies that the \([x \to \tilde{z}_t]\) terms in \( \tilde{E}(u) \) have a vanishing sum. Thus we infer

\[
\tilde{C} = \mathcal{M}(h, u; x) + \mathcal{M}(h, u; -x) - (h \to h')
\]

\[\rightarrow -\frac{\rho}{2} \sum_{i} (p_i(h) - p_i(h'))[L_+(u - x + \omega_2 + \omega_t) + L_+(u + x + \omega_2 - \omega_t)]. \quad (3.56)\]
Now we have
\[ \mathcal{M}(h, 0; x) = 0, \quad \text{(3.57)} \]
cf. (3.47). Therefore, choosing \( u = 0 \) on the rhs of (3.56), we obtain
\[ \tilde{C} = \frac{\rho}{2} \sum_t (p_t(h) - p_t(h'))[L_+(x - \omega_2 - \omega_t) - L_+(x + \omega_2 - \omega_t)] \]
\[ = \frac{\rho}{2} \sum_t (p_t(h) - p_t(h'))[2ir] \]
\[ = 0, \quad \text{(3.58)} \]
where we used (3.48) in the first step and (3.55) in the second one. From this we readily deduce
\[ E(u) = \frac{\rho}{2} \sum_t (p_t(h) - p_t(h'))[L_+(u - z_t + \omega_2 + \omega_t) + L_+(u + z_t + \omega_2 - \omega_t)]. \quad \text{(3.59)} \]
For the \( z_t \)-choice in (3.3) this becomes
\[ E(u) = \rho \sum_{t=0,1} (p_t(h) - p_t(h')) L_+(u + \omega_1 + \omega_2) \]
\[ + \frac{\rho}{2} \sum_{t=2,3} (p_t(h) - p_t(h'))[L_+(u + \omega_1) + L_+(u + \omega_1 + 2\omega_2)]. \quad \text{(3.60)} \]
Finally, using (3.48) this can be simplified as
\[ E(u) = \rho \sum_{t=0,1} (p_t(h) - p_t(h')) L_+(u + \omega_1 + \omega_2) \]
\[ + \rho \sum_{t=2,3} (p_t(h) - p_t(h'))[L_+(u + \omega_1) - ir]. \quad \text{(3.61)} \]

Putting the pieces together, we conclude that we have evaluated the shift \( \sigma_+(h) \):

**Proposition 3.2.** We have the explicit shift formula
\[ \sigma_+(h) = \frac{-i \exp(ra_+/2)}{p_+ R_+(ia_- + ia_+/2)} \left( L_+(ia_-/2 + ia_+/2 + \pi/2r) \sum_{t=0,1} (p_t(h) - p_t(h')) \right. \]
\[ + \left. [L_+(ia_-/2 + \pi/2r) - ir] \sum_{t=2,3} (p_t(h) - p_t(h')) \right). \quad \text{(3.62)} \]

(Of course, \( \sigma_-(h) \) is now given by interchanging \( a_+ \) and \( a_- \).) Moreover, along the way we have obtained the identity (3.55). Introducing the sum function
\[ S_p : \mathbb{C}^8 \to \mathbb{C}, \quad h \mapsto \sum_{t=0}^3 p_t(h), \quad \text{(3.63)} \]
(with \( p_t(h) \) given by (3.5)–(3.7)), it can be reformulated as follows.

**Proposition 3.3.** The function \( S_p \) is invariant under the Weyl group of \( E_8 \).

Admittedly, the above proof of (3.55) is ‘unnatural’, but a direct proof appears elusive.
3.2. The nonrelativistic case. Using several limit relations from [15], we proceed to obtain a ‘nonrelativistic’ (differential operator) counterpart of the above results. First, we set
\[ h = i\gamma_f + ic, \quad c \in \mathbb{R}^8, \quad (3.64) \]
where
\[ \gamma_f \equiv (-a, -a_-/2, -a_+/2, 0, -a + i\omega_1, -a_-/2 - i\omega_1, -a_+/2 - i\omega_1, i\omega_1)^t, \quad (3.65) \]
cf. [15], (3.43)–(3.44). Now we have
\[ -JR\gamma_f = -\gamma_f - a\zeta = \rho\gamma_f, \quad (3.66) \]
where \( \rho \) is the permutation
\[ \rho \equiv \text{diag}(r_4, r_4), \quad (3.67) \]
with \( r_4 \) the reversal permutation on \( \mathbb{C}^4 \). Thus we get
\[ -\rho JRh = i\gamma_f + ic', \quad (3.68) \]
where
\[ c' \equiv -\rho c + (\zeta, c)\zeta/4. \quad (3.69) \]
Second, using (1.27), we can rewrite (3.35) with (3.64) in effect as
\[ S_c(x, y) = \prod_{\delta=+,-} \frac{G(x + \delta y - i(\zeta, c)/4)}{G(x + \delta y + i(\zeta, c)/4)}. \quad (3.70) \]
Third, in view of (3.16) we obtain from (3.36),
\[ (A_+(i\gamma_f + ic; x) - A_+(i\gamma_f + ic'; y))S_c(x, y) = \sigma_+(i\gamma_f + ic)S_c(x, y). \quad (3.71) \]
We are now prepared to study the pertinent limit. Thus we set (cf. [15], (6.33))
\[ c = a_-d, \quad c' = a_-d', \quad a_+ \to \alpha, \quad (3.72) \]
and take the limit \( a_- \to 0 \) for \( d \) fixed. From (1.30) we deduce
\[ \lim_{a_- \to 0} S_c(x, y) = \prod_{\delta=+,-} \exp\left(-[(\zeta, d)/2]\ln R(x + \delta y)\right), \quad R(x) \equiv R(r, \alpha; x). \quad (3.73) \]
To handle the AΔO difference, we introduce
\[ g_0^{(\ell)} = d_0^{(\ell)} + d_2^{(\ell)}, \quad g_1^{(\ell)} = d_4^{(\ell)} + d_6^{(\ell)}, \quad g_2^{(\ell)} = d_1^{(\ell)} + d_3^{(\ell)}, \quad g_3^{(\ell)} = d_5^{(\ell)} + d_7^{(\ell)}, \quad (3.74) \]
and
\[ s_g \equiv \frac{1}{2} \sum_{\ell=0}^3 g_\ell = \frac{1}{2} \sum_{\ell=0}^3 g_\ell'. \quad (3.75) \]
Thus we obtain
\[ g'_t = -g_{t+2} + s_g, \quad g'_{t+2} = -g_t + s_g, \quad t = 0, 1, \]  
(3.76)
or, equivalently,
\[ g' = J_N g, \quad J_N \equiv \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}. \]
(3.77)

Since we have \( g_0 + g_1 = g'_0 + g'_1 \), it now follows from pp. 251–252 in [15] that we have an expansion
\[ A_+ (i \gamma f + i a_- d; x) - A_+ (i \gamma f + i a_- d'; y) = a_+^2 (D(g; x) - D(g'; y) + C(d)) + O(a_+^3), \quad a_- \to 0. \]
(3.78)

Here, \( C(d) \) is a complicated \( d \)-dependent constant, and \( D(g; x) \) denotes the differential operator
\[ D(g; x) \equiv -\frac{d^2}{dx^2} - 2L(g; x) \frac{d}{dx} - L(g; x)^2 + \sum_{t=0}^{3} g_t^2 \wp(x + \omega_t; \pi/2r, i\alpha/2), \]
(3.79)
where
\[ L(g; x) \equiv \sum_{t=0, 1} (g_t L_s (x + \omega_t) + g_{t+2} L_R (x + \omega_t)), \quad \omega_0 = 0, \quad \omega_1 = \pi/2r, \]
(3.80)
\[ L_f (x) \equiv f'(x)/f(x), \quad f = s, R. \]
(3.81)

The constant \( \sigma_+ (i \gamma f + i a_- d) \) must be of the form \( a_+^2 C'(d) + O(a_+^3) \) for \( a_- \to 0 \), but we need not and will not calculate \( C'(d) \). Indeed, we will also ignore the constant \( C(d) \) in (3.78), and work only with \( D(g; x) \). As an explicit check of the final result of the expansion, we present an independent proof. It is to be noted that the shift vanishes in this case. (This amounts to the function on the rhs of (3.87) vanishing identically.)

**Proposition 3.4.** For \( g \in \mathbb{C}^4 \) and \( x, y \in \mathbb{C} \), let
\[ S(g; x, y) \equiv \exp(-s_g \ln(R(x + y)R(x - y))), \quad s_g = \frac{1}{2} \sum_{t=0}^{3} g_t, \]
(3.82)
where the logarithm branch is real for \( x, y \in \mathbb{R} \). Then we have the identity
\[ (D(g; x) - D(g'; y))S(g; x, y) = 0, \]
(3.83)
where the differential operators are given by (3.79)–(3.81) and \( g' \) by (3.76).
Proof. First, we calculate

\[ S^{-1} \partial_x S = -s_g \sum_{\delta = +,-} L_R(x + \delta y), \quad (3.84) \]

\[ S^{-1} \partial_x^2 S = -s_g \partial_x \sum_{\delta = +,-} L_R(x + \delta y) + s_g^2 \left( \sum_{\delta = +,-} L_R(x + \delta y) \right)^2, \quad (3.85) \]

and note that \( L_R(x) \) is odd. Setting

\[ \sigma(g; x, y) = S(g; x, y)^{-1}(D(g; x) - D(g'; y))S(g; x, y), \quad (3.86) \]

we use this to obtain

\[ \sigma(g; x, y) = \sum_{i=0}^{3} (g_i^2 \varphi(x + \omega_i) - g_i'^2 \varphi(y + \omega_i)) \]

\[-L(g; x)^2 + L(g'; y)^2 - 4s_g^2 L_R(x + y)L_R(x - y) \]

\[ +2s_g \left( L(g; x) \sum_{\delta = +,-} L_R(x + \delta y) - L(g'; y) \sum_{\delta = +,-} L_R(y + \delta x) \right). \quad (3.87) \]

To prove that \( \sigma(x, y) \) is constant, we begin by showing it is elliptic in \( x \) with periods \( \pi/r, i\alpha \). Obviously, \( \pi/r \)-periodicity holds true. To show \( i\alpha \)-periodicity, we note that

\[ L_f(x + i\alpha) = L_f(x) - c, \quad c \equiv 2ir, \quad f = s, R, \quad (3.88) \]

cf. (1.22) and (1.25). This implies

\[ L(g; x + i\alpha) = L(g; x) - 2cs_g. \quad (3.89) \]

Therefore, \( \sigma(x + i\alpha, y) - \sigma(x, y) \) is given by

\[ 4cs_g L(g; x) - 4c^2 s_g^2 - 4s_g^2 ((L_R(x + y) - c)(L_R(x - y) - c) - L_R(x + y)L_R(x - y)) \]

\[ +2s_g \left( L(g; x) - 2cs_g \right) \left( \sum_{\delta = +,-} L_R(x + \delta y) - 2c \right) - L(g; x) \sum_{\delta = +,-} L_R(x + \delta y) \right). \quad (3.90) \]

Pairing off terms, we see that (3.90) vanishes. Hence \( \sigma(x, y) \) is elliptic in \( x \). Likewise, we infer \( \sigma \) is elliptic in \( y \).

Next, we consider the residue of \( \sigma \) at the (generically) simple pole

\[ x = y - \gamma, \quad \gamma \equiv i\alpha/2. \quad (3.91) \]

It is proportional to

\[ 2s_g(L(g; y - \gamma) + L(g'; y)) - 4s_g^2 L_R(2y - \gamma). \quad (3.92) \]

Now from the duplication formula (1.32) we deduce

\[ 2L_R(2y - \gamma) = \sum_{t=0,1} (L_R(y + \omega_t) + L_R(y - \gamma + \omega_t)). \quad (3.93) \]
Using also (3.76) and (3.80), we see that (3.92) divided by 2s\_g equals
\[
\sum_{t=0,1} (g_t L_x (y - \gamma + \omega_t) + g_{t+2} L_R (y - \gamma + \omega_t) + s_g (L_x (y + \omega_t) + L_R (y + \omega_t))) - \sum_{t=0,1} (g_t L_R (y + \omega_t) + g_{t+2} L_x (y + \omega_t) + s_g (L_R (y + \omega_t) + L_R (y - \gamma + \omega_t))).
\]
\[\text{(3.94)}\]

Canceling four terms, this can be rewritten as
\[
\sum_{t=0,1} [g_t (L_s (y - \gamma + \omega_t) - L_R (y + \omega_t)) + g_{t+2} (L_R (y - \gamma + \omega_t) - L_s (y + \omega_t))] + s_g \sum_{t=0,1} [L_s (y + \omega_t) - L_R (y - \gamma + \omega_t)].
\]
\[\text{(3.95)}\]

Now from (1.23) we see that the logarithmic derivatives of s and R are related by
\[L_s (x) = L_R (x - \gamma) - ir, \quad L_s (x - \gamma) = L_R (x) + ir.\]
\[\text{(3.96)}\]

Substituting this in (3.95), we see that (3.95) vanishes.

Next we note \(\sigma\) is even in \(x\) and \(y\). Hence the residue at \(x = -y + \gamma\) vanishes, too. Thus it remains to check that the principal parts at the double poles for \(x \equiv \omega_t\) and \(y \equiv \omega_t, t = 0, 1, 2, 3, \) vanish. Now any even elliptic function \(F(z)\) with double poles at the half-period lattice \(k \omega_1 + l \omega_2, k, l \in \mathbb{Z},\) satisfies
\[F(z) = c_{2,1} (z - \omega_t)^{-2} + c_{0,1} + O((z - \omega_t)^2), \quad z \to \omega_t, \quad t = 0, 1, 2, 3, \]
\[\text{(3.97)}\]
as is easily verified. Hence we need only check that the double pole parts of \(g_t^2 \varphi (x + \omega_t)\) and \(L(g; x)^2\) at \(x = \omega_t\) have the same coefficients. Since we have (cf. (1.26) and (1.23))
\[\partial_x L_s (x) = -\varphi (x) - \eta \omega_1, \quad \partial_x L_R (x) = -\varphi (x + \omega_2) - \eta \omega_1,\]
\[\text{(3.98)}\]
this is indeed the case. Therefore \(\sigma (x, y)\) is constant.

It remains to prove that the constant vanishes. To this end, we choose \(x = y\) in (3.87), which yields
\[\sigma = \sum_t (g_t^2 - g_t') \varphi (x + \omega_t) - L(g; x)^2 + L(g'; x)^2 + 2s_g L_R (2x) [L(g; x) - L(g'; x)].\]
\[\text{(3.99)}\]

We now put
\[f_0 (x) = s (x), \quad f_1 (x) = s (x + \omega_1), \quad f_2 (x) = R (x), \quad f_3 (x) = R (x + \omega_1). \]
\[\text{(3.100)}\]
Thus we can rewrite \(L (3.80)\) as
\[L(g; x) = \sum_{t=0}^3 g_t \frac{f_t' (x)}{f_t (x)}, \]
\[\text{(3.101)}\]
and we have
\[
\frac{f'_t(x)^2}{f_t(x)^2} - \frac{f''_t(x)}{f_t(x)} = \wp(x + \omega_t) + d, \quad d \equiv 2\eta r / \pi. \tag{3.102}
\]

Using this, we deduce
\[
\sigma = -\sum_t (g_t^2 - g'_t^2) \left( d + \frac{f''_t(x)}{f_t(x)} \right)
- 2 \sum_{t_1 < t_2} (g_{t_1} g_{t_2} - g'_{t_1} g'_{t_2}) \frac{f'_{t_1}(x)}{f_{t_1}(x)} \frac{f'_{t_2}(x)}{f_{t_2}(x)}
+ \sum_s g_s R'(2x) \left( \sum_t (g_t - g'_t) \frac{f'_t(x)}{f_t(x)} \right). \tag{3.103}
\]

Since \(\sigma\) does not depend on \(x\), we are free to take \(x\) to 0 on the rhs of (3.103). Noting
\[
f_0(x) = x - \eta x^3 r / \pi + O(x^5), \quad x \to 0, \tag{3.104}
\]
(cf. (1.26)) we get
\[
\lim_{x \to 0} f''_0(x)/f_0(x) = -3d. \tag{3.105}
\]

We also infer from (3.100) and (3.102) that we have
\[
f'_j(0) = 0, \quad f''_j(0)/f_j(0) = -e_j - d, \quad j = 1, 2, 3. \tag{3.106}
\]

Using these relations, we get the \(x \to 0\) limit,
\[
\sigma = 2d(g_0^2 - g'_0^2) + \sum_{j > 0} e_j (g_j^2 - g'_j^2)
+ 2g_0 \sum_{j > 0} g_j (e_j + d) - 2g'_0 \sum_{j > 0} g'_j (e_j + d)
- 2(g_0 - g'_0)(e_2 + d) \sum_t g_t. \tag{3.107}
\]

At face value it is still far from obvious that the rhs of (3.107) vanishes. But when we use
\[
\sum_{t=0}^3 (g_t - g'_t) = 0, \tag{3.108}
\]
we see that the coefficient of \(d\) vanishes. Moreover, straightforward calculation shows that the three coefficients of \(e_1, e_2, e_3\) are all given by \(Q(g)/4\), with
\[
Q(g) = -3g_0^2 + g_1^2 + g_2^2 + g_3^2 + 2(g_0 g_1 - g_0 g_2 + g_0 g_3 - g_1 g_2 + g_1 g_3 - g_2 g_3). \tag{3.109}
\]

Using the well-known relation \(e_1 + e_2 + e_3 = 0\), it now follows that \(\sigma\) vanishes, as claimed. \(\Box\)
Using the $G$-duplication formula (1.31), we deduce from (3.13), (3.65) and (1.30) the limit
\[
\lim_{a \to 0} c(i \gamma_f + i a_- d; x) = \exp(-[(d_0 + d_2) \ln R_+(x + i a_+ / 2) + (d_1 + d_3) \ln R_+(x) + (d_4 + d_6) \ln R_+(x - \omega_1 + i a_+ / 2) + (d_5 + d_7) \ln R_+(x - \omega_1)])).
\]
Invoking the reparametrization (3.74) and taking $a_+ \to \alpha$, we therefore obtain the limit function
\[
c_{nr}(g; x) \equiv R(x + i \alpha / 2)^{-g_0} R(x + i \alpha / 2 - \omega_1)^{-g_1} R(x)^{-g_2} R(x - \omega_1)^{-g_3}.
\]
The associated weight function can now be written (cf. (1.16) and (1.23))
\[
w_{nr}(g; x) = p_{2g_0+2g_1}^2 s(x)^{2g_0} s(\omega_1 - x)^{2g_1} R(x)^{2g_2} R(\omega_1 - x)^{2g_3},
\]
where $p(r, \alpha)$ is given by (1.24). With the five obvious branch choices understood, it is positive for $g \in \mathbb{R}^4$ and $x \in (0, \omega_1)$.

Using (3.98), it is straightforward to check the relation
\[
\frac{1}{2} D(g; x) w_{nr}(g; x)^{-1/2} = H_{nr}(g; x) - 2s_g \eta \omega_1,
\]
where $H_{nr}(g; x)$ is the Heun Hamiltonian,
\[
H_{nr}(g; x) \equiv -\frac{d^2}{dx^2} + \sum_{t=0}^{3} g_t (g_t - 1) \wp(x + \omega_t; \pi / 2 r, i \alpha / 2).
\]
Finally, recalling $s_g = s_g'$ (cf. (3.75)), we deduce from (3.83) the eigenfunction identity
\[
(H_{nr}(g; x) - H_{nr}(g'; y)) \Psi_{nr}(g; x, y) = 0,
\]
where
\[
\Psi_{nr}(g; x, y) \equiv (w_{nr}(g; x) w_{nr}(g'; y))^{1/2} S(g; x, y).
\]
It should be noted that when we would work with a shifted $\wp$-function,
\[
\wp_z(z) = \wp(z) + C,
\]
for some constant $C$, then we again get eigenvalue zero on $\Psi_{nr}(x, y)$ for the difference of the corresponding Heun Hamiltonians. This is because we have not only the sum rule (3.108), but also
\[
\sum_{t=0}^{3} (g_t^2 - g_t'^2) = 0.
\]
(Indeed, the matrix $J_N$ in (3.77) is orthogonal.)
4. The Defining $BC_N$ Hamiltonians

4.1. The relativistic case. In this subsection we study arbitrary-$N$ counterparts of the results in Subsect. 3.1. We proceed along the same lines as for the $N = 1$ case, skipping some details that are not substantially different.

The generalization of the $A/\Delta_1 O(3.1)$ reads

$$A_\pm(h, \mu; x) \equiv \sum_{j=1}^{N} \left( V_j(h, \mu; x) \exp(-ia_\pm \partial_{x_j}) + V_j(h, \mu; -x) \exp(i a_\pm \partial_{x_j}) \right) + \mathcal{V}(h, \mu; x),$$

$$\mathcal{V}(h, \mu; x) \equiv \prod_{j=1}^{N} F_j(h, \mu; x),$$

where

$$V_j(h, \mu; x) \equiv V(h; x_j) \prod_{k \neq j} R_\pm(x_j - \delta x_k - \mu + ia_+/2) - R_\pm(x_j - \delta x_k + ia_+/2),$$

$$\mathcal{V}(h, \mu; x) \equiv \frac{\prod_{i=0}^{3} p_i(h)}{2 R_\pm(\mu - i a_+/2) R_\pm(\mu - i a_- - i a_+/2)},$$

cf. also the definitions (3.2)–(3.9). It is related via (1.13) to the defining $BC_N$ Hamiltonian (which differs from the ones in [7, 8] by constants, cf. Appendix B in [15]), with $w(h, \mu; x)$ now given by (1.16) and

$$c(h, \mu; x) \equiv \prod_{j=1}^{N} c(h; x_j) \cdot \prod_{1 \leq j < k \leq N} \frac{G(x_j - \delta x_k - \mu + ia)}{G(x_j - \delta x_k + ia)},$$

where $c(h; x_j)$ is defined by (3.13).

The $A/\Delta O A_-(h, \mu; x)$ obtained from $A_+(h, \mu; x)$ by taking $a_+ \leftrightarrow a_-$ commutes with $A_+(h, \mu; x)$, as is readily verified. The $\Delta O$s $A_\pm$ are clearly $BC_N$ invariant (i.e., invariant under permutations and sign changes of $x_1, \ldots, x_N$), and invariant under $x_j \to x_j + 2\omega_1, h_n \to h_n + 2\omega_1$, and under $h \to wh, w \in S_8$. Moreover, we have

$$A_\delta(h + \omega_1 \xi, \mu; x) = A_\delta(h, \mu; \omega_1 - x_1, \ldots, \omega_1 - x_N), \quad \delta = +, -.$$}

Our eigenfunction Ansatz in this case is the $BC_N$ invariant function

$$S(x, y) \equiv \prod_{j,k=1}^{N} \prod_{\delta_1, \delta_2 = +,-} G(\delta_1 x_j + \delta_2 y_k - ia + \phi).$$

Here we should search for $\phi, h'$ and $\mu'$ such that

$$Q(x, y) = \frac{1}{S(x, y)}(A_+(h, \mu; x)S(x, y) - A_+(h', \mu'; y)S(x, y))$$
is constant. As the generalization of (3.20)–(3.21) we obtain

\[
\mathcal{L}(x, y) = \sum_{j=1}^{N} \mathcal{V}_j(h, \mu; x) \prod_{k=1}^{N} \frac{R_+(x_j - ia_-/2 - \delta y_k + ia - \phi)}{R_+(x_j - ia_-/2 + \delta y_k - ia + \phi)}
\]

\[
+ (x \to -x) + \mathcal{V}(h, \mu; x),
\]

(4.8)

\[
\mathcal{R}(x, y) = \sum_{j=1}^{N} \mathcal{V}(h', \mu'; y) \prod_{k=1}^{N} \frac{R_+(y_j - ia_-/2 - \delta x_k + ia + \phi)}{R_+(y_j - ia_-/2 + \delta x_k - ia + \phi)}
\]

\[
+ (y \to -y) + \mathcal{V}(h', \mu'; y).
\]

(4.9)

For \( \mathcal{L} - \mathcal{R} \) to be independent of \( x_1 \), its residue at the (generically) simple pole

\[
x_1 = -y_1 + ia_- - \phi
\]

must vanish. Requiring from now on \( N > 1 \), it follows that the functions

\[
\mathcal{V}_1(h, \mu; (-y_1 + ia_- - \phi, x_2, \ldots, x_N)) \propto \frac{R_+(2y_1 - ia_- + ia_+/2 - 2\phi)R_+(ia_- + ia_+/2 - 2\phi)}{R_+(2y_1 - ia_- - ia_+/2 + 2\phi)}
\]

\[
\times \prod_{k>1} \frac{R_+(-y_1 + ia_-/2 - \delta y_k + ia - 2\phi)}{R_+(-y_1 + ia_-/2 - \delta y_k - ia)}
\]

(4.11)

and

\[
\mathcal{V}_1(h', \mu'; y_1) \propto \frac{R_+(2y_1 - ia_- + ia_+/2)R_+(ia_- + ia_+/2 - 2\phi)}{R_+(2y_1 - 2ia_- - ia_+/2 + 2\phi)}
\]

\[
\times \prod_{k>1} \frac{R_+(y_1 - ia_-/2 - \delta x_k + ia + \phi)}{R_+(y_1 - ia_-/2 + \delta x_k - ia)}
\]

(4.12)

should be the same. Substituting (4.2), we deduce this amounts to equality of

\[
\prod_{n} \frac{R_+(y_1 + h_n - ia_-/2 + \phi)}{R_+(y_1 - ia_-/2 + \delta y_k - \mu + ia_+/2)}
\]

\[
\times \prod_{k>1} \frac{R_+(-y_1 + ia_- - \phi - \delta x_k - \mu + ia_+/2)}{R_+(-y_1 + ia_- - \phi - \delta x_k + ia_+/2)}
\]

(4.13)

and

\[
\prod_{n} \frac{R_+(y_1 - h_n' - ia_-/2)}{R_+(y_1 - ia_-/2 - \delta y_k + ia - 2\phi)}
\]

\[
\times \prod_{k>1} \frac{R_+(y_1 - ia_-/2 - \delta y_k + ia - 2\phi)}{R_+(y_1 - ia_-/2 - \delta y_k - ia)}
\]

(4.14)

The quotient of these two functions reads

\[
\prod_{n} \frac{R_+(y_1 + h_n - ia_-/2 + \phi)}{R_+(y_1 - h_n' - ia_-/2)} \prod_{k>1} \frac{R_+(y_1 - ia_- + \phi - \delta x_k + \mu - ia_+/2)}{R_+(y_1 + ia_+/2 - \delta x_k - \phi)}
\]

\[
\times \prod_{k>1} \frac{R_+(y_1 - ia_- - \delta y_k - ia_+/2 + 2\phi)}{R_+(y_1 - \delta y_k - \mu + ia_+/2)}
\]

(4.15)
For this function to be identically equal to 1, it clearly suffices to require

\[ h'_n = -h_n - \phi, \quad n = 0, \ldots, 7, \quad (\text{mod } \pi/r), \quad (4.16) \]
\[ -2ia + \phi + \mu = -\phi, \quad -2ia + 2\phi = -\mu', \quad (\text{mod } \pi/r). \quad (4.17) \]

Thus we impose from now on the constraints

\[ \mu' = \mu = 2ia - 2\phi, \quad (4.18) \]

together with (4.16), and note that these conditions are essential for (4.15) to equal 1.

By \( BC_N \) invariance, it now follows that the residues of \( L - R \) at all of the poles

\[ x_j = \pm y_k + ia_+ - \phi \quad (4.19) \]

vanish.

Turning to the \( x_1 \)-dependence of \( R \), we easily verify that \( R \) is elliptic. The same is true for \( V(h, \mu; x) \) and for the \( j > 1 \) summands of the two sums on the rhs of (4.8), cf. also (4.2). But the \( j = 1 \) summand of the first sum has multiplier

\[ \exp(2ir[(\zeta, h) + 2(N - 1)\mu - 4i(N - 1)a + 4N\phi]) \quad (4.20) \]

under \( x_1 \to x_1 + ia_+ \), whereas the \( j = 1 \) summand of the second one has multiplier (4.20) with \( r \to -r \). Using (4.18), we see that these multipliers equal 1, provided

\[ (\zeta, h) + 4\phi = 0, \quad (\text{mod } \pi/r) \quad (4.21) \]

From now on we require (4.21) (in addition to (4.16) and (4.18)), so that only \( h \in \mathbb{C}^8 \) can be freely chosen. Since the functions \( L \) and \( R \) are elliptic in \( x_1, \ldots, x_N, y_1, \ldots, y_N \), and have no poles for \( x \) and \( y \) related by (4.19), it remains to study the residues at the \( y \)-independent \( x_j \)-poles of \( L \) and at the \( x \)-independent \( y_j \)-poles of \( R \). Also, by permutation invariance it suffices to handle the case \( j = 1 \).

Consider first the \( L \)-pole for \( x_1 = x_2 \). It is present in the \( j = 1 \) and \( j = 2 \) summands of the two sums on the rhs of (4.8). Using (1.22), we see that in both sums the residues of the two summands cancel. (Note that this does not involve the above parameter constraints.) Likewise, the residue at the pole \( x_1 = -x_2 \) of the \( j = 1/2 \) term of the first sum cancels against the residue of the \( j = 2/1 \) term of the second one. More generally, there are no \( L \)-poles for \( x_1 = \pm x_k, k > 1 \), and no \( R \)-poles for \( y_1 = \pm y_k, k > 1 \). Moreover, by evenness there are no poles in \( L/R \) for \( x_1/y_1 \) congruent to \( \omega t, t = 0, 1, 2, 3 \).

It remains to consider the poles in \( x_1 \) and \( y_1 \) congruent to \( \pm ia_+/2 + \omega t \). Just as for \( N = 1 \), residue cancellation for \( t = 0, 1 \) is easily verified. For the cases \( t = 2, 3 \) we can adapt our \( N = 1 \) calculations (cf. (3.11)–(3.32)) to deduce residue cancellation, provided we strengthen the constraint (4.21) to

\[ (\zeta, h) + 4\phi = 0, \quad (\text{mod } 2\pi/r). \quad (4.22) \]

(We mention in passing that Lemma 4.2 of [15] contains the same oversight as we already pointed out below (3.32) for the \( BC_1 \) case: the restriction (4.33) must be imposed mod \( 2i\pi/r \), and not mod \( i\pi/r \).)

The upshot is that the constraints (4.16), (4.18) and (4.22) guarantee that \( Q(x, y) \) (given by (4.7)) is constant. As before, the next proposition summarizes these findings and adds some information.
Proposition 4.1. For $h \in \mathbb{C}^8$ and $x, y \in \mathbb{C}^N$ with $N > 1$, let

\[
S(h; x, y) \equiv \prod_{j,k=1}^N G(\delta_1 x_j + \delta_2 y_k - i a - (\zeta, h)/4).
\] (4.23)

Then there exist constants $\sigma_\delta(h)$ such that

\[
(A_\delta(h, \mu_h; x) - A_\delta(-J_R h, \mu_h; y))S(h; x, y) = \sigma_\delta(h)S(h; x, y), \quad \delta = +, -,
\] (4.24)

where

\[
\mu_h \equiv 2ia + (\zeta, h)/2.
\] (4.26)

Moreover, the constants satisfy (3.38)–(3.40).

Proof. The above developments show that (4.24) holds for $\delta = +$. The remaining assertions follow by adapting the proof of Prop. 3.1. In particular, the role of (3.17) is now played by (4.5). □

4.2. The nonrelativistic case. Proceeding as in the $BC_1$ case (cf. (3.64)–(3.69)), we obtain obvious generalizations of (3.70)–(3.71). (Note that $\mu_h$ (4.26) turns into $i(\zeta, c)/2$ under the substitution (3.64).) Also, substituting (3.72), the generalization of (3.73) is clear. Changing parameters according to (3.74)–(3.77), the generalization of (3.78) reads

\[
A_+(i\gamma_f + ia_{-d}, ia_-\lambda; x) - A_+(i\gamma_f + ia_{-d'}, ia_-\lambda; y)
= a_-^2(D(g, \lambda; x) - D(g', \lambda; y) + C(d, \lambda)) + O(a_-^3), \quad a_- \to 0,
\] (4.27)

where

\[
D(g, \lambda; x) \equiv - \sum_{j=1}^N \partial_{x_j}^2 - 2 \sum_{j=1}^N (L(g; x_j) + \lambda \sum_{k \neq j} \sum_{\delta = +, -} L_s(x_j - \delta x_k)) \partial_{x_j}
\]

\[
- \sum_{j=1}^N (L(g; x_j) + \lambda \sum_{k \neq j} \sum_{\delta = +, -} L_s(x_j - \delta x_k))^2
+ \sum_{r=0}^{3} \sum_{j=1}^N g_r(x_j + \omega_r) + \lambda^2 \sum_{j=1}^N \sum_{k \neq j} \sum_{\delta = +, -} g(x_j + \delta x_k).
\] (4.28)

Once more, we verify the results of the expansion by presenting a direct proof, making use of our $N = 1$ calculations. We also show that the shift still vanishes for $N > 1$. (That is, the function on the rhs of (4.34) vanishes identically.)

Proposition 4.2. For $g \in \mathbb{C}^4$ and $x, y \in \mathbb{C}^N$ with $N > 1$, let

\[
S(g; x, y) \equiv \prod_{j,k=1}^N \prod_{\delta = +, -} \exp(-s_g \ln R(x_j + \delta y_k)), \quad s_g = \frac{1}{2} \sum_{r=0}^{3} g_r.
\] (4.29)
where the logarithm branch is real for \( x, y \in \mathbb{R}^N \). Then we have

\[
(D(g, s_g; x) - D(g', s_g; y))S(g; x, y) = 0, \tag{4.30}
\]

where the differential operators are given by (4.28) and \( g' \) by (3.76).

**Proof.** Using obvious abbreviations, we have

\[
S^{-1} \partial_{x_j} S = -s_g \sum_{k, \delta} L_R(x_j - \delta y_k), \tag{4.31}
\]

\[
S^{-1} \partial_{x_j}^2 S = -s_g \partial_{x_j} \sum_{k, \delta} L_R(x_j - \delta y_k) + s_g^2 \left( \sum_{k, \delta} L_R(x_j - \delta y_k) \right)^2.
\]

Therefore,

\[
\sigma(g; x, y) \equiv S(g; x, y)^{-1} (D(g, s_g; x) - D(g', s_g; y))S(g; x, y),
\]

(3.76)

is given by

\[
\sigma(g; x, y) = \sum_{t, j} [g_t^2 \varphi(x_j + \omega_t) - g_t'' \varphi(y_j + \omega_t)]
\]

\[
+ s_g^2 \sum_j \sum_{l \neq j} (\varphi(x_j - \delta x_k) - \varphi(y_j - \delta y_k))
\]

\[
- \sum_j [(L(g; x_j) + s_g \sum_{k \neq j, \delta} L_s(x_j - \delta x_k))^2 - (L(g'; y_j) + s_g \sum_{k \neq j, \delta} L_s(y_j - \delta y_k))^2]
\]

\[
+ 2s_g \sum_j (L(g; x_j) + s_g \sum_{k \neq j, \delta} L_s(x_j - \delta x_k))(\sum_{l, \delta'} L_R(x_j - \delta'y_l))
\]

\[
- 2s_g \sum_j (L(g'; y_j) + s_g \sum_{k \neq j', \delta} L_s(y_j - \delta y_k))(\sum_{l, \delta'} L_R(y_j - \delta'x_l)). \tag{4.34}
\]

Using (3.88)–(3.89), we first calculate the natural generalization of (3.90), namely,

\[
\sigma(x_1 + i \alpha, x_2, \ldots, x_N, y) - \sigma(x, y) = \mathcal{T}(c; x, y) - \mathcal{T}(0; x, y), \tag{4.35}
\]

where

\[
\mathcal{T}(c; x, y) \equiv -(L(g; x_1) + s_g \sum_{k > 1, \delta} L_s(x_1 - \delta x_k) - 2Nc s_g)^2
\]

\[
- s_g^2 \left( \sum_{k, \delta} L_R(x_1 - \delta y_k) - 2Nc \right)^2
\]

\[
+ 2s_g (L(g; x_1) + s_g \sum_{k > 1, \delta} L_s(x_1 - \delta x_k) - 2Nc s_g)
\]

\[
\times \left( \sum_{l, \delta'} L_R(x_1 - \delta'y_l) - 2Nc \right). \tag{4.36}
\]
Clearly, \((4.35)\) is of the form \(Ac^2 + Bc\), and it is easy to check \(A = B = 0\). Hence \(\sigma\) is elliptic in \(x_1\).

Consider next the \(\sigma\)-residue at the \(x_1\)-pole,

\[
x_1 = y_1 - \gamma, \quad \gamma = i\alpha/2.
\] (4.37)

It is proportional to

\[
2s_\gamma (L(g; y_1 - \gamma) + L(g'; y_1) + s_\gamma \sum_{k > 1, \delta} [L_R(y_1 - \gamma - \delta x_k) + L_s(y_1 - \delta y_k)])
\]

\[-2s_\gamma^2 (2L_2(y_1 - \gamma) + \sum_{k > 1, \delta} [L_R(y_1 - \gamma - \delta y_k) + L_R(y_1 - \delta x_k)]).
\] (4.38)

By \((3.96)\) the sums can be canceled. The remaining terms are equal to \((3.92)\) with \(y \to y_1\). We have already shown that \((3.92)\) vanishes, so we deduce that \(\sigma\) has no pole at \((4.37)\). More generally, it follows there are no poles for \(\sigma\) at \((4.37)\). Hence we can pair off terms to deduce that it suffices to show vanishing of the function \((3.92)\), so we deduce that \(\sigma\) has no pole at \((4.37)\).

Turning to the behavior at the poles \(x_1 = \pm x_2\), we first note that the contribution of the penultimate line in \((4.34)\) has no poles for these \(x_1\)-values, since the functions \(L(g; z), L_s(z)\) and \(L_R(z)\) are odd. For the same reason, the only singular terms in the remainder of \((4.34)\) are

\[
2s_\gamma^2 \wp(x_1 \pm x_2) - 2s_\gamma^2 L_s(x_1 \pm x_2)^2.
\] (4.39)

But in view of \((3.98)\) the function \(\wp(z) - L_s(z)^2\) is regular at the origin. Hence \(\sigma\) is regular for \(x_1 = \delta x_2\) and more generally for \(x_1 = \delta x_k\), \(\delta = +, -, k > 1\).

Since \(\sigma\) is elliptic and even in \(x_1\), it follows just as for \(N = 1\) that \(\sigma\) has no poles for \(x_1 = \omega_k\), etc. \((3.97)\)–\((3.98)\). In summary, \(\sigma\) has no poles as a function of \(x_1\), so it does not depend on it. Now independence of \(x_2, \ldots, x_N\) follows by permutation invariance. Moreover, recalling \(s_\gamma = s_\gamma'\), independence of \(y\) follows in the same way.

It remains to prove that the shift \(\sigma\) actually vanishes. To show this, we first choose \(y_j = x_j, j = 1, \ldots, N\), in \((4.34)\). If we now use our previous \(BC_1\) result that the rhs of \((3.99)\) vanishes, then we are left with \(-2s_\gamma^2\) times

\[
\sum_j [L(g; x_j) - L(g'; x_j)] \left( \sum_{k \neq j, \delta = +, -} [L_s(x_j - \delta x_k) - L_R(x_j - \delta x_k)] \right).
\] (4.40)

Hence we can pair off terms to deduce that it suffices to show vanishing of the function

\[
F(u_1, u_2) \equiv [L(g; u_1) - L(g'; u_1)] \sum_{\delta = +, -} [L_s(u_1 - \delta u_2) - L_R(u_1 - \delta u_2)]
\]

\[+ [L(g; u_2) - L(g'; u_2)] \sum_{\delta = +, -} [L_s(u_2 - \delta u_1) - L_R(u_2 - \delta u_1)].
\] (4.41)

Now \(F\) is readily checked to be elliptic and symmetric in \(u_1\) and \(u_2\). Inspecting residues at the \(u_1\)-poles, we see they vanish. Hence \(F\) is constant. Taking \(u_2 \to y\) and letting \(u_1 \to 0\), we obtain

\[
F = 2g_0 - g'_0)\{L_s'(y) - L_R'(y)\} + 2[L(g; y) - L(g'; y)][L_s(y) - L_R(y)].
\] (4.42)
Recalling (3.100)–(3.102), this can be rewritten as

\[
\frac{F}{2} = (g_0 - g_0')[-\wp(y) + \wp(y + \omega_2)] + \left( (g_0 - g_0') \frac{s'(y)}{s(y)} + \sum_{j>0} (g_j - g_j') L_j(y) \right) (L_s(y) - L_R(y)).
\] (4.43)

For \(y \to \omega_1\) we now get

\[
\frac{F}{2} = (g_0 - g_0')(-e_1 + e_3) + (g_1 - g_1') \lim_{y \to \omega_1} \frac{s'(y + \omega_1)}{s(y + \omega_1)} (L_s(y) - L_R(y))
\]

\[
= (g_0 - g_0')(-e_1 + e_3) + (g_1 - g_1') (L_s(y) - L_R'(y))
\]

\[
= (g_0 - g_0')(-e_1 + e_3) + (g_1 - g_1')(-e_1 + e_3).
\] (4.44)

Finally, from (3.77) we obtain

\[
g_0 + g_1 = g_0' + g_1.
\] (4.45)

Therefore \(F\) vanishes, completing the proof. \(\square\)

As the generalization of (3.111)–(3.112) we get the limit functions

\[
c_{nr}(g, \lambda; x) \equiv \prod_{j=1}^{N} c_{nr}(g; x_j) \cdot \left( \prod_{1 \leq j < k \leq N} \prod_{\delta = 0}^{\lambda} R(x_j - \delta x_k + i\alpha/2) \right)^{-\lambda},
\] (4.46)

\[
w_{nr}(g, \lambda; x) \equiv \prod_{j=1}^{N} w_{nr}(g; x_j) \cdot p^{2N(N-1)\lambda} \left( \prod_{1 \leq j < k \leq N} \prod_{\delta = 0}^{\lambda} s(x_j - \delta x_k) \right)^{2\lambda}.
\] (4.47)

Then we readily calculate

\[
w_{nr}(g, \lambda; x)^{1/2} D(g, \lambda; x) w_{nr}(g, \lambda; x)^{-1/2} = H_{nr}(g, \lambda; x) - 2N(s_g + (N-1)\lambda) \eta \omega_1,
\] (4.48)

with \(H_{nr}(g, \lambda; x)\) the Inozemtsev Hamiltonian (1.5). Moreover, we infer

\[
(H_{nr}(g, s_g; x) - H_{nr}(g', s_g; y)) \Psi_{nr}(g; x, y) = 0,
\] (4.49)

where

\[
\Psi_{nr}(g; x, y) \equiv (w_{nr}(g, s_g; x) w_{nr}(g', s_g; y))^{1/2} S(g; x, y).
\] (4.50)

Finally, we point out that when the \(\wp\)-function in the Inozemtsev Hamiltonian is replaced by the shifted \(\wp\)-function (3.117), then the difference of the altered Hamiltonians still annihilates \(\Psi_{nr}(g; x, y)\). (Just as in the \(BC_1\) case, this follows from the sum rules (3.108) and (3.118).)

**Acknowledgements.** We would like to thank F. Nijhoff and A. Veselov for useful discussions concerning the \(E_8\) invariance of the function \(\Sigma_g(z)\) given by (1.17). Also, thanks to critical comments by the referee the exposition of this paper has been improved.
References


Communicated by L. Takhtajan