On the two-point functions of some integrable relativistic quantum field theories

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Two-point functions associated with the Federbush, massless Thirring, and continuum Ising models and their boson analogs are studied. In the Thirring case it is shown that the fields do not define operator-valued distributions, while temperedness of the two-point Wightman function is proved in the Ising case and in the Federbush case for a certain range of coupling constants. By relating the short-distance singularity of the Schwinger functions to the high-energy behavior of the spectral measures it is shown the fields cannot be made to satisfy the CCR/CAR by a rescaling. In the fermionic Federbush case this breakdown of the CAR occurs in spite of the fact that the fields correspond to a local Lagrangian.

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I. INTRODUCTION

This paper is a continuation of previous work of the author on the Federbush, massless Thirring, and continuum Ising models and their boson analogs. In Ref. 1 we showed that the quantum fields of these models are normal ordered quadratic forms that are closely related or equal to the forms implementing improper Bogoliubov transformations generated by local and covariant classical field operators. We also studied the equations of motion and various other aspects, and discussed relations with work of other authors. In Ref. 2 we considered the scattering theory of these models at the classical and at the unphysical and physical quantum levels. The present work, some of whose results were announced in Ref. 1, deals with two-point Schwinger and Wightman functions arising in these models. The main issue we consider is whether the functional \( \langle dx \, F(x) \phi(x) \rangle \) (where \( \phi \) is the vacuum, \( \phi | x \rangle \) the quantum field, and \( F \) a test function in the Schwartz space \( S(\mathbb{R}^d) \)) corresponds to a vector in Fock space. In the case of the Federbush and Ising models and their boson analogs (studied in Secs. II and IV resp.) this is obvious if the Fourier transform \( \hat{F} \) has compact support, but in the case of the Thirring model and its boson analog (Sec. III) we prove that for any \( F \in S(\mathbb{R}^d) \) the functional is either zero or does not correspond to a vector in Fock space. In the former case the main problem is therefore to establish whether the corresponding two-point Wightman distribution in \( \mathcal{D}'(\mathbb{R}^d) \) extends to a tempered distribution. Setting this question is a first step towards verification of the Wightman axioms\(^3\) for these models.

As will be seen, the usual covariance and spectral properties are obvious, so that temperedness of the two-point Wightman function is equivalent to a polynomial short-distance singularity for the two-point Schwinger function. This follows from the theory of Laplace transforms,\(^4\) but in the case at hand this can also be seen in a direct and illuminating way (cf. below). Thus, a large part of the work in Secs. II and IV is concerned with finding the dominant short-distance singularity of the various Schwinger functions. (For the fermionic Ising model of Sec. IV A we refer to the work of McCoy et al.\(^5\)) A very useful tool in this study is a transformation to center of mass variables [cf. (2.22) and its generalization (2.54)], which is also tailor-made to study the measures in the Källen–Lehmann representations of the various two-point functions. [We found a similar transformation some time ago in a rather different context (cf. Ref. 5, pp. 418–9), but the advantage of the present transformation is that its Jacobian equals one.] By relating the short-distance singularity of the Schwinger function to the high-energy behavior of the spectral measure [through formulas like (2.37) below], we study existence of time-zero restrictions, and we find that for all interacting fields below the integral \( \int dp(m) \) diverges, implying that none of these fields can be made to satisfy canonical (anti-) commutation relations by a rescaling. In the case of the fermionic Federbush model this holds true, although the fields are derived from a local Lagrangian. To our knowledge this is the first explicit example of a breakdown of the CAR for a Lagrangian field theory describing massive particles. Also, it follows that the field strength renormalization constant \( Z = \left[ \int dp(m) \right]^{-1} \) vanishes, which indicates nonexistence of time-zero fields according to conventional wisdom. However, Challifour\(^6\) has shown that \( \left[ \text{an infinite resummation of} \right] \) the time-zero field considered in Sec. II A exists in the usual axiomatic sense if the coupling constant is small enough. Thus, this constitutes a counterexample to this lore (provided the resummation involved does not change the fields).

Since we consider here the action of the fields on the vacuum, only the pure creation part of the fields is needed. Therefore, we refrain from giving complete definitions of the fields below, and refer instead to Ref. 1, where more references and background information can also be found. The fields of Secs. II and III act on a Fock space \( \mathcal{F}(\mathcal{H}_+ \oplus \mathcal{H}_-) \), where \( e = a (s) \) stands for antisymmetric (symmetric) in the fermion (boson) case and where \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) are copies of a space \( \mathcal{H} = L^2(\mathbb{R},d\theta) \). Thus we can write \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \), where \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) are copies of \( L^2(\mathbb{R},d\theta) \). The spaces \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) are physically interpreted as state spaces of one-dimensional relativistic particles of positive and negative charge resp., described by rapidity wavefunctions, while the spaces \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) correspond to particles of both charges but of different species ("species"...
meaning left- or right-moving in the Thirring case). In Sec. IV the relevant Fock spaces are $\mathcal{F}_e(\mathcal{H}_e)$, $e = a,s$, corresponding to neutral particles of only one species. Throughout the paper the notation $K^e_\lambda$ is used as shorthand for the Wick monomials $\int d\theta_1 d\theta_2 K_{(\theta_1,\theta_2)} e^{i\theta_1} e^{i\theta_2}$.

II. THE FEDERBUSH CASE

A. Fields on $\mathcal{F}_e(\mathcal{H}_e)$

We define for any $\lambda \in \mathbb{R}$

$$\phi_{\lambda}(x) = e^{i\lambda x} \equiv \exp(K_{(\theta_1,\theta_2)} e^{i\theta_1} e^{i\theta_2})$$

where

$$K_{(\theta_1,\theta_2)} = \exp(i\lambda (p(\theta_1) + p(\theta_2))) K_{(\theta_1,\theta_2)}$$

(2.2)

Here

$$\pi(x, x^{-1})$$

(2.3)

$$p(\theta) = (\cosh \theta, \sinh \theta)$$

(2.4)

the dot denotes the Lorentz inner product, and

$$K_{(\theta_1,\theta_2)} = \frac{\sin \pi \lambda}{2\pi} e^{i\lambda \theta} \sinh \theta$$

(2.5)

Theorem 2A.1: For any $\lambda \in \mathbb{R}$ and $t > 0$ the Schwinger function

$$S_{(\lambda)}(t) = \int_{\mathbb{R}} d\theta \phi_{\lambda}(\theta) = e^{i\lambda \theta}$$

(2.6)

is finite-valued and satisfies

$$S_{(\lambda)}(t) = \exp\left( -\frac{i}{2} \phi_{\lambda}(\theta, \theta) + \phi_{\lambda}(\theta_1, \theta_2) \right)$$

(2.7)

where $A_{(\lambda)}(t)$ is the integral operator on $L^2(\mathbb{R})$ with kernel

$$A_{(\lambda)}(t, t_1, t_2) = \frac{\sin \pi \lambda}{2\pi} e^{i\lambda \theta} \sinh \theta$$

(2.8)

For any $\lambda \in [0, \lambda]$ there is an $N > 0$ so that

$$\lim_{t \to \infty} N \lambda S_{(\lambda)}(t) = 0$$

(2.9)

Furthermore, for any $\lambda \in \mathbb{R}$,

$$S_{(\lambda)}(t) = 1 + O(e^{-2t})$$

(2.10)

and, for any $\lambda \in [0, \lambda]$,

$$\lim_{t \to \infty} \frac{\ln \left| S_{(\lambda)}(t) \right|}{\ln(1/t)} = 2\lambda^2$$

(2.11)

Finally, for any $\lambda \in \mathbb{R}$, $S_{(\lambda)}(t)$ admits a Källén–Lehmann representation

$$S_{(\lambda)}(t) = 1 + \sum_{i=0}^{\infty} \frac{d\rho_{(\lambda)}(m)}{m^2}$$

(2.12)

where $K_{(\lambda)}$ is the modified Bessel function

$$K_{(\lambda)} = \int_0^\infty d\theta \exp\left(-t \cosh \theta + \lambda \theta \right)$$

(2.13)

Here, the measure satisfies for any $\lambda \in [0, \lambda]$

$$\int_0^\infty \frac{d\rho_{(\lambda)}(m)}{m^2} \left\{ \begin{array}{ll} < \infty, & \forall \gamma > 2\lambda^2, \\ \infty, & \forall \gamma < 2\lambda^2. \end{array} \right.$$ 

(2.14)

Theorem 2A.2: For any $\lambda \in [0, \lambda]$ the Wightman function

$$\phi_{(\lambda)}(x) = e^{i\lambda x} \equiv \exp(K_{(\theta_1,\theta_2)} e^{i\theta_1} e^{i\theta_2})$$

(2.15)

extends to a tempered distribution. For any $\lambda \in [0, \lambda]$ its time-zero restriction exists and defines a tempered distribution.

Conjecture 2A.3: The relations (2.11) and (2.14) hold for any $\lambda \in [0, \lambda]$. For any noninteger $\lambda$ with $|\lambda| > 1$ the Schwinger function increases faster than $t^{-N}$ for any $N > 0$ as $t \to \infty$; equivalently, the Wightman function (2.15) does not extend to a tempered distribution.

Proof of Theorem 2A.4: A calculation shows that

$$S_{(\lambda)}(t) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int_0^\infty dx_{n} \sum_{n_0}^\infty \left[ \int_{\mathbb{R}} d\theta \phi_{(\lambda)}(\theta) \right]^{n_0}$$

(2.16)

where $A_{(\lambda)}(t)$ is the symmetric group. Since $A_{(\lambda)}(t)$ is clearly Hilbert–Schmidt for any $t > 0$ and $\lambda \in \mathbb{R}$, convergence of the series and (2.7) follow from Sec. 3 of Ref. 7. The bound (2.10) follows from the estimates

$$\det(1 + T) \leq \exp(||T||_1), \quad T > 0,$$

(2.17)

and

$$||A_{(\lambda)}(t)||_1^2 = O(e^{-2t}), \quad t \to \infty,$$

(2.18)

whose proof is obvious. The assertion (2.9) is a consequence of (2.17) and the fact that the l.h.s of (2.18) is bounded above by $C_1 \ln(1/t)$ for $|\lambda| < 1$, as is readily verified.

To prove (2.11), we recall the well-known fact that

$$\ln(1 + t) = -\sum_{n=1}^\infty \frac{(-1)^n}{n} Tr T^n,$$

(2.19)

provided the rhs converges. To apply this to the case at hand, we shall first derive an upper bound to the function

$$A_{(\lambda)}(t) = \ln \left( \sum_{n=1}^\infty \frac{(-1)^n}{n} Tr T^n \right)$$

(2.20)

where $\theta_{2n+1} = \theta$, and

$$h_{(\lambda)}(\theta) = (1/2\pi)e^{i\theta} \sinh \theta$$

(2.21)

To this end, we introduce a transformation to center of mass variables,

$$y_0 = \ln\left( \sum_{i=1}^\infty e^{i\theta} \right)/M_{(\lambda)}$$

(2.22)

$$y_j = \theta_{2j-1} - \theta_j, \quad j = 1,\ldots,k - 1,$$

(2.23)

where $M_{(\lambda)}$ is the invariant mass,

$$M_{(\lambda)} = \left[ \sum_{i=0}^{k-1} \cosh \theta_i \right]^{-1/2}$$

(2.24)

It is readily verified the Jacobian of this transformation is 1. Hence, setting

$$\tilde{M}_{(\lambda)}(y) = M_{(\lambda)}(\theta, y)$$

(2.25)

and using (2.13), it follows that

$$W_{(\lambda)}(F) = \int \phi_{(\lambda)}(F) \Omega$$

(2.15)
\[ a_{\lambda, \alpha}(t) = 2(\sin \pi \lambda)^2 \int dy K_0(t \hat{M}_2(y)) \times \prod_{j=1}^{n-1} h_{\lambda}(-y_{2j-1})h_{\lambda}(y_{2j}) \]
\[ \times h_{\lambda}(-\sum_{j=1}^{2n-1} y_j). \quad (2.25) \]

Since
\[ \hat{M}_2(y)^k, \quad \forall y \in \mathbb{R}^{k-1}, \]
we now get the upper bound
\[ a_{\lambda, \alpha}(t) < 2(\sin \pi \lambda)^2 I_{n, \alpha}K_0(2nt), \quad (2.27) \]
where \( I_{n, \alpha} \) is the cycle integral
\[ I_{n, \alpha} = (\int d\phi_1 \cdots d\phi_{2n-1} h_{\lambda}(\phi_i)) \times \]
\[ \prod_{j=1}^{n-1} h_{\lambda}(\phi_{2j-1} - \phi_{2j-2} - \phi_{2j})h_{\lambda}(\phi_{2j} - \phi_{2j+1} - \phi_{2j-1})h_{\lambda}(-\phi_{2n-1}), \quad (2.28) \]

obtained from (2.25) by the transformation
\[ \phi_k = \sum_{j=1}^{k} y_j, \quad k = 1, \ldots, 2n - 1, \]
which renders its convolution structure more transparent.

Setting
\[ h_{\lambda}(x) = \int d\theta \exp(i\theta x) h_{\lambda}(\theta), \quad |\lambda| < \frac{1}{2}, \]
one verifies by a contour integration that
\[ h_{\lambda}(x) = \text{sech}[\pi(x - i\lambda)], \quad |\lambda| < \frac{1}{2}, \]
so that
\[ I_{n, \alpha} = \frac{1}{2\pi} \int dx \left[ h_{\lambda}(x)h_{\lambda}(-x) \right]^n = \pi^{-2n-1} \int_0^\infty dx \left[ \cosh x + \cos(2\pi \lambda) \right]^{-n}, \quad |\lambda| < \frac{1}{2}. \]

As a result, the terms of the series \[ \Sigma |1/n| - \sin^2\pi \lambda \] diverge for \( n \to \infty \) if \( |\lambda| > \frac{1}{2} \), so the series diverges for \( |\lambda| > \frac{1}{2} \). However, using the relation
\[ \ln(1 - x) = -\sum_{n=1}^\infty \frac{x^n}{n}, \quad |x| < 1, \]
one infers that the series converges absolutely for \( |\lambda| < \frac{1}{2} \) and that
\[ -\sum_{n=1}^\infty \frac{1}{n} \left( -\sin^2\pi \lambda \right)^n I_{n, \alpha} = \frac{1}{2\pi} \int_0^\infty dx \ln[(\cosh x + 1)(\cosh x + \cos(2\pi \lambda))^{-1}], \]
\[ \lambda > 0, \quad |\lambda| < 1. \]
(Here we used the integral
\[ \int_0^\infty dx \ln[(\cosh x + \cos \alpha)(\cosh x + \cos \beta)^{-1}] = \int_0^\alpha d\alpha \sin \alpha \int_0^\infty \frac{1}{\cosh x + \cos \phi} = \frac{\pi}{2} \left( \alpha^2 - \alpha^2 \right), \quad |\alpha|, |\beta| < \pi, \]
(2.35)

where the last step can be verified by contour integration.)
As a consequence we may write
\[ \ln(S(t)|S(t)) = -\sum_{n=1}^\infty \frac{1}{n} (-\gamma)^n a_{\lambda, \alpha}(t), \]
\[ \forall \tau > 0, \forall \lambda \in \mathbb{R} \backslash \{ -\frac{1}{2}, \frac{1}{2} \}. \]

We finally note that the function \( K_0(mt)/\sqrt{1/t} \) is bounded for \( t > 0 \) and \( m > 2 \) and has limit 1 for \( t \to 0 \) and any \( m > 2 \). The relation (2.11) therefore follows from (2.23), (2.34), and (2.36) by virtue of dominated convergence.

To prove the remaining claims, we first note that (2.12) can be obtained from (2.16) by making the change of variables (2.22) in each term of the series; each term contributes a measure \( dF_2(m) \), where \( F_2(m) \) is the result of omitting the \( \nu_0 \)-dependent exponential factor and then integrating the internal variables over the region \( \hat{M}_2(y)^k \leq m \) in \( R^{2k-1} \) [note that \( F_2(m) = 0 \) for \( m < 2n \)]. Using (2.12), we can now connect the short-distance singularity of the Schwinger function and the high-energy behavior of the spectral measure by observing that we may write
\[ \int_0^\infty dt \int d\phi \left[ S(t) - 1 \right] = \int_0^\infty dx \int dx \hat{F}_2(x)K(x), \]
(2.37)

Assuming from now on that \( |\lambda| < \frac{1}{2} \), it follows from (2.11) that the integral at the rhs converges/diverges for \( \delta \) greater/smaller than \( 2\lambda^2 - 1 \). Since the second factor on the rhs is finite for \( \delta = -1 \), (2.14) results.

**Proof of Theorem 2A.2.** The first statement follows from (2.9) and general results on Laplace transforms, but it is more illuminating to observe that for \( |\lambda| < \frac{1}{2} \) one has
\[ W^\alpha_1(\hat{F}, \hat{F}) - |\hat{F}(0)|^2 = \sum_{n=0}^\infty (2n)!^{-1} \int d\phi_1 \cdots d\phi_n \left[ \hat{F} \left( \sum_{\alpha=1}^{2n} \phi_\alpha \right) \prod_{\alpha=1}^{2n} \left( \sum_{\alpha=1}^{2n} \phi_\alpha \right)^{-\alpha} \right]^2 \]
\[ \times \prod_{\alpha=1}^{2n} K_\alpha(\phi_\alpha - \phi_\beta) \left( \phi_\alpha - \phi_\beta \right)^2 \]
\[ \leq ||F||_{\infty}^2 \sum_{n=0}^\infty \frac{1}{n!} \int d\phi_1 \cdots d\phi_n \left( \sum_{\alpha=1}^{2n} \cosh \phi_{\beta} \right)^{-\alpha} \left( \sum_{\alpha=1}^{2n} \cosh \phi_{\alpha} \right)^{-\alpha} \left( \sum_{\alpha=1}^{2n} \phi_{\alpha} \right)^{-\alpha} \]
\[ \times \prod_{\alpha=1}^{2n} K_\alpha(\phi_\alpha - \phi_\beta) \left( \phi_\alpha - \phi_\beta \right)^2 \left( \phi_\alpha - \phi_\beta \right)^2 \]
\[ \leq ||F||_{\infty}^2 \Gamma(\alpha^{-1})^{-1} \int_0^\infty dt \int d\phi_1 \cdots d\phi_n \left[ S(t) - 1 \right]. \]
(2.38)

Here, \( ||.||_{\infty} \) is a Schwartz norm, and the last step follows from (2.16). In view of (2.9) and (2.10) the integral on the rhs converges for \( |\lambda| < \frac{1}{2} \), provided \( \alpha > N_\lambda \). This clearly implies the assertion.

Finally, we note that
\[ W^\alpha_1(0, \hat{F}[(0, \hat{F})] = \int d\phi \left| \hat{F}(\hat{F}) \right|^2 \]
\[ = \frac{1}{2} \int d\phi \left| \hat{F}(\hat{F}) \right|^2 \int d\phi \ c \left( \cosh \phi \right) \]
\[ \leq \frac{1}{2} \int d\phi \left| \hat{F}(\hat{F}) \right|^2 \int d\phi \ c \left( \cosh \phi \right) \]
(2.39)

where \( \hat{F} \) is the Fourier transform of \( \hat{F} \) on \( S(R) \). By virtue of (2.19)


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the first factor is finite for $|\lambda| < 1/2$, so that the remaining statement follows. □

**B. Fields on $F_{\alpha}(R^4 \otimes R^4)$**

We define for any $\lambda \in R$

$$\psi_{\lambda,1}(x)\Omega = \left(\frac{m(1)}{4\pi}\right)^{1/2} \int d\theta c^*_{\lambda,1}(\theta)$$

$$\times \left(\phi^*_{\lambda,1} \cdots \phi^*_{\lambda,1}\right) = \exp(1/4\varphi^0(\theta) \phi^*_{\lambda,1}(m-1)x) \Omega.$$ (2.40)

Here, $\phi^*_{\lambda,1} \Omega$ is given by (2.1) with $c^*_{\alpha}$ replaced by $c^*_{\lambda,1}$ and the masses $m(s)$ of the two different species $s = \pm 1$ are strictly positive.

**Theorem 2B.1:** For any $\lambda \in R$ and $t > 0$ the Schwinger function

$$\mathcal{S}_{\alpha,1}^{\lambda}(t) = \left\{\left(\frac{1}{2} + it, 0\right) \Omega \right\}^2$$ (2.41)

is finite-valued and satisfies

$$\mathcal{S}_{\alpha,1}^{\lambda}(t) = m(1)S_0(m(1)t)S_\alpha^{\lambda}(m(1)t).$$ (2.42)

Here, $S_\alpha^{\lambda}$ is given by (2.7) and $S_0$ is the Schwinger function of the free Dirac field of mass 1,

$$S_0(t) = \frac{1}{4\pi} \int d\theta \exp(-t \cosh\theta) \left(\frac{e^\theta - 1}{e^\theta - 1}\right).$$ (2.43)

For any $\lambda \in (-1/2, 1/2)$ there is an $N_\lambda > 0$ such that

$$\lim_{t \to 0} N_\lambda^{1/2}S_{\alpha,1}^{\lambda}(t) = 0.$$ (2.44)

Furthermore, for any $\lambda \in R$,

$$\mathcal{S}_{\alpha,1}^{\lambda}(t) = m(1)S_0(m(1)t) + O(\exp(-m(1) + 2m(1)t)),$$

$$t \to \infty,$$ (2.45)

and, for any $\lambda \in (-1/2, 1/2)$,

$$\lim_{t \to 0} \frac{\mathcal{S}_{\alpha,1}^{\lambda}(t)}{\sqrt{1/t}} = \left(1 + 2\lambda^2, 2\lambda^2, 1 + 2\lambda^2\right).$$ (2.46)

Finally, for any $\lambda \in R$, $\mathcal{S}_{\alpha,1}^{\lambda}(t)$ has a spectral representation

$$\mathcal{S}_{\alpha,1}^{\lambda}(t) = m(1) \left( K_0(m(1)t) - K_0(0) \right)$$

$$+ \sum_{m(1) + 2m(-1)} \int d\theta \exp(-t \cosh\theta)$$

$$\times K_0(m(1)t),$$ (2.47)

where

$$K_0(t) = K_0(0) = \int_0^\infty d\theta \cosh\theta \exp(-t \cosh\theta), \quad t > 0.$$ (2.48)

Here, the measures are positive and satisfy the inequalities

$$m(1)dp_1(m) < dp_2(m)$$ (2.49)

and

$$m^2dp_1(m) > m(1)dp_2(m).$$ (2.50)

Moreover, for $\lambda \in [-1/2, 1/2]$ and $j = 1, 2,$

$$\int_{m(1) + 2m(-1)} dp_{\alpha,1}(m) \left| \begin{array}{cc} \lambda \gamma \end{array} \right| < \infty, \quad \forall \gamma > 2\lambda^2,$$

$$= \infty, \quad \forall \gamma < 2\lambda^2.$$ (2.51)

**Theorem 2B.2:** For any $\lambda \in (-1/2)$ the Wightman function

$$\mathcal{W}_{\alpha,1}^{\lambda}(\bar{t}, F) = \int d\theta \left(\frac{1}{4\pi}\right)^{1/2} \int d\theta \exp\left(-t \cosh\theta\right)$$

$$\times \left(\exp(t \cosh\theta)ight)$$

extends to a tempered distribution. For any $\lambda \in [-1/2]$ its off-diagonal elements have a tempered time-zero restriction. Its diagonal elements do not admit a time-zero restriction for any noninteger $\lambda \in R$.

**Remark 2B.3:** As we have shown in Ref. 1, the fields $\psi_{\lambda,1}$ and $\psi_{\lambda,-1}$ fail to satisfy the Federbush equation of motion, but are presumably local in the usual axiomatic sense for $|\lambda| < 1/2$. By omitting the kernel $e^{it\theta}$ in $\phi_{\lambda,1}^{\lambda}$ one obtains fields that solve it, but these fields are very likely nonlocal. The results of Sec. II A and II B can be easily extended to this case, and lead to similar qualitative predictions. We leave the details to the interested reader.

**Proof of Theorem 2B.1:** The only assertions that do not immediately follow from Theorem 2A.1 concern the spectral representation and the measures occurring in it. To prove that they hold true, we note first that we may write $\mathcal{S}_{\alpha,1}^{\lambda}(t)$ as

$$\sum_{\alpha = 0}^\infty \int d\theta \cdots d\theta_{2n+1} \left(\frac{e^{t\cosh\theta} - 1}{e^{t\cosh\theta} - 1}\right)$$

$$\times \exp(-t \left[ \cosh\theta_1 + \cosh\theta_2 \cdots + \cosh\theta_{2n+1} \right]),$$ (2.53)

Next, we introduce a generalization of the transformation (2.22) to the case where the masses $m_i$ corresponding to the rapidities $\theta_i$ are not necessarily equal to 1:

$$y_0 = \ln \left( \prod_{i = 1}^{k} \frac{m_i \theta_i}{M_k(m_i)} \right),$$

$$y_j = \theta_{j+1} - \theta_j, \quad j = 1, \ldots, k - 1,$$ (2.54)

where

$$M_k(m, \theta) = \prod_{i = 1}^{k} m_i \cosh(\theta_i - \theta_j).$$ (2.55)

Since the Jacobian is still equal to 1, the representation (2.47) now follows as in the proof of Theorem 2A.1. $M_{2n+1}(m(1), m(-1), \ldots, m(-1))$ play the role of $M_0(x)$. The inequalities (2.49) and (2.50) are easily seen to hold after one substitutes center of mass variables in (2.53) and (2.53), resp. Finally, (2.51) results from (2.46) by virtue of a formula analogous to (2.37). □

**Proof of Theorem 2B.2:** The first claim follows from the relation

$$\frac{m(1)}{4\pi} \sum_{n = 0}^\infty \int dt \cdots d\theta_{2n}$$

$$\times \left(\cosh\theta_0 + \cosh\theta_2 \cdots + \cosh\theta_{2n}\right)$$

$$\times \left(\frac{e^{t\cosh\theta} - 1}{e^{t\cosh\theta} - 1}\right)$$

$$\times K_0^\alpha(\theta_i - \theta_{i+n}),$$ (2.56)
as in the proof of Theorem 2A.2. The second assertion also follows as in that proof from (2.51) with \( j = 2 \). For \( |\lambda| < \frac{1}{\pi} \) the last statement is a consequence of the fact that the integral of the product \( \Phi_{\lambda,n}(m) \) diverges. To prove it for any noninteger \( \lambda \), we note that if we smear, e.g., the upper component of \( \psi_{\lambda,1}(0,x^1)\Omega \) with some \( f \in \mathcal{S} \mathbb{R} \), then the squared \( L^2 \)-norm of the three-body component is proportional to

\[
\int d\theta_1 d\theta_2 d\varphi_1 e^{i\varphi_1} |\hat{f}(m_1)\sin\theta_1 + m_1 - 1| \times |(\sinh \theta_1 + \sinh \theta_2)|^2 e^{2i(\theta_1 - \theta_2)} \sin \theta_1 \theta_2 \frac{1}{m_1} \int d\theta e^{i\theta} \sinh \theta \sinh \theta \frac{1}{m_1} \int d\phi \sin \phi \cos \phi \int_{-\infty}^{\infty} dp |\hat{f}(p)|^2.
\]

(2.57)

Thus, if \( f \neq 0 \), the \( \varphi \)-integral diverges at \( -\infty \) for any fixed \( \theta \), and therefore the \( L^2 \)-norm is infinite by Fubini's theorem. □

C. Fields on \( \mathcal{F}_s(\mathcal{H}) \)

We define for any \( \lambda \in \mathbb{R} \),

\[
\phi^\lambda_n(x) \Omega := \exp(K_{\lambda,n} x^1 c^* x^1) \Omega,
\]

where

\[
K_{\lambda,n}(\theta_1, \theta_2) := \frac{\sin \lambda \theta}{2\pi} e^{i(\theta_1 - \theta_2)} \sin \theta_1 \theta_2.
\]

(2.58)

Here we have used the same notation as in (2.2), and

\[
\mathcal{K}_{\lambda}(\theta) := \frac{\sin \lambda \theta}{2\pi} e^{i(\theta_1 - \theta_2)} \sin \theta_1 \theta_2.
\]

(2.59)

**Theorem 2C.1:** For any \( \lambda \in [0,1] \) and \( t > 0 \) the Schwinger function

\[
S^\lambda_n(t) := \|\phi^\lambda_n(\{ t, 0 \}) \Omega \|^2
\]

is finite-valued and satisfies

\[
S^\lambda_n(t) = \det(1 - A^\lambda_n(t) A^\lambda_n(t))^\beta,
\]

(2.60)

where \( A^\lambda_n(t) \) is the integral operator on \( L^2(\mathbb{R}) \) with kernel

\[
A^\lambda_n(t, \theta_1, \theta_2) := \frac{\sin \lambda \theta}{2\pi} \exp[-i t \cos \theta_1 + \cos \theta_2 + i(\lambda - 1) \theta_1 - \theta_2] \times \sin \theta_1 \theta_2.
\]

(2.61)

For any \( \lambda \in \mathbb{R} \),

\[
S^\lambda_n(t) = 1 + O(e^{-2t}), \quad t \to 0,
\]

(2.62)

and, for any \( \lambda \in [0,1] \),

\[
\lim_{t \to 0} \frac{\ln S^\lambda_n(t)}{\ln(1/t)} = 2(\lambda - \lambda^2).
\]

(2.63)

Moreover, for any noninteger \( \lambda \) not in \( (0,1) \) there is a \( C_{\lambda} > 0 \) such that

\[
S^\lambda_n(t) = \infty, \quad \forall t \in (0, C_{\lambda}).
\]

(2.64)

Finally, for any \( \lambda \in \mathbb{R} \), \( S^\lambda_n(t) \) admits a spectral representation

\[
S^\lambda_n(t) = 1 + \int_{-\infty}^{\infty} dp \rho^\lambda_n(m) K_0(mt),
\]

(2.65)

where the measure satisfies

\[
\int_{-\infty}^{\infty} dp \rho^\lambda_n(m) = \infty, \quad \forall \gamma > 2(\lambda - \lambda^2),
\]

(2.66)

\[
\forall \gamma > 2(\lambda - \lambda^2).
\]

(2.67)

for any \( \lambda \in (0,1) \).

**Theorem 2C.2:** For any \( \lambda \in (0,1) \) the Wightman function

\[
W^\lambda_n(F,F) := \|\phi^\lambda_n(F) \|_2^2, \quad F \in \mathcal{F}^{\infty}_n(\mathbb{R}^2),
\]

extends to a tempered distribution. Moreover, its time-zero restriction exists and defines a tempered distribution. Finally, if \( \lambda \neq 0 \) and not in \( \mathbb{Z} \), \( W^\lambda_n \) does not extend to a tempered distribution.

**Proof of Theorem 2C.1:** But for the assertion (2.66), the proof proceeds along similar lines as the proof of Theorem 2A.1. The analog of (2.16) is

\[
S^\lambda_n(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots d\theta_{2n}
\]

\[
\times \sum_{m=0}^{\infty} \prod_{i=1}^n A^\lambda_n(t, \theta_i, \theta_{i+n}, \theta_{i+n}) A^\lambda_n(t, \theta_{i+n}, \theta_{i+n})
\]

(2.69)

Again, the Hilbert–Schmidt property of \( A^\lambda_n(t) \) for any \( t > 0 \) and \( \lambda \in \mathbb{R} \) is obvious, but in this case convergence of the series and (2.62) only follow from Sec. 3 of Ref. 7 provided

\[
\|A^\lambda_n(t)\| < 1.
\]

(2.70)

Assuming from now on that \( \lambda \in (0,1) \), we may write

\[
A^\lambda_n(t) = e^{-i\lambda t/2} K_{\lambda,n} e^{-i\lambda t/2}
\]

(2.71)

where \( K_{\lambda,n} \) is multiplication by \( \mathcal{C} \) and \( K_{\lambda,n} \) the convolution operator with kernel \( K_{\lambda,n}(\theta) \), whose norm is 1, since it turns into multiplication by \( \sin \lambda \theta \operatorname{csch}(\pi x - \lambda^2 \theta^2) \) upon Fourier transformation [cf. (2.21) and (2.31)]. Hence, (2.71) is satisfied for any \( t > 0 \), implying the first statement. The bound (2.64) follows in the same way as (2.10), using instead of (2.17) the estimate

\[
|\det(1 - T)|^{-1} \exp(-1) \|T\|_1, \quad 0 < T < 1 < c < 1,
\]

(2.72)

whose proof is easy. [Note that (2.71) holds for any \( \lambda \in \mathbb{R} \) if one chooses \( t \) sufficiently large, since \( \|A^\lambda_n(t)\| \to 0 \) for \( t \to \infty \).]

We proceed to prove (2.65), using the relation

\[
|\det(1 - T)|^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{Tr} T^n, \quad 0 < T < 1.
\]

(2.73)

Introducing

\[
da_n(t) := \operatorname{Tr}[A^\lambda_n(t) A^\lambda_n(t)],
\]

(2.74)

one obtains as the analog of (2.27) the bound

\[
da_n(t) \leq 2(\sin \lambda) \mathcal{C} I_{\lambda,n} - 1/2 K_{\lambda,2n}.
\]

(2.75)

Also, using the method leading to (2.34) and the integral (2.35), one obtains

\[
\sum_{n=1}^{\infty} \frac{1}{n} (\sin \lambda) \mathcal{C} I_{\lambda,n} - 1/2 = \lambda - \lambda^2.
\]

(2.76)

From this, (2.65) follows by the same arguments as those used in the proof of (2.11).

To prove the assertion (2.66), it suffices to show that the bound (2.71) is violated when \( \lambda \) is outside \( (0,1) \) and not an
integer, and when \( t \) is taken small enough. [Indeed, from Sec. 3 in Ref. 7 it is readily seen that this condition is not only sufficient, but also necessary for the convergence of the series at the rhs of (2.70).] To prove this, let us assume that \( \lambda \) is greater than 1 and not an integer (the proof for the other case is similar). Denote the characteristic functions of the intervals \( [\ln(1/t), \ln(1/(1/t^2)) + 1) \) and \( (-\ln(1/t) - 1, -\ln(1/t)) \) by \( \chi^+ \) and \( \chi^- \), resp. Then one has

\[
|\chi^+ A^0 (t) \chi^-| = \left| \frac{\sin \pi \lambda}{2\pi} \int_{\ln(1/t)}^{\ln(1/(1/t^2)) + 1} d\theta_1 d\theta_2 \times \exp\left[-f(\cosh \theta_1 + \cosh \theta_2) \right.ight. \\
\left.\left. + \left(\lambda - \frac{1}{2}\right)(\theta_1 + \theta_2)\right] \right| \leq \left(\frac{1}{t}\right)^{2\lambda - 1} .
\]

Since \( \|\chi^+ \| = 1 \) and the rhs diverges for \( t \to 0 \), it follows that (2.71) is false for small \( t \).

The remaining statements follow as in the proof of Theorem 2A.1, so the theorem is proven. □

**Proof of Theorem 2C.2:** The first claim follows from (2.64) and (2.65) and the relation

\[
\sum_{n=1}^{\infty} \frac{1}{n!} \int d\theta_1 \cdots d\theta_{2n} \left( \sum_{j=1}^{2n} \cosh \theta_j \right)^{-\alpha} \\
\times \prod_{i=1}^{n} K \frac{\partial^n}{\partial \theta_{i+1} \cdots \partial \theta_{i+n}} K \frac{\partial^n}{\partial \theta_{i+1} \cdots \partial \theta_{i+n}} \\
= \Gamma(\alpha)^{-1} \int_0^{\infty} dt \ t^{-\alpha - 1} \left[ S^0 (t) - 1 \right] 
\]

[cf. (2.38)]. The assertion concerning the time-zero function follows as in Theorem 2A.2 from (2.65). The validity of the last statement can be seen as follows: By virtue of the uniform boundedness principle, convergence of the series on the lhs of (2.79) for sufficiently large \( \alpha \) is not only sufficient, but also necessary for the temperedness of \( W^0 \). However, in view of (2.66), the series diverges for any \( \alpha \). Stronger yet, it is readily seen that the expressions corresponding to the identity permutation already diverge if \( n > (4\lambda - 4)^{-\alpha} \) (assuming \( \lambda > 1 \), e.g.). Indeed, one has

\[
\int d\theta_1 \cdots d\theta_{2n} \left( \sum_{j=1}^{2n} \cosh \theta_j \right)^{-\alpha} \\
\times \prod_{i=1}^{n} \exp \left[ (2\lambda - 1)(\theta_i - \theta_{i+n}) \right] \left( \sum_{j=1}^{2n} e^{\theta_j} \right) \\
\geq \int_0^{\infty} d\theta_1 \cdots d\theta_{2n} \left( \sum_{j=1}^{2n} e^{\theta_j} \right) \\
\geq \left(\frac{2n!}{(2n)^n} \right) \int_0^{\infty} d\theta_1 e^\theta \cdots \left( \sum_{j=1}^{n} \frac{1}{j-1} \right) \\
\geq \left(\frac{2n!}{(2n)^n} \right) \frac{\theta_{n+1} e^{\theta_{n+1}} \cdots \theta_2 e^{\theta_2} \cdots \theta_1 e^{\theta_1}}{\theta_1 \theta_2 \cdots \theta_n} \\
= \infty, \ n > (4\lambda - 4)^{-\alpha}, \ \lambda > 1, \ (2.80)
\]

which proves the assertion. □

**D. Fields on \( \mathcal{F}_s (\mathcal{H}^+ \otimes \mathcal{H}^-) \)**

We define for any \( \lambda \in \mathbb{R} \)

\[
\phi_{\lambda, 1}(x) \Omega = - (4\pi)^{-1/2} \\
\times \int d\theta \ c_{\lambda, -1} (\theta) \ e^{ix \theta} \phi^0 (m \! - \! 1) x \Omega, 
\]

where \( \phi^0 \) is given by (2.58) with \( c^0 \) replaced by \( c^\omega \).

**Theorem 2D.1:** For any \( \lambda \in [0, 1] \) and \( t > 0 \) the Schwinger function

\[
\mathcal{S}_{\lambda, 1} (t) = \| \phi_{\lambda, 1}(it, 0) \Omega \|^2 
\]

is finite-valued and satisfies

\[
\mathcal{S}_{\lambda, 1} (t) = S_{KG} (m(1)) S^0 (m - 1) t .
\]

Here, \( S^0 \) is given by (2.62) and \( S_{KG} \) is the Schwinger function of the free Klein–Gordon field of mass 1,

\[
S_{KG} (t) = \frac{1}{4\pi} \int d\theta \| \exp(-t \cosh \theta) \| .
\]

For any \( \lambda \in \mathbb{R} \),

\[
\mathcal{S}_{\lambda, 1} (t) = S_{KG} (m(1)) + O(\exp(-m(1) + 2m(1) - 1) t), \quad t \to \infty ,
\]

and, for any \( \lambda \in [0, 1] \),

\[
\lim_{t \to 0} \frac{\ln \| \mathcal{S}_{\lambda, 1} (t) \|}{\ln(1/t)} = 2(\lambda - \lambda^2) .
\]

Moreover, for any noninteger \( \lambda \) not in \((0, 1)\) there is a \( C_\lambda > 0 \) such that

\[
\mathcal{S}_{\lambda, 1} (t) = \infty, \quad \forall \ t \in (0, C_\lambda) .
\]

Finally, for any \( \lambda \in \mathbb{R} \), \( \mathcal{S}_{\lambda, 1} (t) \) admits a spectral representation

\[
\mathcal{S}_{\lambda, 1} (t) = \frac{1}{2\pi} K_{\alpha}(m(1)t) \\
+ \int_{m(1) + 2m(1)}^\infty d \rho_{\lambda, 1} (m) K_{\alpha}(mt) ,
\]

where the measure satisfies (the analog of) (2.68) for any \( \lambda \in (0, 1) \).

**Proof:** It is readily seen that the assertions follow from Theorem 2C.1 and the behavior of \( K_{\alpha}(t) \) for \( t \to 0 \) and \( t \to \infty \). □

**Theorem 2D.2:** For any \( \lambda \in (0, 1) \) the Wightman function

\[
\mathcal{S}_{\lambda, 1} (\bar{F}, F) = \| \phi_{\lambda, 1}(F) \Omega \|^2, \quad \bar{F}, F \in C_\alpha (\mathbb{R}^2) ,
\]

extends to a tempered distribution, which has a tempered time-zero restriction. For noninteger \( \lambda \) outside \((0, 1)\), \( \mathcal{S}_{\lambda, 1} \) does not extend to a tempered distribution.

**Proof:** Using the analog of (2.56), the proof proceeds along the same lines as the proof of Theorem 2C.2.

**III. THE THRILLING CASE**

**A. Fields on \( \mathcal{F}_s (\mathcal{H}^+, \otimes \mathcal{H}^-) \)**

We define for any \( \lambda \in \mathbb{R} \) and \( s = \pm 1 \)

\[
\phi_{\lambda, s}(x) \Omega = \exp(K_{\alpha, s}^{\lambda} c_{\lambda, -1} c_{\lambda, -1}^* \Omega) A = B, F, \quad (3.1)
\]

where

\[
K_{\alpha, s}^{\lambda}(\theta) = \exp(\pm i(\lambda + 1) x), \quad A = B, F.
\]

Here, the function \( K_{\alpha, s}^{\lambda}(\theta) \) is defined by (2.5) and (2.60) for \( A = F \) and \( A = B \), resp., and

\[
p_{\lambda}(\theta) = e^{i\alpha, s(\theta)} .
\]

We also define the light rays

\[
R_{\alpha} = \{ (p^0, p^1) \in \mathbb{R}^2 | \alpha > 0 \} .
\]
Theorem 3A.1: Let $F \in S(\mathbb{R}^2)$. If $\bar{F}$ vanishes on $R_1$, one has
\[
\|\phi_{\lambda}^{\nu}(F)\Omega\| = 0, \quad A = F, B.
\] (3.5)
If $\bar{F}(\vec{p}_i) \neq 0$ for some $\vec{p}_i \in R_2$, one has
\[
\|\phi_{\lambda}^{\nu}(F)\Omega\| = \infty, \quad \forall A \Omega, \quad A = F, B.
\] (3.6)

Proof: The first statement is clear. To prove (3.6) for $A = F$, we first note that the squared $L^2$-norm of the two-body component of $\phi_{\lambda}^{\nu}(F)\Omega$ is proportional to
\[
\int d\theta_1 d\theta_2 \bar{F}(p_1(\theta_1) + p_2(\theta_2))^2
\]
\[
\times \exp[-2\lambda s(\theta_1 - \theta_2)] \sec^2y \left[ \|\theta_1 - \theta_2\| \right]^2
\]
\[
= \int dxdy f(2e^x \cosh y)e^{i\lambda x} \sec^2y
\]
\[
= \int dxdy f(e^{2x} - e^{-2x})e^{i\lambda x} \sec^2y,
\] (3.7)
where $f > 0$ and $f(\vec{p}_i) > 0$. Thus, if $|\lambda| > |\theta_1 - \theta_2|$, the $y$-integral diverges at $\infty$ for any $\lambda$ in a neighborhood of $\lambda = \ln \vec{p}_i$, so that the integral diverges by Fubini's theorem. We may therefore assume that $|\lambda| < |\theta_1 - \theta_2|$. In this case the integral (3.7) may be convergent, but we claim that the four-body component of the integral is not square integrable. To show this, we note its $L^2$-norm squared is proportional to
\[
\int d\theta_1 \cdots d\theta_4 f(e^{2x_1} + \cdots + e^{2x_4})
\]
\[
\times \|h_{\lambda}(\theta_1 - \theta_2)h_{\lambda}(\theta_1 - \theta_4) - h_{\lambda}(\theta_1 - \theta_2)h_{\lambda}(\theta_3 - \theta_4)\|^2
\]
where $h_{\lambda}$ is defined by (2.21). One obtains a lower bound to this integral if one replaces $f(x)$ by $g_\epsilon f(x)$, where $g_\epsilon$ is continuous, $0 < g_\epsilon < 1$, $g_\epsilon(x) = 1$ for $x > 2\epsilon$ and $g_\epsilon(x) = 0$ for $x < \epsilon$. Hence, to prove it diverges, we may as well assume that $f(x)$ vanishes for $x < \epsilon$ and that $p_0 > 2\epsilon$. We now write the integral as
\[
2 \int d\theta_1 \cdots d\theta_4 f(e^{2x_1} + \cdots + e^{2x_4}) (h_{\lambda}(\theta_1 - \theta_2)h_{\lambda}(\theta_3 - \theta_4) - h_{\lambda}(\theta_1 - \theta_2)h_{\lambda}(\theta_1 - \theta_4))
\]
\[
\times \exp(4\lambda x) \sec^2y_1 \sec^2y_2
\]
\[
- \int dy_1 dy_2 f(M_\lambda(y) e^{2x}) h_{\lambda}(\theta_1 - \theta_2)h_{\lambda}(\theta_1 - \theta_4)
\]
\[
\times \int dy_1 dy_2 h_{\lambda}(\theta_1 - \theta_2)h_{\lambda}(\theta_1 - \theta_4)
\]
\[
\equiv I_1 - I_2,
\] (3.9)
where we have changed to the center of mass variables (2.22) in $I_1$. If $x_1, x_2$ in $I_1$ are chosen such that $2e^{x_1} \cosh y_1$ is in a neighborhood of $p_0$, the $x_1$-integration diverges at $-\infty$ for any $y_1$. Hence, $I_1 = \infty$ by Fubini's theorem. Thus, it suffices to show $I_2$ is convergent. But by assumption $\sup_{F} \subset [e, \infty)$, so that
\[
I_2 \leq C_1 \int dy_1 (y_1)^3 \int_0^\infty \frac{dx}{x} f(x)
\]
\[
< \infty,
\] (3.10)

since $|\lambda| < 1$ by assumption. This proves the theorem for $A = F$. The proof for $A = B$ can be simplified by noting that no minus sign occurs in the analog of (3.8), so that its divergence for any $\lambda$ is a direct consequence of the divergence of the analog of $I_1$. \(\square\)

B. Fields on $\mathcal{F}_s(\mathbb{R}^2 \otimes \mathbb{R}^2)$, $\epsilon = a, s$

We define for any $\lambda \in \mathbb{R}$
\[
\psi_{\lambda,a}(x)\Omega
\]
\[
= \frac{i}{4\pi} \int_\mathbb{R} e^{i(x-p)(\theta)} \phi_{\lambda,a}^{\nu}(x)\Omega,
\] (3.11)
which holds on $\mathcal{F}_s$ and
\[
\phi_{\lambda,a}^{\nu}(x)\Omega = \frac{i}{4\pi} \int_\mathbb{R} e^{i(x-p)(\theta)} \phi_{\lambda,a}^{\nu}(x)\Omega
\]
(3.12)
which holds on $\mathcal{F}_s$. Here, $\phi_{\lambda,a}^{\nu}(x)\Omega, A = F, B$, is given by (3.1). We also define the light cone
\[
V_+ = \{(p^0, p) \in \mathbb{R}^3 | p^0 > \|p\| \}
\] (3.13)
and denote the interior of $V_+$ by $V_+^0$.

Theorem 3B.1: Let $F \in S(\mathbb{R}^4)$. If $\bar{F}$ vanishes on $V_+$, one has
\[
\|\psi_{\lambda,a}^{\nu}(F)\Omega\| = \|\phi_{\lambda,a}^{\nu}(F)\Omega\| = 0.
\] (3.14)
If $\bar{F}(\vec{p}) \neq 0$ for some $\vec{p} \in V_+^0$, one has
\[
\|\psi_{\lambda,a}^{\nu}(F)\Omega\| = \|\phi_{\lambda,a}^{\nu}(F)\Omega\| = \infty, \quad \forall A \Omega.
\] (3.15)

Remark 3B.2: For $\lambda \in [0,1]$ and $s = 1$ the fields considered here and in the preceding section coincide with the fields of Ref. 1. Analytic continuation in $\lambda$ and the analogous definition for $s = -1$ lead to the above fields, which are, however, slightly different from the fields obtained from a consideration of the underlying Bogoliubov transformations. We introduced these fields to ease the notation and because it is natural to consider the analytically continued fields (as we also did in Sec. II). The fields of Ref. 1 and the fields obtained by omitting the kernel $e^{i\theta}$ lead to the same results, as is easily verified.

Proof: We argue in a similar way as in the proof of Theorem 3A.1. The $L^2$-norm squared of the three-body component of $\psi_{\lambda,a}^{\nu}(F)\Omega$ is proportional to
\[
\int d\theta d\phi G(e^{i\theta} - e^{-i\phi}) + e^{i\theta} + e^{-i\phi}
\]
\[
+ e^{i \theta} - e^{-i \phi} \|e^{i\theta} + 4\lambda \| \sec^2\phi
\] (3.16)
where $G > 0$ and $G(\vec{p}^0_1, -\vec{p}^0_2) > 0$. Hence, for $|\lambda| > |\theta_1 - \theta_2|$ the $y$-integral diverges at $\infty$ if one chooses $\theta$ and $\phi$ such that $e^{i\theta} + e^{i\phi} - e^{i\theta} - e^{i\phi}$ is in a neighborhood of $(-\vec{p}^0_1, -\vec{p}^0_2)$, implying divergence of (3.16). For $|\lambda| < |\theta_1 - \theta_2|$, the $L^2$-norm squared of the five-body component is bounded below by
\[
\int d\theta d\phi e^{i\theta} \left| \bar{F}(p_0(\theta_0) + \sum_{i=1}^2 p_{-i}(\theta_0)) \right|^2 |h_{\lambda}(\theta_1 - \theta_2)h_{\lambda}(\theta_1 - \theta_4)
\]
\[
\times |h_{\lambda}(\theta_1 - \theta_4) - h_{\lambda}(\theta_1 - \theta_2)h_{\lambda}(\theta_3 - \theta_4)|^2,
\] (3.17)
where \( f^\theta \) stands for the integral over the set of \( \theta \) for which the argument of \( \tilde{F} \) belongs to a closed ball \( B \subset V^0_+ \) around \( \tilde{p} \). As this implies \( \theta_i \) ranges over a bounded interval, divergence of (3.17) follows in the same way as the divergence of (3.8), proving the theorem in the fermion case. Again, the proof is somewhat shorter in the boson case, since no minus sign occurs in the analog of (3.17), which immediately results in its divergence for any \( \lambda \in \mathbb{R} \). □

IV. THE ISING CASE

A. Fields on \( \mathcal{F}_s(\mathcal{K}^0) \)

We define

\[
\phi_F^0(x)\Omega = \exp\{K_F^0 e^0\Omega\} \Omega ,
\]

(4.1)

where

\[
K_F^0(\theta_1, \theta_2) = \exp\{\text{det} \{ p(\theta_1) + p(\theta_2) \} \} K_F^0(\theta_1 - \theta_2) .
\]

(4.2)

Here, the first factor on the rhs is the same as in (2.2), and

\[
K_F^0(\theta) = \frac{i(2\pi)}{\sinh_2 \theta} .
\]

(4.3)

We also define

\[
\phi_F^+ (x)\Omega = \exp\{x \cdot \text{det} \{ p(\theta_1) + p(\theta_2) \} \} K_F^0(\theta_1 - \theta_2) \Omega .
\]

(4.4)

Theorem 4A.1: For any \( t > 0 \) the Schwinger function

\[
S_F^r (t) = \| \phi_F^r (jt, 0)\Omega \|^2
\]

is finite and satisfies

\[
S_F^r (t) = (\text{det} \{ 1 + A^r (t) A^r (t) \})^{1/2} ,
\]

(4.6)

where \( A^r (t) \) is the integral operator on \( L^2(\mathbb{R}) \) with kernel

\[
A^r (t, \theta_1, \theta_2) = (1/2\pi) \exp\{ - it(\cos \theta_1 + \cos \theta_2)\}
\]

\[
\times \tanh\{i(\theta_1 - \theta_2)\} .
\]

(4.7)

Moreover,

\[
S_F^r (t) = 1 + O(e^{-2t}) , \quad t \to \infty ,
\]

(4.8)

and

\[
\lim_{t \to \infty} \frac{\ln \| S_F^r (t) \|^2}{\ln(1/t)} = \frac{1}{4} .
\]

(4.9)

Finally, \( S_F^r (t) \) admits a spectral representation

\[
S_F^r (t) = \sum_{j=1}^\infty d\tilde{p}_j^r (m) K_0 (mt) ,
\]

(4.10)

where the measure satisfies

\[
\int_2^\infty \frac{d\tilde{p}_j^r (m)}{m} = \left\{ \begin{array}{ll} 0 & \text{if} \quad \gamma > \frac{1}{2}, \\
\infty & \text{if} \quad \gamma < \frac{1}{2}. \end{array} \right.
\]

(4.11)

Theorem 4A.2: For any \( t > 0 \) the Schwinger function

\[
S_F (t) = \| \phi_F^r (jt, 0)\Omega \|^2
\]

(4.12)

satisfies the bound

\[
S_F^r (t) < (1/2\pi) K_0 (t) S_F^r (t) .
\]

(4.13)

For \( t \) sufficiently large it admits a representation

\[
S_F^r (t) = \frac{1}{2} G (t) S_F^r (t) ,
\]

(4.14)

where

\[
G (t) = 2 \sum_{n=-\infty}^{\infty} \frac{1}{(2\pi)^{2n+1}} \int d\theta_1 \cdots d\theta_{2n+1} \exp\{ - t \left( \sum_{n=0}^{2n} \cosh \theta_n \right) \} \prod_{n=1}^{2n} \tanh \{ j(\theta_{n+1} - \theta_n) \} .
\]

(4.15)

Moreover,

\[
S_F^r (t) = S_{KG} (t) + O(e^{-2t}) , \quad t \to \infty ,
\]

(4.16)

and

\[
\lim_{t \to \infty} \frac{\ln \| S_F^r (t) \|^2}{\ln(1/t)} = \frac{1}{4} .
\]

(4.17)

Finally, (4.10) and (4.11) hold true with \(-\) replaced by \(+\).

Theorem 4A.3: The Wightman functions

\[
W = \| \phi_F^r (t)\Omega \|^2 , \quad \tilde{F}_0 \in C^0_\infty (\mathbb{R}^2) ,
\]

(4.18)

extend to tempered distributions that admit tempered time-zero restrictions.

Proof of Theorem 4A.1: Since \( \| A^r (t) \| < \infty \) for any \( t > 0 \), and, moreover, \( A^r (t, \theta_2, \theta_1) = - A^r (t, \theta_1, \theta_2) \), (4.6) follows from Sec. 4 in Ref. 8. The work of McCoy et al.\(^4\) implies (4.9), and the remaining statements then follow as in the proof of Theorem 2A.1. □

Proof of Theorem 4A.2: To prove (4.13), we observe that

\[
S_F^r (t) = \| e^{it \theta_1} \Omega \|^2 ,
\]

(4.19)

where

\[
f_t (\theta) = (4\pi)^{-1/2} \exp\{ - i\theta \cos \theta \} .
\]

(4.20)

Since \( \| e^{it \theta_1} \| = \| g \| \) for \( g \in L^2 (\mathbb{R}) \), (4.13) follows. Next, we note that if \( A \) is a Hilbert–Schmidt operator for which \( A(\theta_2, \theta_1) = - A(\theta_1, \theta_2) \), then we may write, using the CAR,

\[
\| e^{it \theta_1} \| \exp\{i\theta^2 / 2A(\theta_1) \Omega \|^2
\]

\[
= \sum_{n=-\infty}^{\infty} \| \frac{(e^{it} \theta_1)^{n+1}}{n+1} \|^2 \exp\{i\theta^2 / 2A(\theta_1) \Omega \|^2
\]

\[
= \sum_{n=-\infty}^{\infty} \| \frac{(e^{it} \theta_1)^{n+1}}{n+1} \|^2 \exp\{i\theta^2 / 2A(\theta_1) \Omega \|^2
\]

\[
= \sum_{n=-\infty}^{\infty} \| \frac{(e^{it} \theta_1)^{n+1}}{n+1} \|^2 \exp\{i\theta^2 / 2A(\theta_1) \Omega \|^2
\]

\[
= \sum_{n=-\infty}^{\infty} \| \frac{(e^{it} \theta_1)^{n+1}}{n+1} \|^2 \exp\{i\theta^2 / 2A(\theta_1) \Omega \|^2
\]

(4.21)

where the last step holds true if in addition \( \| A^{-1} \| = 0 \) for \( N \to \infty \). Consequently, (4.14) follows from the fact that

\[
\| A^r (t) \| < \infty \quad t \to \infty .
\]

(4.17)

Finally, (4.17) follows from Ref. 4, and the remaining assertions then follow as before. □

Proof of Theorem 4A.3: The proof is similar to that of the analogous claims in Theorems 2A.2 and 2D.2 and will therefore be omitted. □

B. Fields on \( \mathcal{F}_s(\mathcal{K}^0) \)

We define

\[
\phi_F^B (x)\Omega = \exp\{ K_F^B e^0 \Omega \},
\]

(4.22)

where \( K_F^B \) is given by the rhs of (4.2) with \( F \to \mathbb{B} \), and

\[
K_F^B (\theta) = (1/2\pi) \sec^2 \frac{\theta}{4} .
\]

(4.23)

We also define

\[
\Psi_F^B (x)\Omega = (4\pi)^{-1/2} \int d\theta e^{i\theta} \left( - e^{-\theta^2 / 2} \right) \phi_F^B (x)\Omega ,
\]

(4.24)
Theorem 4B.1: For any \( t > 0 \) the Schwinger function
\[
S^B(t) = \| \Phi^B(t) \|^2
\]
is finite-valued and satisfies
\[
S^B(t) = \left| \text{det} \left( 1 - A^B(t)^* A^B(t) \right) \right|^{-1/2},
\]
where \( A^B(t) \) is the integral operator on \( L^2(\mathbb{R}) \) with kernel
\[
A^B(t, \theta_1, \theta_2) = (1/2\pi) \exp \left\{ - \frac{1}{2} t (\cosh \theta_1 + \cosh \theta_2) \right\} \times \text{sech} \left( \frac{1}{2} \theta_1 - \theta_2 \right).
\]
Moreover,
\[
S^B(t) = 1 + O(e^{-2t}), \quad t \to \infty,
\]
and
\[
\lim_{t \to \infty} \frac{\ln \|S^B(t)\|}{\ln(1/t)} = \frac{1}{4}.
\]
Finally, \( S^B(t) \) admits a spectral representation
\[
S^B(t) = 1 + \int_{2}^{\infty} \frac{d\rho_B(m)}{m^2} K_0(mt),
\]
where the measure satisfies
\[
\int_{2}^{\infty} \frac{d\rho_B(m)}{m^2} \left\{ \begin{array}{ll}
< \infty, & \forall \gamma > \frac{1}{2} \\
= \infty, & \forall \gamma < \frac{1}{2}.
\end{array} \right.
\]

Theorem 4B.2: For any \( t > 0 \) the Schwinger function
\[
\mathcal{S}^B(t) = \| \Phi^B(t) \|^2
\]
is finite-valued and satisfies
\[
\mathcal{S}^B(t) = \| G \mathcal{S} \mathcal{S}^B(t) \|^2,
\]
where
\[
G_{\mathcal{S}}(t) = \delta \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^\infty d\theta_1 \cdots d\theta_{2n+1} e^{\theta_1 \cdots \theta_{2n+1}} \times \exp \left\{ - t \sum_{j=1}^{2n+1} \cosh \theta_j \right\} \prod_{j=1}^{2n+1} \text{sech} \left( \frac{1}{2} \theta_j + \theta_j \right).
\]
Moreover, \( \mathcal{S}^B(t) \) satisfies
\[
\mathcal{S}^B(t) = S^B(t) + O(e^{-2t}), \quad t \to \infty,
\]
its diagonal elements satisfy
\[
\lim_{t \to \infty} \frac{\ln \|\mathcal{S}^B(t)\|}{\ln(1/t)} = \frac{5}{4},
\]
and its off-diagonal elements satisfy
\[
\int_0^\infty dt \mathcal{S}^B_{\alpha, \beta}(t) \left\{ \begin{array}{ll}
< \infty, & \forall \alpha > \frac{1}{2} \\
= \infty, & \forall \alpha < \frac{1}{2}.
\end{array} \right.
\]
Finally, \( \mathcal{S}^B(t) \) admits a spectral representation
\[
\mathcal{S}^B(t) = \frac{K_0(t)}{2\pi} \left( K_0(t) - K_0(t) \right)
\]
\[
+ \int_{2}^{\infty} m \, d\rho_B(m) \left( K_0(mt) 0 \quad 0 \quad K_0(mt) \right)
\]
\[
\times \left( 0 \quad K_0(mt) \quad 0 \quad 0 \right).
\]
Here, the measures are positive and satisfy the inequalities (2.49) and (2.50) with \( m(1) = 1 \); furthermore,
\[
\int_{j}^{\infty} d\rho_B(m) m^{-\gamma + 2 - \frac{1}{2}} \left\{ \begin{array}{ll}
< \infty, & \forall \gamma > \frac{3}{2} \\
= \infty, & \forall \gamma < \frac{3}{2},
\end{array} \right. \quad j = 1, 2.
\]

Theorem 4B.3: The Wightman functions
\[
\mathcal{W}^B(F,F) = \| \Phi^B(F) \|^2,
\]
and
\[
\mathcal{W}^B(F,F) = \| \Phi^B(F) \|^2,
\]
extend to tempered distributions. \( \mathcal{W}^B \) admits a tempered time-zero restriction, while \( \mathcal{W}^B \) does not admit a time-zero restriction.

Proof of Theorem 4B.1: Since \( \| A^B(t) \| < \infty \), \( \| A^B(t) \| < 1 \), and \( \| A^B(t, \theta_1, \theta_2, \theta_3) \| = A^B(t, \theta_1, \theta_2) \) for any \( t > 0 \), (4.26) follows from Sec. 4 in Ref. 8.

But in view of (2.62) and (2.63) this implies
\[
S^B(t) = S^B(t, 1/2)\| t \|^{1/2}.
\]
Hence, (4.29) is a consequence of (2.65), and the other assertions can then be proven as before.
Combing this with (4.29), we obtain (4.36).

To prove (4.37), we write

\[
- G_- (t) = - \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \frac{1}{(2\pi)^{2n+1}} \partial_1 \int \ldots \partial_{2n+1} \sum_{j=1}^{\infty} \frac{1}{\sinh \theta_j} \times \exp \left( - \sum_{j=1}^{\infty} \cosh \theta_j \right) \prod_{j=1}^{\infty} \text{sech} \left( \frac{1}{2} \sum_{j=1}^{\infty} \left( \theta_j + \theta_{2n+1} \right) \right) \times \text{sech} \left( \frac{1}{2} \left( \theta_1 + \theta_{2n+1} \right) \right) + \frac{K(t)}{\pi}.
\]

(4.45)

\[
= \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \frac{1}{(2\pi)^{2n+1}} \int dy_1 \ldots dy_{2n+1} \tilde{M}_{2n+1} \times \exp \left( - i \tilde{M}_{2n+1} \right) \prod_{j=1}^{\infty} \text{sech} \left( \frac{1}{2} \sum_{j=1}^{\infty} y_j \right) \times \text{sech} \left( y_0 + f_n(y) \right) + \frac{K(t)}{\pi}.
\]

(4.46)

where

\[
f_n(y) = \ln \left( \frac{\tilde{M}_{2n+1}(y)}{\left[ 1 + \sum_{j=1}^{\infty} \exp \left( \sum_{j=1}^{\infty} y_j \right) \right]^{\frac{1}{2}}} \right) + \frac{1}{2} \sum_{j=1}^{\infty} y_j.
\]

(4.47)

Hence, for any \( \epsilon > 0 \) we have, using (2.26),

\[
\int_0^\infty dt \left( - G_- (t) \right) \leq \frac{\Gamma \left[ 1 + \frac{1}{2} \right]}{2\pi} \int dy_0 \text{sech} \left( \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{2n+1} \int dy_1 \ldots dy_{2n+1} \tilde{M}_{2n+1} \times \text{sech} \left( \frac{1}{2} \sum_{j=1}^{\infty} y_j \right) - \epsilon \right) \times C e \sum_{n=1}^{\infty} (2n+1)^{-1} \times C e < \infty.
\]

(4.48)

On the other hand, by virtue of (4.46),

\[
\int_0^\infty dt \left( - G_- (t) \right) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty.
\]

(4.49)

Combining this with (4.48) and (4.29), (4.37) follows. \( \square \)

**Proof of Theorem 4B.3.** The statements follow from Theorems 4B.1 and 4B.2 as in preceding proofs. \( \square \)

**Remark 4B.4.** It is of interest to point out that nonexistence of a time-zero restriction for the diagonal elements already follows from the fact that the three-particle component of \( \Psi_\phi (0, f) \) [where \( f \in S[R] \)] is not square integrable, which can be seen as in the proof of Theorem 2B.2. Note, however, that for the off-diagonal elements all \( (2n+1) \)-particle contributions are finite, since they are bounded above by a multiple of the integral

\[
I_n \equiv \int \ldots \int d\theta_1 \ldots d\theta_{2n+1} \left| \tilde{f} \left( \sum_{j=1}^{2n+1} \sinh \theta_j \right) \right|^2.
\]

(4.50)

To see that \( I_n \) is finite, one can either change variables \( \theta_j \to p_j = \sinh \theta_j \) and use Young’s and Hölder’s inequality, or change to the center of mass variables (2.22), from which one infers directly that

\[
I_n = \int dp | \tilde{f} (p) |^2 \int dy \tilde{M}_{2n+1} \times (y) | \sinh \theta_0 |^2 < \infty,
\]

(4.51)

where the last step follows by using (2.24) and, e.g., the arithmetic–geometric mean inequality. Thus, in this case information on the sum of the terms is essential to conclude no time-zero restriction exists.

**Remark 4B.5.** If one combines the results of Sato et al. [9] (who introduced the bosonic Ising model considered in this section) with Ref. 4, one can get more detailed information on the short-distance behavior of \( S^a \) and \( G_{ij} \), but this only follows after the introduction of considerable machinery. It would be of interest to reobtain such results (and their analogs in the fermion case) in a simpler and more direct way by using only the underlying operators \( A^a(t) \) and \( A^f(t) \).

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