Integrable Quantum Field Theories and Bogoliubov Transformations

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The Federbush, massless Thirring and continuum Ising models and related integrable relativistic quantum field theories are studied. It is shown that local and covariant classical field operators exist that generate Bogoliubov transformations of the annihilation and creation operators on the Fock spaces of the respective models. The quantum fields of these models are closely related or equal to quadratic forms implementing these transformations, and hence formally inherit the covariance and locality of the underlying classical field operators. It is proved that the Federbush and massless Thirring fields on the physical sector do not satisfy the equation of motion. Closely related fields are defined that do satisfy it, and which lead to the same S-matrix, but these fields are presumably non-local. Bethe transforms are constructed for the various models, and on the unphysical sector the relation with the field theory approach is established.

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1. Introduction

The main purpose of this paper is to present a unified treatment of a number of integrable one-dimensional relativistic quantum field theories, viz., the Federbush [1-3], massless Thirring [4-6] and continuum Ising [7-15] models, and field theories related to these in the sense that they lead to the same S-matrix. These models are the simplest examples of integrable field theories: the quantum fields can be found explicitly and the two-body S-matrix (which determines the complete S-matrix) is a rapidity-independent phase factor in the Thirring and Ising cases, and only depends on the sign of the rapidity difference in the Federbush case.

As we shall show in this paper, these theories are all associated with certain Bogoliubov transformations of the annihilation and creation operators on the Fock spaces that describe the particles of the respective models. The fact that Bogoliubov transformations play a role in these models is implicit in earlier discussions, and explicit in the work of Onsager [16] and Schultz et al. [17] on the lattice Ising model, and the work of Berezin [6] on the massless Thirring model. Our point of view has been in particular inspired by the recent work of Sato et al. [13-15] on the continuum Ising model, and by our previous work on Bogoliubov transformations [18, 19].

It seems, however, not to have been realized before that the quantum fields of these models can all be described in terms of quadratic forms that implement Bogoliubov transformations generated by local and covariant classical field operators, the locality and covariance of the quantum fields being a consequence of this. We show that such classical field operators are associated with the (one-dimensional) massive Dirac and Klein-Gordon equations, and lead to the Federbush and Ising quantum field operators and bosonic analogs thereof, and also with the massless Dirac and Klein-Gordon equations, where they lead to the quantum fields of the massless Thirring model and its boson analog. We explain in detail how the covariance and locality of the classical field operators is inherited by the quantum fields, and discuss the scattering content of the various field theories.

Subsequently, we investigate whether the fermionic Federbush and Thirring fields satisfy the equation of motion derived from the respective Lagrangeans (it seems the bosonic versions and the continuum Ising model do not correspond to a Lagrangean). It turns out that on the physical sector the Federbush and Thirring fields do not satisfy the equation of motion, provided one accepts our definition of field products. However, they fail to do so by a narrow margin, and correspondingly one can easily find other fields that do satisfy the equation of motion as defined by us. The latter fields are covariant and lead to the same S-matrix, but are most likely non-local. The paper is concluded with an account of an approach to these models that deals only with the dynamics and scattering content.
We shall now outline the plan of the paper in more detail. Chapter 2 begins with a section that describes the common features of the classical field operators introduced in the remainder of the chapter, and summarizes some material from Refs. [18, 19]. In Sections 2B–2E we then consider the massive and massless Dirac and Klein–Gordon cases separately. In Chapter 3 corresponding quantum fields are introduced (without reference to specific Lagrangeans) and their locality and scattering properties are considered. Section 3A explains the main ideas that are relevant in this connection, while Sections 3B and 3C contain an explicit study of the various field theories on the unphysical and physical sectors, respectively. These two chapters are more or less self-contained and may be skipped on first reading by readers who are mainly interested in the Federbush, massless Thirring and continuum Ising models, which are the subject of Chapters 4, 5 and 6, respectively.

After some preliminaries we first study the Federbush model on the unphysical sector in Section 4A. The relation with the relevant field of Section 3B is made clear, and the equation of motion is studied. We show that it can be satisfied on this sector provided one changes the coupling constant dependence in the obvious quantum analog of the classical solution. The positive energy Federbush model is the subject of Section 4B. We begin by comparing our fields with those of other authors, and then study the equation of motion, which leads to the conclusion we have already mentioned above. Finally, we consider the question whether the Hamiltonian can be expressed in terms of the nonperturbative field operators in the usual fashion of Lagrangean field theory, and arrive at a negative answer.

In Chapter 5 we treat the massless Thirring model along the same lines as the Federbush model, and reach essentially the same conclusions. Again, the unphysical and physical sectors are studied separately, in Sections 5A and 5B resp. We consider the question whether the Federbush model reduces to two decoupled massless Thirring models in the mass-zero limit, and obtain a negative answer. We also consider the analogous question for the *massive* Thirring model on the physical sector, which leads to the same negative result. Section 5B ends with a comparison of the "Glaser approach" [5] we have followed, and the more widely known approach exemplified by the work of Klaiber [20).

The continuum Ising model and its boson version [14, 15] are studied in Chapter 6. Section 6A summarizes some general features, and contains an explanation of the relation between the usual Ising model and its doubled version [8–11] in terms of Bogoliubov transformations on "neutral" and "charged" Fock spaces, as discussed in Refs. [18, 19]. In Section 6B we explicitly define various quantum fields connected with the Ising model, and in Section 6C we study their locality and scattering properties. We explain in particular how asymptotic free boson/fermion fields can arise in a fermion/boson Fock space, a phenomenon that has been signaled before in Refs. [13, 14], but that has been insufficiently clarified.

The paper concludes with Chapter 7, which deals with a Hamiltonian approach to the field theories under consideration. After an introduction summarizing the main features, we discuss the unphysical sector in Section 7A and the physical sector in Section 7B. We show that on the unphysical sector the two approaches are unitarily
equivalent, which sheds considerable light on several questions discussed in earlier chapters. Section 7A is concluded with remarks on the connection with Thirring's work [4], the massive Thirring model on the unphysical sector, coupling constant dependence, and the unphysical chiral Gross-Neveu model. Chapter 7 ends with Section 7B, where we simplify and extend previous results on Bethe transforms for the Federbush and Ising models [21].

This paper is mathematically rigorous, except where explicitly indicated otherwise. However, we have only occasionally made use of notions from functional analysis that may not be familiar to the less mathematically inclined physicist, and we have tried to isolate and signpost these places, so that they might be skipped by the reader who does not want to familiarize himself with these notions (by spending some time on Ref. [22], for example). We should add, however, that quite a few points made in this paper, in particular as regards obscurities, errors, and conflicting viewpoints in the literature, hinge on a precise definition of the various operators and limiting relations involved.

Finally, we should like to mention that a number of results of a purely technical nature will be quoted without proof in the remainder of the paper. These results concern scattering properties and properties of the two-point functions of the various quantum fields introduced below. Their proofs involve rather long estimates, and will therefore be relegated to other papers, viz., Ref. [23] and Ref. [24], respectively.

2. CLASSICAL FIELD OPERATORS

2A. Preliminaries

In all four cases below the classical field operators act on a one-body space $\mathcal{H} = L^2(R, d\theta)^2$, which is the direct sum $\mathcal{H}_+ \oplus \mathcal{H}_-$ of spaces $\mathcal{H}_+, \mathcal{H}_- = L^2(R, d\theta)$ with corresponding projections $P_+, P_-$ resp. Thus, any $f \in \mathcal{H}$ can be written $f = (f_+(\theta), f_-(\theta))$. The space $\mathcal{H}$ carries a representation $U(a, \Lambda)$ of the Poincaré group given by

$$ (U(a, \Lambda)f)_\delta(\theta) = \exp(ia \cdot \delta p(\theta))f_\delta(\theta - \alpha), \quad \delta = +, -, \quad (2.1) $$

where $\alpha$ is the rapidity corresponding to $\Lambda$, i.e.,

$$ \Lambda = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix}. \quad (2.2) $$

In the massive case (Sections 2B, C) one has

$$ p(\theta) = m(\cosh \theta, \sinh \theta), \quad (2.3) $$

while in the massless case (Sections 2D and E)

$$ p(\theta) = (e^{is\theta}, se^{is\theta}), \quad (2.4) $$

where $s = 1(-1)$ for right- (left-) moving particles. Thus, $\mathcal{H}_+(\mathcal{H}_-)$ describes a positive
(negative) energy relativistic particle in terms of a square-integrable wave function depending on its rapidity. In all four cases the charge conjugation operator $C$ is defined by

$$(Cf)_0(\theta) = \tilde{f}_- (\theta),$$

where the bar denotes complex conjugation. Note that

$$[C, U(a, \Lambda)]_+ = 0.$$  \hfill (2.6)

The classical field operator is defined by

$$\phi_{el}(x) = U(x, 1)U_U(x, 1)^*,$$  \hfill (2.7)

where $U$ is a unitary operator in the Dirac case, a pseudo-unitary operator in the Klein–Gordon case, i.e.,

$$U_UqU^* = U^*qU = q, \quad q \equiv p_+ - p_-.$$  \hfill (2.8)

The operator $U$ has two properties that imply covariance and locality of $\phi_{el}(x)$, i.e.,

$$U(a, \Lambda) \phi_{el}(x)U(a, \Lambda)^* = \phi_{el}(\Lambda x + a)$$  \hfill (2.9)

and

$$[\phi_{el}(x), \phi_{el}(y)]_+ = 0, \quad (x - y)^2 < 0.$$  \hfill (2.10)

These two properties are the “covariance relation”

$$[U, U((0, a^1), 1) U((0, a^1), 1)^*]_+ = 0, \quad \forall a^1 \in R.$$  \hfill (2.11)

and the “locality relation”

$$[U, U((0, \Lambda_0), 1) U((0, \Lambda_0), 1)^*]_+ = 0, \quad \forall \Lambda_0 \in \mathbb{R}.$$  \hfill (2.12)

To see that (2.11) and (2.12) imply (2.9) and (2.10), first note that by (2.11) and (2.7)

$$U(a, \Lambda) \phi_{el}(x)U(a, \Lambda)^* = U(a, \Lambda)U(x, 1) \phi_{el}(0)U(x, 1)^*U(a, \Lambda)^* = U(a + \Lambda x, 1)U(0, \Lambda)U(0, \Lambda)^*U(a + \Lambda x, 1)^* = \phi_{el}(\Lambda x + a),$$

which is (2.9), and then observe that for a space-like vector $y - x$ one can find $\Lambda_0$ such that $\Lambda_0(y - x) = (0, a^1)$. Hence by (2.7), (2.9) and (2.12)

$$[\phi_{el}(x), \phi_{el}(y)]_+ = U(x, 1)[\phi_{el}(0), \phi_{el}(y - x)]_+ U(x, 1)^*$$

$$= U(x, 1)U(0, \Lambda_0^{-1})[\phi_{el}(0), \phi_{el}(0, a^1)]_+ U(0, \Lambda_0^{-1})^* U(x, 1)^*$$

$$= 0,$$
proving (2.10). Note also that (2.9) and (2.10) trivially imply (2.11) and (2.12), so that these relations are equivalent.

In the next chapter we shall use these classical field operators to generate Bogoliubov transformations on the anti-symmetric and symmetric Fock spaces $F_a(\mathcal{H})$ and $F_s(\mathcal{H})$ over $\mathcal{H}$ in the Dirac case and Klein–Gordon case, respectively. (If $U$ commutes with $C$ one can also consider the neutral particle Fock spaces $F_a(\mathcal{H}_a)$ and $F_s(\mathcal{H}_s)$, cf. Section 6A.) These transformations are not unitarily implementable, but one can introduce quantum fields $\phi(x)$ on the various Fock spaces that implement the transformation in the sense of quadratic forms, and therefore inherit the covariance and locality properties of $\phi(x)$, as we shall show in the next chapter. We shall have occasion to make extensive use of the lore on the unitarily implementable transformations. In particular, provided that $P_+ U P_+$ and $P_- U P_-$ have inverses (as operators on $\mathcal{H}_+, \mathcal{H}_+$ resp.), the implementing operator may be concisely expressed in terms of an operator $\Lambda$ that we have termed the associate of $U$ (cf. Refs. [18, 19]). However, instead of $\Lambda$ one may also use an operator $Z$, defined by

$$Z = 1 + q\Lambda,$$

if $U$ is unitary, or

$$Z = 1 + \Lambda,$$

if $U$ is pseudo-unitary. This choice is more natural at the classical level, since $Z$ is pseudo-unitary/unitary if and only if $U$ is unitary/pseudo-unitary, as may be checked from the relations defining $Z$:

$$Z_{++} = U_{++} - U_{+-} U_{--}^{-1} U_{-+}, \quad Z_{+-} = U_{+-} U_{--}^{-1},$$

$$Z_{-+} = -U_{++}^{-1} U_{+-}, \quad Z_{-} = U_{--}^{-1}$$

(cf. Eq. (7.11) in Ref. [19]). Here, we have used the notation

$$A_{\delta \delta'} = P_{\delta} A P_{\delta'}, \quad \delta, \delta' = +, -$$

for an operator $A$ on $\mathcal{H}$. We shall refer to $Z$ as the conjugate of $U$. Note that relations (2.15) imply that the "double conjugate" of $U$ is $U$, i.e., $U$ is the conjugate of $Z$.

In the remainder of this chapter we shall explicitly define operators $U$ satisfying (2.11) and (2.12) and then we shall determine the conjugate $Z$ of $U$. The conjugate is found in the following way: Under a Fourier transformation $F: \mathcal{H} \rightarrow \mathcal{H} \equiv L^2(R, dy)^2$, defined by

$$(Ff)_{\delta}(y) = \frac{1}{\sqrt{2\pi}} \int d\theta \exp(i\theta y) f_\delta(\theta),$$

one clearly has $\tilde{U}(A) \equiv FU(A) F^{-1} = \exp(i\alpha y)$, so that $\tilde{U} \equiv FUF^{-1}$ is multiplication
by a $2 \times 2$ matrix function $\tilde{u}_{\theta \theta}(y)$ in view of (2.11). Equivalently, $U$ is a $2 \times 2$ matrix convolution operator, whose kernels satisfy

$$u_{\theta \theta}(\theta) = \frac{1}{2\pi} \int dy \exp(-i\theta y) \tilde{u}_{\theta \theta}(y).$$

(2.18)

The kernels $u_{\theta \theta}(\theta)$ occurring in the following are expressed in hyperbolic functions, and one can explicitly find the corresponding $\tilde{u}_{\theta \theta}(y)$. But then it is trivial to find $\tilde{Z} = FZF^{-1}$ from (2.15). Again, it is possible to explicitly perform the Fourier transformations leading from the matrix elements $\tilde{Z}_{\theta \theta}(y)$ to the convolution kernels $z_{\theta \theta}(\theta)$, so that $Z$ can be explicitly found.

We shall close this section by listing the integrals needed for the procedure just sketched. It turns out that one only needs two integrals, viz.,

$$\frac{1}{2\pi} \int d\theta \exp(i\theta y) \frac{\exp(\lambda \theta)}{\cosh \theta/2} = \frac{1}{\cosh \pi(y - i\lambda)}, \quad |\lambda| < \frac{1}{2},$$

(2.19)

and

$$\frac{1}{2\pi} \int d\theta \exp(i\theta y) \frac{\exp(\lambda \theta)}{\sinh \theta/2} = i \tanh \pi(y - i\lambda), \quad |\lambda| < \frac{1}{2}.$$  

(2.20)

Both integrals can be easily verified by a contour integration.

2B. The Massive Dirac Case

The one-dimensional Dirac operator is the ordinary differential operator

$$\tilde{H}_0 = -i\gamma^5 \frac{d}{dx^1} + \gamma^0 m, \quad m > 0,$$

(2.21)

on $\mathcal{H} = L^2(dx^1)^2$, whose domain consists of absolutely continuous $f \in \mathcal{H}$ with square-integrable derivatives. (Throughout this paper we use

$$\gamma^\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(2.22)

as representation of the $\gamma$-algebra, and we use $x^1$ to avoid confusion with the 2-vector $x = (x^0, x^1)$.) We diagonalize $\tilde{H}_0$ on the Hilbert space $\mathcal{H} = L^2(d\theta)^2$ by means of the unitary operator $W: \mathcal{H} \to \mathcal{H}$, defined by

$$(Wg)(x^1) = (2\pi)^{-1/2} \sum_{\delta = \pm} \int d\theta \exp(i \delta mx^1 \sinh \theta) w_{\delta}(\theta) g_{\delta}(\theta),$$

(2.23)

with inverse

$$(W^{-1}f)_{\delta}(\theta) = (2\pi)^{-1/2} \int dx^1 \exp(-i \delta mx^1 \sinh \theta) \bar{w}_{\delta}(\theta) \cdot f(x^1).$$

(2.24)
Here, the \( w_\delta(\theta) \) are the usual positive and negative energy spinors
\[
 w_+(\theta) = (m/2)^{1/2}(e^{i\theta/2}, e^{-i\theta/2}), \quad w_-(\theta) = i(m/2)^{1/2}(e^{i\theta/2}, \quad e^{-i\theta/2}),
\] (2.25)
which are related by
\[
 w_\delta(\theta) = \hat{C} w_{-\delta}(\theta),
\] (2.26)
where \( \hat{C} \) is the charge conjugation operator,
\[
(\hat{C}f)(x^1) = i\gamma_4 f(x^1).
\] (2.27)
Using the convention \( A = W^{-1} \tilde{A} W \) if \( \tilde{A} \) is an operator on \( \mathcal{H} \) we now get
\[
 (H_{0,f})_\delta(\theta) = \delta m \cosh \theta f_\delta(\theta),
\] (2.28)
\[
 (Cf)_\delta(\theta) = f_{-\delta}(\theta).
\] (2.29)
More generally, the representation of the Poincaré group on \( \mathcal{H} \) is given by (2.1), i.e.,
\[
 (U(a,1)f)_\delta(\theta) = \exp(i \delta m(a^0 \cosh \theta - a^1 \sinh \theta)) f_\delta(\theta - \infty).
\] (2.30)
Consider now multiplication by \( \epsilon(x^1) \) on \( \mathcal{H} \), where \( \epsilon(x^1) \) equals 1 (\(-1\), if \( x^1 > 0 \) \( x^1 < 0 \). This is clearly a unitary operator (denoted henceforth by \( \hat{U}_F \)), and a calculation using (2.23) and (2.24) shows that \( \hat{U}_F \) acts as convolution with
\[
 \frac{1}{2\pi i} \begin{pmatrix}
 P \\
 \frac{\sinh \theta/2}{\cosh \theta/2} \\
 \frac{i}{\cosh \theta/2} \\
 \frac{1}{\sinh \theta/2}
\end{pmatrix},
\] (2.31)
where \( P \) denotes the principal value. As a result, \( \hat{U}_F \) satisfies the covariance relation (2.11). However, it is clear that \( \hat{U}_F \) also satisfies the locality relation (2.12), since \( \hat{U}(0, a^1), 1) \hat{U}_F \hat{U}(0, a^1), 1) \ast \) acts as multiplication by \( \epsilon(x^1 - a^1) \), which commutes with multiplication by \( \epsilon(x^1) \). Hence, the field operator
\[
 \phi^F_{ei}(x) = U(x, 1) \hat{U}_F U(x, 1) \ast
\] (2.32)
is covariant and local. This field operator corresponds to the Ising model in the scaling limit (cf. Chapter 6).

To find the conjugate of \( \hat{U}_F \), which we shall need in Chapter 6, we proceed as sketched in Section 2A: From Eqs. (2.19)-(2.20) we get that
\[
 \hat{U}_F = \begin{pmatrix}
 \frac{\tanh \pi y}{\cosh \pi y} & \frac{1}{\cosh \pi y} \\
 \frac{1}{\cosh \pi y} & -\tanh \pi y
\end{pmatrix},
\] (2.33)
so that from (2.15) we have

\[ Z_F = \begin{pmatrix} \coth \pi y & -\frac{1}{\sinh \pi y} \\ \frac{1}{\sinh \pi y} & -\coth \pi y \end{pmatrix}. \]  

(2.34)

Clearly, \( Z_F \) is an unbounded pseudo-unitary operator whose domain includes in particular all Schwartz space test functions satisfying \( g_+(0) = g_-(0) \). One can extend the matrix elements \( \hat{z}_F^e(\gamma) \) to tempered distributions by taking the principal value at the origin. Now from (2.19) and (2.20) one sees that

\[ \frac{1}{2\pi i} \int d\theta \exp(i\theta y) \tanh \frac{\theta}{2} = \frac{P}{\sinh \pi y}, \]  

(2.35)

and that

\[ \frac{1}{2\pi i} \int d\theta \exp(i\theta y) \coth \frac{\theta}{2} = \frac{1}{2\pi i} \int d\theta \exp(i\theta y) \tanh \frac{\theta}{4} \]

\[ + \frac{1}{2\pi i} \int d\theta \exp(i\theta y) \frac{1}{\sinh \frac{\theta}{2}} = P \coth \pi y, \]  

(2.36)

where the integrals stand for Fourier transforms of tempered distributions. As a result, \( Z_F \) acts as convolution with

\[ \frac{1}{2\pi i} \begin{pmatrix} P \coth \frac{\theta}{2} & -\tanh \frac{\theta}{2} \\ \tanh \frac{\theta}{2} & -P \coth \frac{\theta}{2} \end{pmatrix} \]  

(2.37)

on test functions \( f = (f_+(t), f_-(t)) \in S(\mathbb{R})^2 \) satisfying \( \int d\theta f_+(\theta) = \int d\theta f_-(\theta) \).

The classical field operator associated with the Federbush model is defined by

\[ \phi^F_{\pi, \text{el}}(x) = U(x, 1) U^F_{\pi}(x, 1)^*, \quad \lambda \in \mathbb{R}, \]  

(2.38)

where

\[ U^F_{\pi} = \cos \pi \lambda + iU^F \sin \pi \lambda, \quad \lambda \in \mathbb{R}. \]  

(2.39)

Hence, \( \hat{U}^F_{\pi} \) is multiplication by \( \exp(i\pi \lambda \hat{x}) \). It follows as before that \( \phi^F_{\pi, \text{el}}(x) \) is covariant and local. From (2.31) it follows that \( U^F_{\pi} \) is convolution with

\[ \cos \pi \lambda \delta(\theta) + \frac{\sin \pi \lambda}{2\pi} \begin{pmatrix} \frac{P}{\sinh \theta/2} & \frac{i}{\cosh \theta/2} \\ \frac{i}{\cosh \theta/2} & -\frac{P}{\sinh \theta/2} \end{pmatrix}, \]  

(2.40)

and from (2.33) we get

\[ \hat{U}^F_{\pi} = \frac{1}{\cosh \pi y} \begin{pmatrix} \cosh \pi (y + i\lambda) & i \sin \pi \lambda \\ i \sin \pi \lambda & \cosh \pi (y - i\lambda) \end{pmatrix}. \]  

(2.41)
Using (2.15) we then have
\[
\hat{Z}_\lambda^F = \frac{1}{\cosh \pi (y - i\lambda)} \begin{pmatrix}
\cosh \pi y & i \sin \pi \lambda \\
-i \sin \pi \lambda & \cosh \pi y
\end{pmatrix}.
\] (2.42)

Clearly, if \( \lambda \neq \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots \), \( \hat{Z}_\lambda^F \) is a bounded pseudo-unitary operator, and using Eqs. (2.19) and (2.20) one sees that for these values \( \hat{Z}_\lambda^F \) is convolution with
\[
\cos \pi \lambda \delta(\theta) + \frac{\sin \pi \lambda}{2\pi} \exp \left( (\lambda - \left[ \lambda + \frac{1}{2} \right]) \theta \right) \begin{pmatrix}
P \\
\sinh \theta/2 \\
\frac{(-)^{[\lambda+1/2]} i}{\cosh \theta/2} \\
\frac{(-)^{[\lambda-1/2]} i}{\sinh \theta/2}
\end{pmatrix}. \] (2.43)

Here, \([x]\) denotes the greatest integer less than or equal to \(x\). Also, using the relation \( U_{\pm 1/2}^F = \pm i U_{\pm 1/2}^F \) and Eqs. (2.15) and (2.37), one sees that \( \hat{Z}_{\pm 1/2}^F \) is convolution with
\[
\frac{1}{2\pi} \begin{pmatrix}
+P \cosh \frac{\theta}{2} & i \tanh \frac{\theta}{2} \\
-i \tanh \frac{\theta}{2} & \pm P \cosh \frac{\theta}{2}
\end{pmatrix}, \] (2.44)

which is a well-defined unbounded operator on functions \( f = (f_+(\theta), f_-(\theta)) \in S(R)^2 \) satisfying
\[
\int d\theta f_+(\theta) = \mp i \int d\theta f_-(\theta). \] (2.45)

But on such functions convolution with (2.44) equals convolution with
\[
\pm \frac{1}{2\pi} \exp \left( -\frac{\theta}{2} \right) \begin{pmatrix}
P \\
\sinh \theta/2 \\
\pm i \\
\frac{\cosh \theta/2}{\sinh \theta/2}
\end{pmatrix}, \] (2.46)

which is just (2.43) for \( \lambda = \pm \frac{1}{2} \). Thus, (2.43) holds for all \( \lambda \in \mathbb{R} \), provided the condition (2.45) is met for \( \lambda = 2n \pm \frac{1}{2}, n \in \mathbb{Z} \).

2C. The Massive Klein–Gordon Case

Up to minor changes, we formulate the free Klein–Gordon equation as in Ref. [25]. Thus we take as the one-dimensional Klein–Gordon operator the Hamiltonian
\[
\hat{H}_\lambda = \begin{pmatrix}
0 & \frac{1}{\sqrt{m}} \\
\frac{d}{dx} \cdot \frac{1}{\sqrt{m}} & 0
\end{pmatrix}, \quad m > 0 \] (2.47)
on the Hilbert space $\tilde{\mathcal{H}} \simeq H_{1/2}(R) \oplus H_{-1/2}(R)$, which is the Fourier transform of the space $\mathcal{H}_p = L^2(R, E_p \, dp^1) \oplus L^2(R, E^{-1}_p \, dp^1)$, where
\[ E_p = [(p^1)^2 + m^2]^{1/2}. \] (2.48)

By definition, $\tilde{H}_0$ is the Fourier transform of the self-adjoint multiplication operator
\[
\begin{pmatrix}
0 & 1 \\
E_p & 0
\end{pmatrix}
\]
on $\mathcal{H}_p$. We diagonalize $\tilde{H}_0$ on the Hilbert space $\mathcal{H} = L^2(d\theta)^2$ by means of the unitary operator $W: \mathcal{H} \to \mathcal{H}$, defined by
\[
(Wg)(x^1) = (2\pi)^{-1/2} \sum_{\delta = \pm} \int d\theta \exp(i \delta m \, x^1 \sinh \theta) \, w_\delta(\theta) \, g_\delta(\theta), \tag{2.49}
\]
with inverse
\[
(W^{-1}f)_\delta(\theta) = (2\pi)^{-1/2} \int dx^1 \exp(-i \delta m \, x^1 \sinh \theta) \, w^*_\delta(\theta) \cdot f(x^1). \tag{2.50}
\]

Here the $w^*_\delta(\theta)$ are defined by
\[
\begin{align*}
w_\delta(\theta) &= 2^{-1/2}(\delta, m \cosh \theta), \\
w^*_\delta(\theta) &= 2^{-1/2}(\delta m \cosh \theta, 1).
\end{align*} \tag{2.51}
\]

Defining the charge conjugation operator by
\[
(Cf)_j(x^1) = (-i)^j f_j(x^1), \quad j = 1, 2, \tag{2.52}
\]
the relations (2.28)–(2.30) hold true in this case as well.

In view of the nonlocality of the inner product in $\mathcal{H}$, the operator $U^p$ of multiplication by $e(x^1)$ is not defined for all $f \in \mathcal{H}$, i.e., it may happen that $e(x^1)f(x^1) \notin \mathcal{H}$. We shall therefore proceed formally at first. Using (2.49) and (2.50) one sees that $U^B \equiv W^{-1} \tilde{U}^B W$ acts as convolution with
\[
\frac{1}{2\pi i} \begin{pmatrix}
P \coth \frac{\theta}{2} & -\tanh \frac{\theta}{2} \\
\tanh \frac{\theta}{2} & -P \coth \frac{\theta}{2}
\end{pmatrix}. \tag{2.53}
\]
Comparing with Eq. (2.37) we see that $U^B$ equals the operator $Z^F$ of the previous section. Hence, $Z^B$ equals $U^F$, so that $Z^B$ acts as convolution with
\[
\frac{1}{2\pi i} \begin{pmatrix}
P \sinh \frac{\theta}{2} & \frac{i}{\cos \frac{\theta}{2}} \\
\frac{i}{\cosh \frac{\theta}{2}} & P \sinh \frac{\theta}{2}
\end{pmatrix}. \tag{2.54}
\]
by virtue of Eq. (2.31). Moreover, $U^B$ is pseudo-unitary, and satisfies the covariance and locality relations (2.11) and (2.12) in view of (2.53) and the commutativity of multiplication by $\epsilon(x^1)$ and $\epsilon(x^3-a^3)$. Hence, the field operator

$$\phi^B_\epsilon(x) = U(x, 1) U^B U(x, 1)^*$$

(2.55)

is covariant and local. This field operator corresponds to the “bosonic Ising model” (cf. Chapter 6).

Let us now briefly digress to indicate how the above arguments may be made rigorous. First we note that $FW^{-1}$ maps all test functions $f = (f_1(x^1), f_2(x^1)) \in S(\mathbb{R}^2) \subset \mathcal{H}$ satisfying $f_1(0) = 0$ onto a set of test functions $g = (g_+(y), g_-(y)) \in S(\mathbb{R}^2) \subset \mathcal{H}$ satisfying $g_+(0) = g_-(0)$. These dense subspaces are in the domains of $\hat{U}^B$ and $\hat{U}^B$ resp. (observe that $\epsilon(x^1)f_1(x^1) \in H_1(\mathbb{R}) \subset H_{1/2}(\mathbb{R})$) and are left invariant by $\hat{U}(0, \Lambda)$ and $\hat{U}(0, \Lambda)$ resp., so that (2.53) and (2.11) hold true on the corresponding subspace $\mathcal{D}$ in $\mathcal{H}$. However, since $\mathcal{D}$ is not invariant under $U(a, 1)$ and functions with jump discontinuities are not in $H_{1/2}(\mathbb{R})$, it becomes a little awkward to specify the dense sets on which Eqs. (2.9), (2.10) and (2.12) hold. It can be shown that these difficulties disappear if one smears the field operators with functions $F \in S(\mathbb{R}^2)$: The resulting operators are bounded and covariant, and commute if $\text{supp} \; F$ and $\text{supp} \; G$ are spacelike separated (for the proof of the last assertion, use the finite propagation speed of $\exp(-i\mathcal{H}_{\text{d}})$).

Proceeding again as in Section 2B, we define the classical field operator for the “bosonic Federbush model” by

$$\phi^B_{\text{cl}}(x) = U(x, 1) U^B U(x, 1)^*, \quad \lambda \in \mathbb{R},$$

(2.56)

where

$$U^B_\lambda = \cos \pi \lambda \pm i U^B \sin \pi \lambda, \quad \lambda \in \mathbb{R}.$$  

(2.57)

Thus, $U^B_\lambda$ acts as convolution with

$$\delta(\theta) \cos \pi \lambda + \frac{\sin \pi \lambda}{2 \pi} \begin{pmatrix} P \tanh \theta/2 & -\tanh \theta/2 \\ \tanh \theta/2 & -P \tanh \theta/2 \end{pmatrix},$$

(2.58)

so that using (2.34) we have

$$\hat{U}^B_\lambda = \frac{1}{\sinh \pi y} \begin{pmatrix} \sinh \pi (y + i\lambda) & -i \sin \pi \lambda \\ i \sin \pi \lambda & \sinh \pi (y - i\lambda) \end{pmatrix}.$$  

(2.59)

The conjugate is then the unitary operator

$$Z^B_\lambda = \frac{1}{\sinh \pi (y - i\lambda)} \begin{pmatrix} \sinh \pi y & -i \sin \pi \lambda \\ -i \sin \pi \lambda & \sinh \pi y \end{pmatrix}.$$  

(2.60)
Using Eqs. (2.19) and (2.20) it follows that $Z_{\beta}$ is convolution with
\[
\cos \pi \lambda \delta (\theta) + \frac{\sin \pi \lambda}{2 \pi} \exp \left( \left( \lambda - \frac{1}{2} - [\lambda] \right) \theta \right) \begin{pmatrix}
\frac{1}{\sinh \theta/2} & \frac{(-)^{[\lambda]}}{\cosh \theta/2} \\
\frac{(-)^{[\lambda]}}{\cosh \theta/2} & \frac{1}{\sinh \theta/2}
\end{pmatrix}.
\] (2.61)

2D. The Massless Dirac Case

In the massless case $\tilde{H}_0$ is the operator (2.21) on $\tilde{H} = L^2(dx^1)^2$ with $m = 0$. It is clear that $\tilde{H}_0$ leaves the subspaces $\tilde{H}_1$ of “positive chirality” wave functions $(f_1(x^1), 0)$ and $\tilde{H}_{-1}$ of “negative chirality” wave functions $(0, f_{-1}(x^1))$ invariant. The spaces $\tilde{H}_s$ describe both positive and negative energy massless particles which move to the right and to the left with unit group speed for $s = 1$ and $s = -1$ resp. This can be seen by diagonalizing $\tilde{H}_0$ on the Hilbert space $\tilde{H}_1 \oplus \tilde{H}_{-1}$ (where $\tilde{H}_1$, $\tilde{H}_{-1}$ are two copies of $\tilde{H} = L^2(dx^1)^2$) by means of the unitary operator

\[
(W_{n})_{s} f(x^1) = (2\pi)^{-1/2} \int d\theta \ e^{i\theta \epsilon \theta} \left[ \exp(i x^1 \epsilon \theta) \ G_{s,+}(\theta) + i \exp(-i x^1 \epsilon \theta) \ G_{s,-}(\theta) \right]
\] (2.62)

with inverse

\[
(W_{-1})_{s} f(x^1) = (2\pi)^{-1/2} \int dx^1 \ e^{i\theta \epsilon \theta} \left[ \exp(-i x^1 \epsilon \theta) \ f_s(x^1) + i \exp(i x^1 \epsilon \theta) \ f_s(x^1) \right].
\] (2.63)

(It is to be noted that this spectral representation would be the $m \downarrow 0$ limit of the one in Section 2B if we had employed wave functions depending on the momenta $p = \epsilon \theta$ and $p = m \ sinh \theta$ resp., instead of the more convenient and natural rapidities.) One easily verifies that

\[
(H_{0})_{s,\theta}(\theta) = \delta_{\epsilon \theta} f_{s,\theta}(\theta),
\] (2.64)

and that

\[
(Cf)_{s,\theta}(\theta) = f_{s,-\theta}(\theta),
\] (2.65)

if we define $\mathcal{C}$ again by Eq. (2.27). In this case the representation of the Poincaré group is given by

\[
(U(a, \Lambda)f)_{s,\theta}(\theta) = \exp(i\delta(a^0 - sa^1) \epsilon \theta) f_{s,\theta}(\theta - \alpha).
\] (2.66)

The unitary multiplication operator

\[
\tilde{U}^{\epsilon}_{\theta} = \begin{pmatrix}
-i\epsilon(x^1) & 0 \\
0 & \epsilon(x^1)
\end{pmatrix}
\] (2.67)
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on $\mathcal{H}$ clearly leaves $\mathcal{H}_s$ invariant, and therefore leads to an operator $U_{F_0} = U_{F_0} \oplus U_{F_1}$ on $\mathcal{H}_1 \oplus \mathcal{H}_{-1}$. A calculation shows that $U_{F_0}$ acts on $\mathcal{H}_s$ as convolution with

$$
\frac{-s}{2\pi i} \left( \begin{array}{ccc} P & i & \cosh \theta/2 \\ -i & \cosh \theta/2 & -P \\ \cosh \theta/2 & -\sinh \theta/2 & \end{array} \right).
$$

The classical field operators associated with the massless Thirring model are the operators

$$
\phi_{\lambda,s,cl}^F(x) := U_s(x, 1) U_{\lambda,0}^F U_s(x, 1)^*,
$$

where

$$
U_{\lambda,0}^F := \cos \pi \lambda + iU_{\lambda,0}^F \sin \pi \lambda.
$$

Note that by virtue of Eq. (2.66) $\phi_{\lambda,s,cl}^F(x)$ acts as multiplication by $\exp[-i\pi\lambda \varepsilon(x^1 - y^1 + x^0)]$ on $\mathcal{H}_s$. Thus, $\phi_{\lambda,s,cl}^F$ is not only covariant and local, but also satisfies the stronger commutation relation

$$
[\phi_{\lambda,s,cl}^F(x), \phi_{\lambda,s,cl}^F(y)]_{-} = 0, \quad \forall x, y \in \mathbb{R}^2.
$$

Finally, comparing Eqs. (2.31) and (2.68), we see that $U_{F_0}^F = -sU^F$. As a result,

$$
U_{\lambda,s}^F = U_{-\lambda,s}^F.
$$

so that

$$
Z_{\lambda,s}^F = Z_{-\lambda,s}^F.
$$

2E. The Massless Klein–Gordon Case

The massless Klein–Gordon equation is of course just the wave equation. To formulate it as a Hilbert space evolution equation we again adopt the two-component form of Section 2C. (Some small snags are encountered in doing so rigorously, but we shall disregard these at first and obviate them at the end of this section.) Now, $\tilde{\mathcal{H}}_0$ is the operator

$$
\tilde{\mathcal{H}}_0 = \begin{pmatrix} 0 & 1 \\ -\left(\frac{d}{dx^1}\right)^2 & 0 \end{pmatrix},
$$

acting on a Hilbert space $\tilde{\mathcal{H}}$, which in this case is the Fourier transform of $\mathcal{H}_{p1} = L^2(\mathbb{R}, |p^1| dp^1) \oplus L^2(\mathbb{R}, |p^1|^{-1} dp^1)$; $\tilde{\mathcal{H}}_0$ is by definition the Fourier transform of the self-adjoint multiplication operator

$$
\begin{pmatrix} 0 & 1 \\ (p^1)^2 & 0 \end{pmatrix}
$$
on \( \mathcal{H}_{\rho t} \). As in the massless Dirac case, \( \mathcal{H} \) can be decomposed into subspaces \( \mathcal{H}_s \), corresponding to particles with positive and negative energy, moving to the right \( (s = 1) \) and to the left \( (s = -1) \) resp. Indeed, we may write

\[
\begin{pmatrix} f \cr g \end{pmatrix} = F^{-1} \begin{pmatrix} f + \hat{g}/p^1 \cr \hat{g} + p^1 f \end{pmatrix} + F^{-1} \begin{pmatrix} f - \hat{g}/p^1 \cr \hat{g} - p^1 f \end{pmatrix},
\]

and this decomposition is clearly left invariant by \( \tilde{H} \). Note that

\[
\begin{pmatrix} f_1 \cr f_2 \end{pmatrix} \in \mathcal{H}_s \iff f_2 = -i sf_1.
\]

Correspondingly, we can define a spectral representation of \( \tilde{H}_0 \) on the Hilbert space \( \mathcal{H}_1 \oplus \mathcal{H}_{-1} \) through the unitary operator

\[
(Wg)(x^1) = (2\pi)^{-1/2} \sum_{s,\delta = \pm 1, -} \int d\theta \exp(i \delta x^1 s e^{a\theta}) w_{s,\delta}(\theta) g_{s,\delta}(\theta)
\]

with inverse

\[
(W^{-1}f)_{s,\delta}(\theta) = (2\pi)^{-1/2} \int d x^1 \exp(-i \delta x^1 s e^{a\theta}) w^*_{s,\delta}(\theta) \cdot f(x^1).
\]

Here,

\[
w_{s,\delta}(\theta) = 2^{-1/2}(\delta, e^{a\theta}), \quad w^*_{s,\delta}(\theta) = 2^{-1/2}(\delta e^{a\theta}, 1).
\]

If one defines the charge conjugation operator again by Eq. (2.52), Eqs. (2.64)–(2.66) hold true in this case too.

In view of Eq. (2.76), the multiplication operator

\[
U^\rho_\alpha = -s \begin{pmatrix} e(x^1) & 0 \\ 0 & e(x^1) \end{pmatrix}
\]

leaves \( \mathcal{H}_s^\rho \) invariant, and hence gives rise to an operator \( U^\rho_\alpha \) on \( \mathcal{H}_s \). Using (2.77) (2.78) one verifies that \( U^\rho_\alpha \) acts as convolution with

\[
-\frac{s}{2\pi i} \begin{pmatrix} P \cotanh \theta/2 & -\tanh \theta/2 \\ \tanh \theta/2 & -P \cotanh \theta/2 \end{pmatrix}.
\]

Defining classical field operators

\[
\phi^{\rho_\alpha}_{\lambda, s, e_1}(x) = U_\lambda(x, 1) U^{\rho_\alpha}_{\lambda, s} U_\lambda(x, 1)^*,
\]

where

\[
U^{\rho_\alpha}_{\lambda, s} = \cos \pi \lambda + i U^\rho_\alpha \sin \pi \lambda,
\]

one concludes as in Section 2D that \( \phi^{\rho_\alpha}_{\lambda, s, e_1}(x) \) is a covariant family of commuting
field operators. Also, a comparison of Eqs. (2.53) and (2.81) shows that $U_\alpha^B = -s U^B$, so that

$$U_\alpha^B = U_{-\alpha}^B,$$

and therefore,

$$Z_\alpha^B = Z_{-\alpha}^B.$$

Let us finally add some comments concerning mathematical rigor. Firstly, $L^2(R, |p|^1 dp^1)$ contains functions with non-integrable singularities at the origin. A priori, these functions do not define tempered distributions, so that their Fourier transform is ill defined. But if we define Fourier transformation to be isometric on the dense set of Schwartz space test functions we may complete the resulting pre-Hilbert space, thereby obtaining $\langle 0, 0 \rangle \mathcal{H}$. Equations (2.77)-(2.78) should be understood in this sense.

Secondly, $U^B$ is of course not defined on all of $\mathcal{H}_s$. Initially, one can take as its domain the vectors $(f, -isf')$, where $f \in S(R)$ vanishes at the origin. This dense subspace of $\mathcal{H}_s$ is mapped into $\mathcal{H}_s$ by $\varepsilon(x^2)$, and for such vectors one can verify Eq. (2.81) rigorously. Since $U^B$ is a multiplication operator on $\mathcal{H} = L^2(R, dy)^2$ its domain can be extended in an obvious way, leading to an extension of $U^B$.

Thirdly, since the domain of $U^B$ is not invariant under $U_s(a, 1)$, Eq. (2.9) only holds on a dense set depending on $x$ and $a$, while Eq. (2.71) for $\phi^B_{\lambda, s, c}$ only holds on the intersection of the respective domains (which is however still dense in $\mathcal{H}_s$). It is not hard to see that these difficulties get worse if one smears $\phi^B_\alpha$ with, e.g., functions $F \in S(R^2)$, in contrast to the massive case: Given $F \neq 0$, it is very likely that there is no dense subspace $\mathcal{D} \subset \mathcal{H}_s$ such that $\int_{\mathcal{D}} (\phi^B_\alpha F) \mathcal{D} \subset \mathcal{H}_s$, and it is even plausible that no dense subspace of $\mathcal{H}_s$ exists on which all smeared field operators can be defined.

3. Quantum Field Operators

3A. Preliminaries

This section serves a similar purpose as Section 2A: We give an account of the general features of the field theories that will be considered in the following sections, and summarize some formalism and results from Refs. [18–19] we have occasion to use.

There are two distinct second-quantized many-particle theories associated with the "first-quantized," single-particle theory of Section 2A. In both cases one deals with the Fock space $\mathcal{F}(\mathcal{H})$ over $\mathcal{H}$, where $\varepsilon$ stands for symmetric (anti-symmetric) in the boson (fermion) case. The first theory is considered unphysical, since it is still afflicted with the negative energies of the one-particle theory: one simply associates product operators $I(U)$ on $\mathcal{F}(\mathcal{H})$ to one-particle operators $U$ on $\mathcal{H}$. If one uses this quantization to get quantum field operators $\phi(x)$ on $\mathcal{F}(\mathcal{H})$ corresponding to the classical field
operators $\phi_{\text{op}}(x)$, it is obvious that the operators $\phi(x)$ will also be covariant and local, since the $\Gamma$-operation satisfies

$$\Gamma(U_1)\Gamma(U_2) = \Gamma(U_1U_2).$$

By definition these field operators leave the vacuum invariant, and therefore the field theory does not have a particle interpretation. However, on the Fock space $F(H_1 \oplus H_1)$, where $H_2$ is a copy of $H$, it is possible to define covariant field operators that do couple the vacuum to all states and that possess asymptotic free fields in the $LSZ$ sense, connected by a unitary $S$-matrix. These field operators are the product of a free field acting on one factor of $F(H_1) \otimes F(H_1) \cong F(H_1 \oplus H_1)$ and a field $\phi(x)$ acting on the other factor. In this way we shall obtain in Section 3B four distinct asymptotically complete field theories on $F(K \oplus H)$, corresponding to the operators $\phi^F$, and $\phi^F$ of Chapter 2. The theories associated with $\phi^F$ and $\phi^F$ are closely connected with the Federbush and Thirring models, as will be shown in Sections 4A, 5A and 7A.

Let us now consider the second, positive energy theory. Noting that $E(2) \equiv E(2) \oplus E(2)$, one here interprets $E(2)$ as a particle Fock space, $E(2)$ as the Fock space of its antiparticle, both having only positive energies. The physical Fock space analogs of the unphysical one-particle operators can be obtained as follows. Define a field operator on $E(2) \otimes E(2)$ by

$$\Phi(v) = a(P_+v) + b^*(P_-v), \quad v \in H,$$

where the bar denotes complex conjugation, and where $a$ is a particle annihilation operator, $b^*$ an antiparticle creation operator (for precise definitions the reader could consult Ref. [18], Section 2). Transformations $\Phi(v) \rightarrow \Phi(U^*v)$, where $U$ is a unitary (pseudo-unitary) operator in the fermion (boson) case, leave the commutation relations

$$[\Phi(v), \Phi(w)]_+ = (v, w) \quad \text{(fermions),}$$

$$[\Phi(v), \Phi(w)]_- = (v, qw) \quad \text{(bosons)}$$

invariant, and amount to a special kind of Bogoliubov transformation of the annihilation and creation operators (remember that $q = P_+ - P_-$). By definition such a transformation is unitarily implementable if there exists a unitary operator $U$ on $E(2)$ such that

$$\mathcal{U}\Phi(U^*v) - \Phi(v)\mathcal{U}, \quad \forall v \in H.$$  

One can show that $\mathcal{U}$ is unique up to a phase factor if it exists. For operators $U$ that commute with $P_+$ and $P_-$ one can take for $\mathcal{U}$ the operator $\Gamma(U_{++}) \otimes \Gamma(U_{--})$. In particular, for the representation $U(a, \Lambda)$ defined by (2.1) one gets

$$(\mathcal{U}(a, \Lambda) \psi)^{n.r}(\theta_1, \ldots, \theta_n; \theta_{n+1}, \ldots, \theta_{n+r})$$

$$= \exp \left[ ia \cdot \sum_{j=1}^{r+n} p_j(\theta) \right] \psi^{n.r}(\theta_1 - \alpha, \ldots, \theta_{n+r} - \alpha).$$

(3.6)
Thus, the second-quantized Hamiltonian (i.e., the generator of time translations) is a positive operator, which is zero on the vacuum $\Omega$.

More generally, we regard the unitary operators $\mathcal{H}$ as the physical, second-quantized analogs of the unphysical (pseudo-)unitary classical operators $U$. At this point, this interpretation may seem to be at best unorthodox, since so far we have only shown that it leads to the same results as the customary "text-book" approach to second quantization in the case of the representation of the Poincaré group. However, in the case of massive spin-0 and spin-½ particles interacting with time-dependent external fields, the present scheme leads in a quick and smooth way to a Fock space $S$-operator that is equal to the usual Feynman–Dyson $S$-operator, except that no divergent vacuum amplitudes occur [26]. Also, by requiring that Eq. (3.5) only be satisfied in the sense of quadratic forms, we shall readily get normal ordered quantum fields that are essentially equal to the fields introduced by Wightman [2] and Glaser [5] in their study of the Federbush and massless Thirring models resp. We shall come back to this in more detail in Sections 4B and 5B.

If $U$ does not commute with $P_\pm$ and $P$ the transformation is not unitarily implementable in general. The necessary and sufficient condition for unitary implementability is that $U_\pm$ and $U_\pm$ be Hilbert–Schmidt. If this condition is satisfied the normal form of the implementing operator can be explicitly found. In the boson case it can be written

$$\mathcal{H} = \det(1 - Z_+^*Z_+)^{1/2} \exp(Z_+a^*b^* + (Z_+ - 1) a^*a + (Z_- - 1) bb^* + Z_-ba);$$

(3.7)

where $Z$ is the conjugate (2.15) of $U$ and where the arbitrary phase factor is chosen such that $(\Omega, \mathcal{H}\Omega) > 0$. In the fermion case one gets

$$\mathcal{H} = \det(1 + Z_+^*Z_+)^{-1/2} \exp(Z_+a^*b^* + (Z_+ - 1) a^*a - (Z_- - 1) bb^* - Z_-ba);$$

(3.8)

provided there are no non-zero vectors $f_\pm \in \mathcal{H}_\pm$ satisfying $U_-f_- = 0$ or $U_+f_+ = 0$; if this condition is relaxed the normal form is more involved. In Eqs. (3.7)–(3.8) the expression $Z_+a^*b^*$ (e.g.) is shorthand for $\int d\theta_1 d\theta_2 Z_+ (\theta_1, \theta_2) a^*(\theta_1) b^*(\theta_2)$, and the expansion of the exponential strongly converges on algebraic tensors (i.e., finite linear combinations of vectors of the form $\prod_{i=1}^n a^*(f_i) \prod_{j=1}^m b^*(g_j)\Omega$). For more details and rigorous proofs the reader may consult Refs. [18, 19].

Assume now that $U_1$, $U_2$ are two unitary or pseudo-unitary operators satisfying the above restrictions. If they moreover commute it follows that $\mathcal{H}_1 \mathcal{H}_2 = e^{i\Phi} \mathcal{H}_2 \mathcal{H}_1$. Indeed, $\mathcal{H}_1 \mathcal{H}_2$ and $\mathcal{H}_2 \mathcal{H}_1$ both implement the Bogoliubov transformation generated by $U_1U_2 = U_2U_1$, so that the assertion follows. Thus, if one can show that $\mathcal{H}_1 \mathcal{H}_2$ and $\mathcal{H}_2 \mathcal{H}_1$ have the same vacuum expectation value, one must have $\mathcal{H}_2 \mathcal{H}_1 = \mathcal{H}_1 \mathcal{H}_2$. (In the case of Eq. (3.1) this was evident, since $\Gamma(U)\Omega = \Omega$.) Likewise, assuming that $U_1$ and $U_2$ are related by $U_2 = U(a_0, A_0) U_1 U(a_0, A_0)^*$, it follows that $\mathcal{H}_2 = \mathcal{H}(a_0, A_0) \times \mathcal{H}_1 \mathcal{H}(a_0, A_0)^*$, since $\mathcal{H}(a_0, A_0)\Omega = \Omega$, and $(\Omega, \mathcal{H}\Omega) > 0$ by convention.
Consider now the transformation generated by the classical field operators $\phi_0(x), x \in \mathbb{R}^2,$ given by Eq. (2.7). If these transformations were unitarily implementable the resulting second-quantized field operator would be covariant and local. Indeed, its covariance then follows immediately from the argument in the preceding paragraph; as a result of covariance, its two-point function has a Källen–Lehmann representation $i \int dp(m) \Delta_+(m, x - y),$ which together with (2.10) and the preceding paragraph implies locality.

Unfortunately, it easily follows from the covariance relation (2.11) that the operators $\phi_0(x)_{\pm}$ are not even compact, so that the transformation generated by $\phi_0(x)$ cannot be unitarily implemented in $\mathcal{F}(\mathcal{H}).$ However, as long as $Z$ is a bounded operator, the expressions at the right-hand sides of Eqs. (3.7) and (3.8) can still be defined as quadratic forms on algebraic tensors, provided one omits the determinantal factor, which does not make sense if $Z_{++}$ is not Hilbert–Schmidt. (In the case of the quantum fields below one can regard this procedure as a wave function renormalization.) Denoting this form by $E(Z),$ one can show that

$$E(Z)\Phi(U^*v) = \Phi(v)E(Z), \quad \forall v \in \mathcal{H},$$

in the sense of quadratic forms. Indeed, a proof of this can be given along the same lines as the proof of Eqs. (4.21) and (4.22) in Ref. [18], since only the conjugacy relations (2.15) between $U$ and $Z$ enter in the form case (i.e., no convergence and domain difficulties occur).

The quadratic form on algebraic tensors corresponding in this way to $\phi_0(x)$ will be denoted by $\phi(x).$ It is easy to see that the covariance relation

$$\mathcal{H}(a, A)\phi(x)\mathcal{H}(a, A)^* = \phi(Ax + a)$$

holds in the form sense, but locality cannot be formulated rigorously for the unsmeared quantum fields $\phi(x),$ since the formal product $\phi(x)\phi(y)$ does not define a quadratic form. However, one may hope that the quadratic form $\int dx F(x)\phi(x),$ where $F \in S(\mathbb{R}^2),$ is the form of an operator $\phi(F)$ whose domain includes the vacuum $\Omega$ and vectors that are finite linear combinations of vectors of the form $\prod_{i=1}^n \phi(F_i)\Omega.$ Proving this is a nontrivial problem to which we intend to come back elsewhere. (It can be shown that this actually holds true for the field operators associated with the Federbush and Ising models, provided the test functions have compact support in momentum space. However, this is false for the field operators associated with the massless Thirring model: For any $F \in S(\mathbb{R}^2),$ $\phi(F)\Omega$ either vanishes or is not in Fock space [24].) If this would be true, it is plausible in view of the above that these smeared field operators satisfy $[\phi(F), \phi(G)]_+ = 0$ for supp $F$ spacelike w.r.t. supp $G.$ In any case, as regards locality properties of the quantum fields we only intend to argue heuristically in this paper.

It is clear from formulas (3.7) and (3.8) that the quantum fields just introduced do not couple the vacuum to the one-particle states and that they leave the charge-zero sector invariant. However, in analogy to the unphysical sector, one may define fields...
on the larger Fock space $F(\mathcal{H}_1 \oplus \mathcal{H}_-)$ that do give rise to a field theory that can be interpreted in terms of incoming and outgoing asymptotic particle states. Thus, we obtain in Section 3C two distinct asymptotically complete field theories on $F(\mathcal{H}_1 \oplus \mathcal{H}_-)$, corresponding to the classical operators $\phi_{\lambda,\epsilon}^{F(0)}$ for $\epsilon = a$, to $\phi_{\lambda,\epsilon}^{B(0)}$ for $\epsilon = s$. The connection of the fermionic theories with the positive energy Federbush and massless Thirring models will be established in Sections 4B and 5B resp.

3B. The Unphysical Sector

The Fock space employed throughout this section and the next will be $F(\mathcal{H}_1 \oplus \mathcal{H}_-)$ (abbreviated $F$), where $\mathcal{H}_\pm$ are copies of the space $\mathcal{H} = L^2(d\theta)^2$ of Chapter 2. It is, however, often convenient to think of $F$ as the tensor product $F(\mathcal{H}_1) \otimes F(\mathcal{H}_-)$, to which it is naturally isomorphic. Thus, $\Gamma(U)$ will denote the operator that equals $\Gamma(U)$ on the factor $F(\mathcal{H}_\pm)$ and the identity on the other factor, if $U$ is an operator on $\mathcal{H}$, but also the corresponding operator on $F$, which in unabbreved but clumsy notation would be written $\Gamma(U \otimes 1)$ in the case of $\Gamma(U)$. We shall denote by $c_{s,\theta}^{(s)}(\theta)$ the annihilation and creation operators on $F$, physically interpreted as corresponding to particles of species $s$ and sign of energy $\delta$.

On $F$ we define two field theories describing massive particles and two describing massless particles. In both cases the representation of the Poincaré group can be written

$$\psi(a, \Lambda) \psi(\theta_1, s_1, \delta_1; \ldots; \theta_N, s_N, \delta_N)$$

$$= \exp \left( ia \cdot \sum_{j=1}^{N} \delta_j p_{s_j}(\theta_j) \right) \psi(\theta_1 - \alpha, s_1, \delta_1; \ldots; \theta_N - \alpha, s_N, \delta_N),$$

where

$$p_s(\theta) \equiv m(s)(\cosh \theta, \sinh \theta), \quad m(s) > 0,$$

in the massive case,

$$p_s(\theta) \equiv (e^{s\theta}, se^{s\theta}),$$

in the massless case. The field operators corresponding to $\phi_{\lambda,\epsilon}^{F(0)}$ are defined by

$$\psi_{\lambda,s}(0) = \psi_{0,s}^{F(0)} \phi_{\lambda,s}^{F(0)}$$

and those corresponding to $\phi_{\lambda,\epsilon}^{B(0)}$ by

$$\phi_{\lambda,s}(0) = \phi_{0,s}^{B(0)} \phi_{\lambda,s}^{B(0)}.$$

Here, the free Dirac and Klein–Gordon fields $\psi_{0,s}(x)$ and $\phi_{0,s}(x)$ are defined by

$$\psi_{0,s}(x) = \left( \frac{m(s)}{4\pi} \right)^{1/2} \int d\theta \left[ c_{s,\lambda}(\theta) \left( e^{\theta/2} - e^{-\theta/2} \right) e^{-ix \cdot p_s} + c_{s,\lambda}^{\dagger}(\theta) \left( e^{\theta/2} - e^{-\theta/2} \right) e^{ix \cdot p_s} \right].$$
\[ \psi_{0,s}(x) = \left( \frac{1}{2\pi} \right)^{1/2} \int d\theta \ e^{i\theta/2} [c_{s,1}(\theta) e^{-i\pi p_+} + is c_{s,-1}(\theta) e^{i\pi p_+}], \]  
\eqref{3.17}

\[ \phi_{0,s}(x) = \left( \frac{1}{4\pi} \right)^{1/2} \int d\theta [c_{s,1}(\theta) e^{-i\pi p_+} - c_{s,-1}(\theta) e^{i\pi p_+}], \]  
\eqref{3.18}

\[ \phi^0(x) = \left( \frac{1}{2\pi} \right)^{1/2} \int d\theta [c_{s,1}(\theta) e^{-i\pi p_+} - c_{s,-1}(\theta) e^{i\pi p_+}], \]  
\eqref{3.19}

where \( p_s \equiv p_s(\theta) \) is defined by Eqs. \eqref{3.12} and \eqref{3.13} resp. Also,

\[ \phi^A_{\lambda,s}(x) = \Gamma_s(\phi^A_{\lambda,s,cl}(x)), \quad A = F(0), B(0), \]  
\eqref{3.20}

where

\[ \phi^A_{\lambda,s,cl}(x) = U_s(x, 1) U^A_{\lambda,\theta} U_s(x, 1)^* , \quad A = F, B. \]  
\eqref{3.21}

More explicitly, let us specify the normal form of \( \phi^A_{\lambda,s}(x) \) \( (A = F(0), B(0)) \), using the results of Chapter 2 and the easily verified fact that the normal form of \( \Gamma(U) \) is \( \text{exp}(U - 1) c^+ c^- \). One gets

\[ \phi^A_{\lambda,s}(x) = \text{exp}[U^A_{\lambda,s} c^*_{s,1} c_{s,-1} + (U^A_{\lambda,s} c^*_{s,1} c_{s,1} \quad \text{and} \quad \text{and} \]

\[ U^A_{\lambda,s}(\theta_1, \theta_2) = \text{exp}[i x \cdot \left( \delta p_s(\theta_1) - \delta p_s(\theta_2) \right)] U^A_{\lambda,s}(\theta_1 - \theta_2), \]  
\eqref{3.23}

and

\[ U^A_{\lambda}(\theta) = \cos \pi \lambda \delta(\theta) + \frac{\sin \pi \lambda}{2\pi} \begin{pmatrix} P \theta & i \\ \sinh \frac{\theta}{2} & \cosh \frac{\theta}{2} \end{pmatrix}, \quad A = F(0), \]  
\eqref{3.24}

(cf. Eqs. \eqref{2.40}, \eqref{2.72}), while

\[ U^A_{\lambda}(\theta) = \cos \pi \lambda \delta(\theta) + \frac{\sin \pi \lambda}{2\pi} \begin{pmatrix} P \cotanh \frac{\theta}{2} & -\tanh \frac{\theta}{2} \\ \tanh \frac{\theta}{2} & -P \cotanh \frac{\theta}{2} \end{pmatrix}, \quad A = B(0) \]  
\eqref{3.25}

(cf. Eqs. \eqref{2.58}, \eqref{2.84}). All field operators introduced above are well defined as quadratic forms on the dense subspace \( \mathcal{D} \) of \( \mathcal{F} \), where \( \mathcal{D} \) is the space of algebraic tensors whose constituent functions are in \( C_0^\infty \). That is, \( \psi \in \mathcal{D} \) if and only if it is a finite linear combination of vectors of the form \( \prod_{n=1}^N c^*_{n,\delta_n}(f_n) \Omega \), where \( f_n(\theta) \in C_0^\infty \).
The fields \( \phi^{A}_{\lambda,s} \) are covariant by construction, and using this it is readily verified that

\[
\mathcal{U}(a, A) \phi^{A}_{\lambda,s}(x) \mathcal{U}(a, A)^* = \left( \begin{array}{cc} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{array} \right) \phi^{A}_{\lambda,s}(Ax + a),
\]

\[
(3.26)
\]

\[
\mathcal{U}(a, A) \phi^{0}_{\lambda,s}(x) \mathcal{U}(a, A)^* = e^{i\alpha} \phi^{0}_{\lambda,s}(Ax + a),
\]

\[
(3.27)
\]

\[
\mathcal{U}(a, A) \phi^{(o)}_{\lambda,s}(x) \mathcal{U}(a, A)^* = \phi^{(o)}_{\lambda,s}(Ax + a).
\]

\[
(3.28)
\]

As regards locality of the above field operators, we know already that the fields \( \phi^{A}_{\lambda,s} \) commute at spacelike separations. Also, one easily sees that the free fields \( \phi^{0}_{\lambda,s} \) are local in the sense that

\[
[\phi^{0}_{\lambda,s}(x), \phi^{0}_{\lambda,s}(y)^{(*)}]_\epsilon = 0, \quad (x - y)^2 < 0,
\]

\[
(3.29)
\]

where \( \epsilon = \mp (\pm) \) if \( \phi^{0}_{\lambda,s} \) is considered as an operator on \( \mathcal{F}_x(\mathcal{F}_y) \). More generally, we expect that

\[
[\phi^{(o)}_{\lambda,s}(x), \phi^{(o)}_{\lambda,s}(y)^{(*)}]_\epsilon = 0, \quad (x - y)^2 < 0,
\]

\[
(3.30)
\]

in the customary smeared sense. It is convenient to postpone our arguments for this to Section 7A. In contrast, the fields \( \phi^{(o)}_{\lambda,s} \) are not local, since already the free fields \( \phi^{0}_{\lambda,s} \) are non-local. The definitions (3.18)--(3.19) have been made in analogy with the free fields on the physical sector, and are sensible since a satisfactory scattering theory results for the fields \( \phi^{(o)}_{\lambda,s} \).

The main interest of these field theories derives from the fact that they admit a particle interpretation, in the sense that asymptotic free fields exist, connected by a unitary S-matrix. We shall now detail this. Define unitary operators

\[
(S_{\lambda}\psi)(\theta_1, s_1, \delta_1; \ldots; \theta_N, s_N, \delta_N) = \exp \left[ -i\pi\lambda \sum_{1 \leq i < j \leq N} \epsilon(\theta_i - \theta_j)(s_i - s_j) \right]
\]

\[
\psi(\theta_1, s_1, \delta_1; \ldots; \theta_N, s_N, \delta_N)
\]

\[
(3.31)
\]

and

\[
(S^{\dagger}_{\lambda}\psi)(\theta_1, s_1, \delta_1; \ldots; \theta_N, s_N, \delta_N) = \exp \left[ -i\pi\lambda \sum_{1 \leq i < j \leq N} |s_i - s_j| \right]
\]

\[
\psi(\theta_1, s_1, \delta_1; \ldots; \theta_N, s_N, \delta_N)
\]

\[
(3.32)
\]

on \( \mathcal{F}_x \). Note that these operators are well defined, since the term in square brackets is symmetric under permutations of \( (\theta_i, s_i, \delta_i) \). Introduce in- and out-fields by setting

\[
\psi^{(o)}_{\lambda,s}^{\text{in}}(x) = S^{\dagger}_{\lambda/2}\phi^{(o)}_{\lambda,s}(x) S^{(o)}_{\lambda/2}
\]

\[
(3.33)
\]

\[
\phi^{(o)}_{\lambda,s}^{\text{out}}(x) = S^{(o)}_{\lambda/2}\phi^{(o)}_{\lambda,s}(x) S^{(o)}_{\lambda/2}.
\]

\[
(3.34)
\]
Now let $F_s(t, x)$ be smooth solutions of the free Dirac (Klein–Gordon) equation with mass $m(s)$, and let $g_s$ be a smooth solution of the wave equation, moving to the right for $s = 1$, to the left for $s = -1$ (i.e., $g_s(t, x)$ is a function of $t-sx$); "smooth" meaning that the time-zero functions are $C_0^\infty$ in rapidity space. Then one has

$$\lim_{t \to \pm \infty} \int dx' F_s(t, x') \cdot \psi_{\lambda, s}(0, x') = \int dx' \tilde{F}_s(0, x') \cdot \psi_{\lambda, s}^\text{out}(0, x'),$$  

(3.35)

$$\lim_{t \to \pm \infty} \int dx' g_s(t, x') \psi_{\lambda, s}^0(0, x') = \int dx' g_s(0, x') \tilde{\psi}_{\lambda, s}^\text{out}(0, x'),$$  

(3.36)

$$\lim_{t \to \pm \infty} \int dx' f_s(t, x') \tilde{\phi}_{\lambda, s}(0, x') = \int dx' f_s(0, x') \tilde{\phi}_{\lambda, s}^\text{out}(0, x'),$$  

(3.37)

$$\lim_{t \to \pm \infty} \int dx' \tilde{g}_s(t, x') \tilde{\phi}_{\lambda, s}(0, x') = \int dx' \tilde{g}_s(0, x') \tilde{\phi}_{\lambda, s}^\text{out}(0, x').$$  

(3.38)

in the sense of quadratic forms on $\mathcal{D} \times \mathcal{D}$, where $\mathcal{D}$ is the dense subspace of $\mathcal{F}$ defined below Eq. (3.25). As a result, the field theories are asymptotically complete, and $S_\lambda$ is the $S$-matrix for the theories $\psi_{\lambda, s}$, $\phi_{\lambda, s}$ on $\mathcal{F}$, while $S_\lambda^0$ is the $S$-matrix for the theories $\psi_{\lambda, s}^0$, $\phi_{\lambda, s}^0$ on $\mathcal{F}^0$.

The validity of Eqs. (3.35)–(3.38) follows from rather involved arguments that will be presented elsewhere [23]. However, in the case of Eqs. (3.35) and (3.36) a simpler proof can be given, and will be sketched in Section 7A.

3C. The Physical Sector

In this section we proceed along the same lines as in Section 3B. It is therefore convenient to use the same notation for corresponding objects. Here, $\mathcal{H}(a, A)$ is again defined by Eq. (3.11), except that the factor $\delta_j$ is omitted in the exponential. The fields to be studied are again defined by Eqs. (3.14)–(3.19), but now one should replace $c_{s-1}(\theta)$ in Eqs. (3.16)–(3.19) by $c_{s-1}(\theta)$. Also, $\phi_{\lambda, s}(x)$ is here defined as explained in Section 3A (with $\phi_{\lambda, s, s'}(x)$ defined by Eq. (3.21) for $A = F, B$). Explicitly,

$$\phi_{\lambda, s}(x) = \exp\left[Z_{\lambda, s, ++}^{A, x} c_{s, -1}^* c_{s, -1}^{\dagger} + (Z_{\lambda, s, +-}^{A, x} - 1) c_{s, 1}^* c_{s, 1}^{\dagger}\right] \cdot \{F(0), B(0)\},$$  

(3.39)

where

$$Z_{\lambda, s, \delta \delta'}(\theta_1, \theta_2) = \exp[i \chi \cdot (\delta \rho_{\delta}(\theta_1) - \delta' \rho_{\delta}(\theta_2))].$$  

(3.40)

and

$$Z_{\lambda}(\theta) = \cos n \lambda \delta(\theta) + \frac{\sin \pi \lambda}{2\pi} \exp \left((\lambda - \left[\lambda + \frac{1}{2}\right]) \theta\right) \begin{pmatrix} P \sinh(\theta/2) & (-)^{(\lambda+1/2)} i P \\ (-)^{(\lambda-1/2)} i P & \cosh(\theta/2) \end{pmatrix},$$  

$$A := F(0) \quad (3.41)$$
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(cf. Eqs. (2.43), (2.73)), whereas

\[ Z_\lambda^A(\theta) = \cos \pi \lambda \delta(\theta) + \frac{\sin \pi \lambda}{2\pi} \exp \left( \left( \lambda - \frac{1}{2} \right) \theta \right) \frac{P}{\sinh(\theta/2)} \frac{(-1)^{|A|}}{\cosh(\theta/2)} \frac{(-1)^{|A|}}{\cosh(\theta/2)} \frac{P}{\sinh(\theta/2)}. \]

A = B_\omega \quad (3.42)

(cf. Eqs. (2.61), (2.85)). As in Section 36, the fields are well-defined quadratic forms on \( D \times D \), but now we shall only consider \( \psi^{(0)}_\lambda, s \) on \( \mathcal{F}_\lambda \) and \( \phi^{(0)}_\lambda, s \) on \( \mathcal{F}_\lambda \). (Note that the free fields \( \psi^{(0)}_\lambda, s \) and \( \phi^{(0)}_\lambda, s \) only satisfy equal time canonical (anti)commutation relations on \( \mathcal{F}_\lambda \) and \( \mathcal{F}_\lambda \) resp.; the fields are non-local on the “wrong statistics” Fock space.)

After observing that the fields defined above again satisfy the covariance relations (3.26)–(3.28), we now proceed to show heuristically that

\[ [\psi^{(+)}_\lambda, s(x), \psi^{(+)}_\lambda, s(y)] = 0, \quad (x - y)^2 < 0, \quad (3.43) \]

and that

\[ [\phi^{(+)}_\lambda, s(x), \phi^{(+)}_\lambda, s(y)] = 0, \quad (x - y)^2 < 0. \quad (3.44) \]

In Section 3A we have already argued that \( \phi^{(0)}_{\lambda, s} (A = F, B) \) satisfies (3.44). Taking this for granted, an easy calculation, using the fact that Eqs. (3.43)–(3.44) obviously hold for \( h = 0 \), shows that these equations are implied by the equations

\[ \psi^{(0)}_{\lambda, s}(x) \phi^F_{\lambda, s}(y) = \exp[-i\pi \lambda s(x^1 - y^1)] \phi^F_{\lambda, s}(y) \psi^{(0)}_{\lambda, s}(x), \quad (x - y)^2 < 0 \quad (3.45) \]

and

\[ \phi^{(0)}_{\lambda, s}(x) \phi^B_{\lambda, s}(y) = \exp[-i\pi \lambda s(x^1 - y^1)] \phi^B_{\lambda, s}(y) \phi^{(0)}_{\lambda, s}(x), \quad (x - y)^2 < 0. \quad (3.46) \]

respectively. We shall now prove Eqs. (3.45)–(3.46) for \( x = (0, x^1), \ y = (0, y^1) \). More precisely, we shall show that if one puts \( x^0 = y^0 = 0 \) in (3.45) and (3.46), and integrates the left- and right-hand sides with a test function in \( x^1 \), the resulting equations hold true in the sense of quadratic forms. The general case then formally follows from covariance.

First, let \( f \in S(R)^2 \). In obvious notation one then has

\[ \int dx^1 \bar{f}(x^1) \cdot \psi^{(0)}_{\lambda, s}(0, x^1) = \Phi_\lambda(W_s^{-1}f). \quad (3.47) \]

This easily follows from the definitions of \( \psi^{(0)}_{\lambda, s} \) (Eq. (3.16) with \( c_{s-1} \mapsto c^*_n-1 \), \( W \) (Eq. (2.23)), and the field operator \( \Phi_\lambda \) (Eq. (3.2); of course, \( a \mapsto c_{s+1} \), \( b^* \mapsto c^*_{n+1} \)). Now from the definition of \( \Phi^F_{\lambda, s}(y) \) (cf. Section 3A) one has

\[ \Phi_\lambda(W_s^{-1}f) \phi^F_{\lambda, s}(y) = \phi^F_{\lambda, s}(y) \Phi_\lambda(W_s^{-1} c^*_n) \Phi^F_{\lambda, s}(y)f. \quad (3.48) \]
But if \( y = (0, y') \), \( \phi_{s, \text{el}}^R(y) \) is multiplication by \( \exp[i \pi \lambda \epsilon (\cdot - y')] \) on \( \mathcal{H}_s \), so that Eq. (3.45) follows. Next, let \( g \in S(R) \). Then

\[
\int dx^1 \tilde{g}(x^1) \phi_{0, s}(0, x^1) = \Phi_s(\mid H_0 \mid^{-1} W_s^{-1} \tilde{g}),
\]

where \( \tilde{g} \) denotes \( (g, 0) \in \mathcal{H}_s \). (Again, this easily follows from the definition of the free field, Eq. (3.18) with \( c_{s,-1} \rightarrow c_{s,-1}^+ \), and Eq. (2.50).) Also, by definition,

\[
\Phi_s(\mid H_0 \mid^{-1} W_s^{-1} \tilde{g}) \phi_{n, s}^R(y) = \phi_{n, s}^R(y) \Phi_s(\phi_{s, \text{el}}^R(\cdot)^* \mid H_0 \mid^{-1} W_s^{-1} \tilde{g}).
\]

But a pseudo-unitary operator \( U \) satisfies

\[
U^* = qU^{-1}q.
\]

If, moreover, \( \tilde{U} \) acts as multiplication by a function on \( \mathcal{H} \) one has, using Eq. (2.47),

\[
U^* \mid H_0 \mid^{-1} W^{-1} \tilde{g} = qU^{-1}W^{-1} \tilde{H}_0^{-1} \begin{pmatrix} \tilde{g} \\ 0 \end{pmatrix} = \mid H_0 \mid^{-1} W^{-1} \tilde{H}_0 \tilde{U}^{-1} \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}.
\]

Now \( \phi_{s, \text{el}}^R(0, y)^{-1} \) acts as multiplication by \( \exp[i \pi \lambda \epsilon (\cdot - y')] \) on \( \mathcal{H}_s \) and is pseudo-unitary, so that Eq. (3.46) results.

Only the last part of this proof is rigorous, but we feel that the arguments presented above are a strong indication that the fields \( \psi_{s, s} \) and \( \phi_{s, s} \) are local in the usual smeared sense, given that the smeared fields make sense on the usual polynomial domain (cf., e.g., Ref. [37]). In any case, they explain why such a result is to be expected. We have restricted ourselves in the above to the massive fields, although similar arguments would "prove" locality for the massless fields \( \psi_{s, s} \) and \( \phi_{s, s} \). Our reason for this is that there is no hope of giving any precise sense to this notion in the latter case, since the fields either annihilate the vacuum or map it out of Fock space after smearing with any \( F \in S(R^2) \). (In contrast, \( \psi_{s, s, s}(F) \Omega \in \mathcal{F}_s \) and \( \phi_{s, s, s}(F) \Omega \in \mathcal{F}_s \) for all \( F \in S(R^2) \). We shall prove these assertions elsewhere [24].)

Nevertheless, the massless fields admit asymptotic fields in the LSZ sense, just like the massive ones, and this is our main reason for considering them (apart from the connection \( \psi_{s, s} \) has with the massless Thirring model, cf. Section 5B). Indeed, let us define the multiplication operators

\[
(S\lambda \psi)(\theta_1, s_1, \delta_1 ; \ldots; \theta_N, s_N, \delta_N) = \exp \left[ -i\pi \lambda \sum_{i<j} \epsilon(\theta_i - \theta_j)(s_i - s_j) \delta_i \delta_j \right] \psi(\theta_1, s_1, \delta_1 ; \ldots; \theta_N, s_N, \delta_N)
\]

(3.53)
and introduce in- and out-fields by Eqs. (3.33)-(3.34), where the free fields are now of course the physical free fields. Then Eqs. (3.35)-(3.38) hold true in the same sense as in Section 3B, so that $S_\lambda$ is the $S$-matrix for the physical sector fields $\psi_{\lambda,s}$ and $\phi_{\lambda,s}$, and $S_\lambda^0$ is the $S$-matrix for the physical sector fields $\psi_{0,s}^0$ and $\phi_{0,s}^0$. Of course, these facts imply that these field theories are asymptotically complete. Again, we shall relegate the rather technical proof of these assertions to Ref. [23].

4. THE FEDERBUSH MODEL

The Lagrangean of the classical Federbush model reads

$$\mathcal{L} = \sum_{s=\pm 1} \bar{\psi}_s (i\gamma^\mu \partial_\mu - m(s)) \psi_s - 2\pi \lambda \epsilon_{\mu\nu} J^\mu J^\nu_\perp,$$ (4.1)

where

$$\epsilon_{10} = -\epsilon_{01} = 1, \quad \epsilon_{00} = \epsilon_{11} = 0, \quad m(s) > 0, \quad s = \pm 1,$$ (4.2)

$$J^\mu_s = \bar{\psi}_s \gamma^\mu \psi_s, \quad \bar{\psi}_s = \bar{\psi}_s^0.$$

(It is to be noted that we use a tilde to denote the Pauli adjoint instead of the usual bar, since a bar denotes complex conjugation throughout this paper.) From this we obtain the equations of motion

$$\left( i(\partial_0 + \partial_1) - m(s) \right) \psi_s = 4\pi \lambda \epsilon_{\mu\nu} \left( | \psi_{-s,2} |^2 - | \psi_{-s,1} |^2 \right) \psi_s.$$ (4.3)

It easily follows that

$$\psi_{\lambda,s}(t, x^1) = \psi_{0,s}(t, x^1) \exp \left( i\pi \lambda s \int dy^1 \epsilon(y^1 - x^1) | \psi_{0,-s}(t, y^1) |^2 \right)$$ (4.4)

is a solution to (4.3) if $\psi_{0,s}$ is a free solution. To see this, note that (4.3) implies

$$\partial_0 (| \psi_{s,1} |^2 + | \psi_{s,2} |^2) = -\partial_1 (| \psi_{s,1} |^2 - | \psi_{s,2} |^2)$$ (4.5)

(current conservation), and use integration by parts. For a solution of the form (4.4) the Hamiltonian density corresponding to $\mathcal{L}$,

$$\mathcal{H} = \sum_{s=\pm 1} \bar{\psi}_s \left( -i\gamma^\mu \partial_\mu + \gamma^0 m(s) \right) \psi_{\lambda,s} + 2\pi \lambda \epsilon_{\mu\nu} J^\mu J^\nu_\perp,$$ (4.6)
can be written

$$\mathcal{H} = \sum_{s=\pm 1} \psi_{0,s}(-i\gamma^5 \partial_1 + \gamma^0 m(s)) \psi_{0,s}.$$  \hspace{1cm} (4.7)

Thus, the interacting Hamiltonian for $\psi_\lambda$ equals the free Hamiltonian for $\psi_0$.

Actually, it is not hard to see that the Cauchy problem may be uniquely solved for the nonlinear PDE (4.3), and that any solution may be written in the form (4.4). Also, the wave and scattering maps can be explicitly found. For precise statements the reader may consult Ref. [23]. Here, we shall only deal with the two second-quantized theories associated with the Lagrangean (4.1), viz., the unphysical, negative energy theory (Section 4A), and the physical positive energy theory (Section 4B). However, for this purpose it is convenient to add one more remark about the classical theory: it has infinitely many conserved polynomial “charges,” corresponding to the infinite number of conserved quadratic charges of the non-interacting theory. This readily follows from the fact that

$$\psi_{0,s}(t, x^1) = \psi_{\lambda,s}(t, x^1) \exp \left[ -i\pi \lambda s \int dx^1 \epsilon( y^1 - x^1) \psi_{\lambda,-s}(t, y^1) \right]$$  \hspace{1cm} (4.8)

is a free solution if $\psi_{\lambda,s}$ solves Eq. (4.3). Consider, e.g., the sequence of free conserved higher order Hamiltonians

$$H_n(\psi_0) = \sum_{s=\pm 1} m(s)^{1-n} \int dx^1 \psi_{0,s}(t, x^1) \left[ -i\gamma^5 \partial_1 + \gamma^0 m(s) \right] \psi_{0,s}(t, x^1).$$  \hspace{1cm} (4.9)

If one uses Eq. (4.8) this obviously leads to a sequence of conserved Hamiltonians $\tilde{H}_n(\psi_0)$ that are polynomial in $\psi_{\lambda,s}$ and $\psi_{\lambda,-s}$. (Of course, for these expressions to make sense, one needs to impose regularity and decay conditions on the initial data, e.g., that they be in Schwartz space.) At the quantum level, the existence of the corresponding conserved charges is a trivial consequence of the structure of the $S$-matrix. However, the question whether these charges admit expressions in terms of the non-perturbative field operators that have the same polynomial character as at the classical level needs to be reconsidered.

4A. The Unphysical Sector

We shall now introduce field operators on the unphysical sector that are the obvious analogs of the classical solution (4.4). On this sector the question of statistics is immaterial, and correspondingly we take as Hilbert space the Fock space $\mathcal{F}_{\epsilon}(\mathcal{H}_0 \oplus \mathcal{H}_{-1})$, where $\epsilon$ stands for a (antisymmetric) or $s$ (symmetric), and where $\mathcal{H}_\epsilon$ are copies of the space $\mathcal{H}$ of Section 2B. On $\mathcal{F}_\epsilon$ we define a Hamiltonian $\tilde{H}_0$ as the second quantization of $\tilde{H}_{0,m(\epsilon)} \oplus \tilde{H}_{0,m(-1)}$, where $\tilde{H}_{0,m(\epsilon)}$ is the free Dirac operator of mass $m(s)$. Now set

$$\tilde{\psi}_{\lambda,s}(0, x^1) = c_s(x^1) \exp \left[ i\pi \lambda s \int dx^1 \epsilon( y^1 - x^1) c_s^*(y^1) c_s(y^1) \right]$$  \hspace{1cm} (4.10)
and
\[ \psi_{\lambda,s}(t, x^1) = \exp(i\hat{H}_0 t) \psi_{\lambda,s}(0, x^1) \exp(-i\hat{H}_0 t). \]
(4.11)

Here,
\[ c_s^\dagger(x^1) = \left( \begin{array}{c} c_{s,1}^\dagger(x^1) \\ c_{s,2}^\dagger(x^1) \end{array} \right) \]
are annihilation (creation) operators on \( \mathcal{F} \). The above objects are all well defined as quadratic forms on a dense subspace of \( \mathcal{F}_c \), e.g., finite particle Schwartz space test functions. Since
\[ \psi_{0,s}(t, x^1) = \exp(i\hat{H}_0 t) c_s(x^1) \exp(-i\hat{H}_0 t) \]
(4.12)
satisfies the free Dirac equation with mass \( m(s) \), Eqs. (4.10) and (4.11) are the analogs of Eq. (4.4).

If one disregards non-commutation and domain problems, Eq. (4.3) is easily verified for \( \psi_{\lambda,s} : \) one simply uses
\[ \partial_x \exp(f(x)) = \exp(f(x)) f'(x) \]
(4.13)
for \( f(x) \) an \( x \)-dependent operator, and proceeds as in the verification that (4.4) is a solution to the classical PDE. Unfortunately, in our case the operator \( f(x) \) is only differentiable as a quadratic form on a dense set, and the result \( f'(x) \) is only a form. Moreover, the question whether \( \psi_{\lambda,s} \) and \( f(x) \) commute (which is necessary for (4.13) to hold) has no unambiguous meaning, since the product is ill defined to begin with. These mathematical remarks may not appeal to some readers, but here is a final and perhaps more convincing argument that not all is well with the “solution” (4.10)–(4.11): it is periodic in \( \lambda \), while the equation of motion (4.3) is not.

Let us therefore proceed more carefully. To obtain the precise equation of motion, and also to establish the connection with the field \( \psi_{\lambda,s} \) of Section 3B, it is convenient to transform to the rapidity Fock space \( \mathcal{F}_c(\mathcal{H}_1 \oplus \mathcal{H}_s) \) by using the unitary operator
\[ \mathcal{W} = \Gamma(W_{m(1)} \oplus W_{m(-1)}), \]
(4.14)
where \( W_m \) is defined in Section 2B. For the free field one gets
\[ \mathcal{W}^{-1} \psi_{0,s}(t, x^1) \mathcal{W} = \psi_{0,s}(t, x^1), \]
(4.15)
where \( \psi_{0,s} \) is defined by Eq. (3.16). Also a moment’s thought shows that the second factor on the r.h.s. of Eq. (4.10) is just the operator \( \Gamma_{--}(\psi_{\lambda,s}^F(0, x^1)) \), so that in view of Eqs. (3.14) and (3.20) and the covariance of the classical field operator one obtains more generally
\[ \mathcal{W}^{-1} \psi_{\lambda,s}(t, x^1) \mathcal{W} = \psi_{\lambda,s}(t, x^1). \]
(4.16)
As in Section 3B we shall regard $\psi_{\lambda,s}(x)$ as a quadratic form on $\mathcal{D} \times \mathcal{D}$, where $\mathcal{D}$ is the dense subspace of $\mathcal{F}$ introduced below Eq. (3.25). For any $\phi_1, \phi_2 \in \mathcal{D}$ the function

$$\alpha(x) = \langle \phi_1, \psi_{\lambda,s}(x) \phi_2 \rangle$$  \hspace{1cm} (4.17)$$

is $C^\infty$ in $x^0$ and $x^1$, so that it makes rigorous mathematical sense to act on it with the differential operator at the l.h.s. of Eq. (4.3). We claim that one obtains

$$\left( i(\partial_0 + \partial_1) \begin{pmatrix} -m(s) \\ -m(s) \end{pmatrix} \begin{pmatrix} -m(s) \\ i(\partial_0 - \partial_1) \end{pmatrix} \right) \alpha(x)$$

$$= 4s \sin \pi \lambda \left\langle \phi_1, \begin{pmatrix} \psi_{\psi_{-s,2}}(x) & \psi_{\psi_{-s,2}}(x) \\ -\psi_{\psi_{-s,1}}(x) & \psi_{\psi_{-s,1}}(x) \end{pmatrix} \psi_{\lambda,s}(x) : \phi_2 \right\rangle.$$  \hspace{1cm} (4.18)

To see this, note that $\phi_1(x)$ satisfies Eq. (4.3) with $h = 0$ in the sense of forms, so that in view of Eq. (3.14) one only needs to verify that

$$i(\partial_0 \pm \partial_1) \psi_{\lambda,s}(x) = \mp 4s \sin \pi \lambda : \psi_{\psi_{-s,2}}(x) : \psi_{\psi_{-s,2}}(x) : \psi_{\psi_{-s,1}}(x) : \psi_{\psi_{-s,1}}(x):$$  \hspace{1cm} (4.19)

as forms on $\mathcal{D} \times \mathcal{D}$. But Eq. (4.19) follows from the explicit formulas (3.16) and (3.22)-(3.24) and a very useful property of normal ordered exponentials: they do satisfy Eq. (4.13) if $A(x)$ is a quadratic form in annihilation and creation operators, provided that the l.h.s. and r.h.s. are understood as normal ordered. The proof of this fact is easy and will be omitted.

Summarizing, we have proved that

$$\left( i(\partial_0 + \partial_1) \begin{pmatrix} -m(s) \\ -m(s) \end{pmatrix} \begin{pmatrix} -m(s) \\ i(\partial_0 - \partial_1) \end{pmatrix} \right) \psi_{\lambda,s} = 4s \sin \pi \lambda : \psi_{\psi_{-s,2}}(x) : \psi_{\psi_{-s,2}}(x) : \psi_{\psi_{-s,1}}(x) : \psi_{\psi_{-s,1}}(x):$$  \hspace{1cm} (4.20)

in the sense of quadratic forms on $\mathcal{D} \times \mathcal{D}$. Let us compare this with the classical equation of motion (4.3). Apart from the conspicuous replacement of $\pi \lambda$ by $\sin \pi \lambda$, Eq. (4.20) is an eminently reasonable quantum “translation” of Eq. (4.3): the field $\psi_{\lambda,s}(x)$ equals $\psi_{\psi_{0,s}}(x)$ times an $x$-dependent unitary operator, so that the obvious definition of the a priori undefined quadratic form product $\psi_{\psi_{-s,2}}(x) \psi_{\psi_{-s,2}}(x)$ is that it equal $\psi_{\psi_{-s,1}}(x) \psi_{\psi_{-s,1}}(x)$. Moreover, Wick ordering is a well-known feature of the translation of classical products into quantum products, so that the field $\psi_{\lambda,s}$ seems an acceptable analog of the classical solution $\psi_{\lambda,s}$ but for the change in coupling constant. One may argue that this change is immaterial, since one can multiply $\psi_{\lambda,s}$ by $(\pi \lambda / \sin \pi \lambda)^{1/2}$ to get the desired coupling constant in Eq. (4.20). However, if one also requires canonical anti-commutation relations at equal time this is not allowed, since $\psi_{\lambda,s}(0, x^1)$ is unitarily equivalent to $\psi_{\psi_{0,s}}(0, x^1)$ (cf. Section 7A), and therefore canonical.

Nevertheless, it is possible to get the correct coupling constant and a canonical solution, provided $|\lambda| \leq 1/\pi$: One simply replaces the $\lambda$ occurring in $\psi_{\lambda,s}$ by

$$f(\lambda) = \frac{1}{\pi} \arcsin \pi \lambda.$$  \hspace{1cm} (4.21)
Thus, for a restricted range of coupling constants one can solve Eq. (4.3) at the quantum level. We consider it very likely that no canonical solution exists for $\lambda$ outside this range.

As we have just shown, the field operator $\psi_{\lambda,s}$ as defined in Subsection 3B and considered above, does not satisfy the quantum analog of Eq. (4.3). It is however an interesting object in its own right, and we shall reconsider it in Section 7A. In particular we shall study the question raised at the end of the introduction of this chapter.

4B. The Physical Sector

It is convenient to begin this section by repeating the definition of the physical sector field operator $\psi_{\lambda,s}$ of Section 3B for $|\lambda| < \frac{1}{2}$. It reads

$$\psi_{\lambda,s} = \psi_{0,s} \phi_{\lambda,-s}^F .$$

(4.22)

where $\psi_{0,s}$ is the free Dirac field

$$\psi_{0,s}(x) = \left( \frac{m(s)}{4\pi} \right)^{1/2} \int d\theta \left[ c_{s,1}(\theta) \left( \frac{e^{\theta/2}}{e^{-\theta/2}} \right) e^{-ix_{\mu,\nu}} + c^*_{s,-1}(\theta) \left( \frac{i e^{\theta/2}}{-i e^{-\theta/2}} \right) e^{ix_{\mu,\nu}} \right].$$

(4.23)

and

$$\phi_{\lambda,s}^F(x) = \exp[Z_{\lambda,s,+}^F c_{s,1}^* c_{s,-1}^* - (Z_{\lambda,s,+}^F - 1) c_{s,1}^* c_{s,-1}^* - (Z_{\lambda,s,-1}^F - 1) c_{s,-1}^* c_{s,1}] .$$

(4.24)

In Eq. (4.24) the kernels are given by

$$Z_{\lambda,s,\delta\delta'}^F(\theta_1, \theta_2) = \exp[ix \cdot (\delta p_s(\theta_1) - \delta' p_s(\theta_2))] Z_{\lambda,s,\delta\delta'}^F(\theta_1 - \theta_2),$$

(4.25)

where

$$Z_\lambda^F(\theta) = \cos \pi \lambda \delta(\theta) + \frac{\sin \pi \lambda}{2\pi} e^{i\theta} \begin{pmatrix} P & i \\ \sinh \theta/2 & \cosh \theta/2 \end{pmatrix}, \quad |\lambda| < \frac{1}{2} .$$

(4.26)

Up to irrelevant constants the above fields $\psi_{\lambda,s}$ are equal to the fields employed by Schroer et al. [3] in their study of the Federbush model, except that they use the kernels (4.26) without the restriction $|\lambda| < \frac{1}{2}$. Their use of these fields was based on previous work of Wightman [2] and Lehmann and Stehr [28]. Wightman had started from the fields $\psi_{\lambda,s}$ on the unphysical sector (cf. Section 3B), and followed Thirring [4] and Glaser [5] in making the "hole theory" substitution $c_{s,-1}(\theta) \rightarrow c_{s,-1}(\theta)^*$ not only in the free unphysical field $\psi_{0,s}$ but also in the unphysical field $\phi_{\lambda,s}^F$, so as to obtain a solution on the physical sector. More in detail, we have seen above that on the unphysical sector

$$\phi_{\lambda,s}^F(x) = \exp \left[ i\pi \lambda s \sum_{\delta,\delta' \neq \pm 1} U_{\delta\delta'}^F c_{s,\delta}^* c_{s,\delta'} \right] ,$$

(4.27)
where
\[ U^F_{66'}(\theta_1, \theta_2) = \exp \left[ i x \cdot \left( \delta p_s(\theta_1) - \delta p_s(\theta_2) \right) \right] U^F_{\delta \delta'}(\theta_1 - \theta_2) \quad (4.28) \]

and
\[ U^F(\theta) = \frac{1}{2\pi i} \begin{pmatrix} P & i \cosh \theta/2 & i \sinh \theta/2 \\ -i \cosh \theta/2 & P & -i \sinh \theta/2 \\ -i \sinh \theta/2 & i \cosh \theta/2 & P \end{pmatrix}. \quad (4.29) \]

Making the substitution mentioned above, and normal ordering the resulting exponent so as to turn it into a well-defined quadratic form, Wightman arrived at the physical sector field
\[ \phi^W_{\lambda,s}(x) = \exp(i\pi\lambda s) \left[ U^F_{-+}c_{s,-1}^*c_{s,s} + U^F_{++}c_{s,s}^*c_{s,-1}^* + U^F_{-+}c_{s,-1}^*c_{s,s} + U^F_{++}c_{s,s}^*c_{s,-1} \right] \quad (4.30) \]

Here, the triple dots stand for vacuum subtractions [2], without which the exponential is meaningless. Lehmann and Stehr [28] then argued that the normal form of \( \phi^W_{\lambda,s}(x) \) is just \( \phi^F_{\lambda,s}(x) \).

This result can be easily understood in the framework of Bogoliubov transformations. Indeed, the field \( r_{-,-} \) satisfies the formal commutation relations (cf. Ref. [2], Eq. (4.107)), and has vacuum expectation value one, just like the field \( +I,B \) (cf. Eq. (3.45)). Thus, if the quadratic form implementing the improper Bogoliubov transformation generated by \( \phi^F_{\lambda,0}(x) \) is unique up to a constant (just like the unitary operators implementing proper Bogoliubov transformations), then the two fields must coincide. We do not know whether this is generally true in the form case, though this seems very plausible if \( U \) and its conjugate \( Z \) are bounded. (If \( Z \) is unbounded, uniqueness does not necessarily hold, but this is due to the restriction on the domain. An example of this occurs in the continuum Ising model, cf. Section 6C.) However, we do know that if a quadratic form is of the form of Eq. (4.24) and satisfies the commutation relations (4.31), the kernels \( Z^F_{\lambda,s,\delta \delta',s} \) are uniquely determined. (More generally, if Eq. (3.9) holds, the kernels \( Z_{\delta \delta'} \) in the normal ordered exponential \( E(Z) \) are uniquely determined by \( U \), if \( U \) admits a bounded conjugate given by (2.15). For instance, let \( f, v \in H_\lambda \) and evaluate the expectation \( \langle b^* f, \Omega \rangle \) of (3.9). This gives \( \langle v, U_- Z_- f \rangle = \langle v, f \rangle \), implying \( Z_- = U_-^{-1} \).) Thus, if one can argue that the normal form of \( \phi^W_{\lambda,s} \) should be of the form of Eq. (4.24) (as Lehmann and Stehr do), the kernels must be given by Eqs. (4.25)–(4.26) for \( |\lambda| < \frac{1}{2} \), and more generally by Eqs. (3.40)–(3.41) for any \( \lambda \). This “explains” the result of Lehmann and Stehr, except that they get Eq. (4.26) for any \( \lambda \), instead of the correct Eq. (3.41). This is presumably due to non-convergence of the exponential at the right-hand side of Eq. (4.30) for \( |\lambda| > \frac{1}{2} \).
it could also be that the formal arguments of Ref. [28] are not valid for \( |\lambda| > \frac{1}{2} \) for other reasons. In this connection we should like to mention that Glaser [5], who solved essentially the same problem for the massless Thirring model (cf. Section 5B) did get the correct periodic dependence on \( \lambda \).

The fields \( \phi^{\lambda}_{\alpha}(x) \) with kernels (4.26) for any \( \lambda \) were also recently studied by Sato et al. (cf. Ref. [14] and references given there). Likewise, they studied \( \phi^{\lambda}_{\alpha}(x) \), for which

\[
Z^{R}(\theta) = \cos \pi \lambda \delta(\theta) + \frac{\sin \pi \lambda}{2\pi} \left( \frac{1}{\cosh \theta/2} \right) \left( \frac{1}{\cosh \theta/2} \right), \quad 0 < \lambda < 1
\]

(4.32)

by using the right-hand side for any \( \lambda \), instead of the periodic kernels (2.61). (Actually, their fields are slightly different, but in an irrelevant way.) The preceding paragraph shows that a number of their formal commutation relations [14] are in error for \( |\lambda| > \frac{1}{2} \) and \( \lambda \notin [0, 1] \) resp. We should also like to point out that it is by no means obvious that convolution with the right-hand side of Eqs. (4.26) and (4.32) is a densely defined operator on \( \mathcal{H} \) for \( |\lambda| > \frac{1}{2} \) and \( \lambda \notin [0, 1] \) resp. (apart from the case \( \lambda \in Z \) of course). Indeed, the kernels do not define tempered distributions, and therefore do not admit a Fourier transform for these \( \lambda \)-values. Also, the convolution operation is of the form \( A^{-1}BA \), where \( B \) is a bounded convolution operator, and \( A \) the unbounded operator of multiplication by \( e^{i\theta} \), and it is not clear to us that this product makes sense on a dense subspace.

As we have just described, the above fields with kernels (4.26) and (4.32) for any \( \lambda \) have been used by several authors. Let us therefore add some remarks on these fields. Firstly, it is easy to see that they are well defined as quadratic forms on \( \mathcal{D} \times \mathcal{D} \), where \( \mathcal{D} \) is the subspace defined below Eq. (3.25), and their “matrix elements” \( \langle \psi_1, ..., \psi_2 \rangle \) (where \( \psi_1, \psi_2 \in \mathcal{D} \)) are entire in \( \lambda \), in contrast to those of our periodic fields, which are discontinuous for \( \lambda \in Z + \frac{1}{2} \) and \( \lambda \in Z \) resp. Secondly, they lead to the same LSZ \( S \)-matrix as the one found in Section 3C. This can be seen from the results of Ref. [23]. Thirdly, if one smears the fields with \( F \in S(R^3) \) for which the Fourier transform is compact support, the result is an operator-valued distribution (in the \( \mathcal{D} \)-topology on \( \mathcal{F} \)) defined on the vacuum and the ensuing polynomial domain. Also, in the boson case one can show that \( \phi^{\lambda}_{\alpha}(F)\Omega \) and \( \phi^{\lambda}_{\alpha}(F)\Omega \) are in Fock space for all \( F \in S(R^3) \) only if \( 0 \leq \lambda \leq 1 \); for other \( \lambda \)-values the extra factor \( e^{i\lambda \theta} \) arising when Eq. (4.32) is used restricts the admissible test functions (cf. Ref. [24]; we believe that the situation is similar in the fermion case, but have not proved this so far).

Let us now proceed to establish whether the field \( \psi^{\alpha}(x) \) satisfies the equations of motion (4.3) in any reasonable sense. Proceeding as in Section 4A, a calculation shows that for \( |\lambda| < \frac{1}{2} \) one has

\[
\left( \begin{array}{cc}
(i\partial_{0} + \partial_{1}) & -m(s) \\
-m(s) & i(-\partial_{0} - \partial_{1})
\end{array} \right) \psi^{\alpha} = 4s \sin \pi \lambda \left( \begin{array}{c}
\psi^{*} \psi^{*} \psi^{*} \psi^{*} \psi^{*} \\
\psi^{*} \psi^{*} \psi^{*} \psi^{*} \psi^{*}
\end{array} \right) \psi^{\alpha}.
\]

(4.33)
in the sense of quadratic forms on $\mathcal{D} \times \mathcal{D}$, where we have introduced

$$\psi_{0,\lambda,s}(x) := \left(\frac{m(s)}{4\pi}\right)^{1/2} \int d\theta \, e^{i\theta} \left[ c_{\lambda,1}(\theta) \left( e^{\theta/2} e^{-i\pi \cdot \nu_s} + c_{\lambda,-1}(\theta) \left( -ie^{\theta/2} e^{i\pi \cdot \nu_s} \right) \right] \right].$$

(4.34)

Thus, this new field equals the free field $\psi_{0,\lambda}(x)$ (cf. Eq. (4.23)), except that an additional factor $e^{i\theta}$ occurs in the integrand, which renders it a non-local free field.

We feel that Eq. (4.33) is a most disturbing result if one insists on regarding $\psi_{\lambda,s}$ as a quantum solution to the nonlinear PDE (4.3), as it seems hard to believe that any rigorous definition of the formal product $\mid \psi_{\lambda,-s,i} \mid^2 \psi_{\lambda,s,i}$ occurring in (4.3) can amount to the right-hand side of (4.33). In our opinion, just like on the unphysical sector, the obvious and natural definition would be $\psi_{0,-s,i}^* \psi_{0,-s,i} \psi_{\lambda,s,i}^*$. Note that this amounts to putting

$$r_{\lambda}(x) \ast r_{\lambda}(x) = 1.$$ 

(4.35)

We have two reasons for this definition. Firstly, this would reflect a property of the classical solutions, viz., that $\mid \psi_{\lambda,-s,i} \mid^2 = \mid \psi_{0,-s,i} \mid^2$. Secondly, for $\mid \lambda \mid < \frac{1}{2}$ there exist classical unitary operators $U^F_{\lambda,\epsilon}$ with the properties:

(i) $s \cdot \lim_{\epsilon \to 0} U^F_{\lambda,\epsilon} = U^F_{\lambda}$;

(ii) $(U^F_{\lambda,\epsilon})_{++}$ are Hilbert–Schmidt;

(iii) $(\Omega, \mathcal{U}_{-s,\lambda} \Omega)^{-1} \mathcal{U}_{-s,\lambda} \to U^F_{\lambda,\epsilon}(0)$ for $\epsilon \downarrow 0$ in the sense of forms on $\mathcal{D} \times \mathcal{D}$, where $\mathcal{U}_{\lambda,\epsilon}$ is the unitary operator implementing the transformation generated by $U^F_{\lambda,\epsilon}$. (For example, one can show (i), (ii)/(iii) are satisfied for $\mid \lambda \mid < \frac{1}{2}$ by the unitary multiplication operator

$$U^F_{\lambda,\epsilon} = \frac{1}{\sinh \pi(x^1/\epsilon - i\lambda)} \left[ \sinh \frac{x^1}{\epsilon} - i \sin \pi \lambda \right]$$

(4.36)

on $\mathcal{H}$. However, we are not certain that (iii) holds for $\frac{1}{4} \leq \mid \lambda \mid < \frac{1}{2}$.) Then the obvious cutoff field operator $\phi^F_{\lambda,s,e}(x)$ satisfies

$$\phi^F_{\lambda,s,e}(x)^* \phi^F_{\lambda,s,e}(x) = (\Omega, \mathcal{U}_{-s,\lambda,\epsilon} \Omega)^{-2}.$$ 

(4.37)

For $\epsilon \downarrow 0$ the constant diverges, but if we renormalize it to one, it is clear that our definition of the triple product results after normal ordering.

If one agrees that the field products should be defined in the way just suggested, it is obvious that $\psi_{\lambda,s}$ is not a solution. Thus, one is naturally led to the question whether the equation of motion so defined can be solved at all. This question has an affirmative answer: a solution is obtained by simply omitting the factor $e^{i\theta}$ in Eq. (4.26) and the corresponding kernels, since this factor was responsible for “changing” $\psi_{0,s}$ to $\psi_{0,\lambda,s}$. (However, note that without this factor in $\phi^F_{\lambda,\epsilon}$ it is unlikely that one can find a cutoff field equal to a unitary $x$-dependent operator times an $x$-independent constant, so that our product definition is less cogent.) From the results of Ref. [23] one sees that
the corresponding field $\xi_{A,s}$ leads to the same $S$-matrix as $\psi_{A,s}$, so that $\xi_{A,s}$ is physically equivalent to $\psi_{A,s}$. However, we see no reason why $\xi_{A,s}$ should be local, since the argument showing locality of $\psi_{A,s}$ (cf. Section 3C) is clearly false for $\xi_{A,s}$.

In summary, it seems that the obvious local quantum field associated with (4.3) is not a solution, while the obvious quantum field solution is not local. We have disregarded the coupling constant dependence we discussed in some detail in Section 4B, since there is now no reason why one could not multiply the field by a scale factor, thereby obtaining any arbitrary coupling constant. (This is why we restricted ourselves to $|\lambda| < \frac{1}{2}$.) The point is that in this case the fields $\psi_{A,s}$ and $\xi_{A,s}$ clearly do not satisfy equal-time canonical anticommutation relations, since the measure $p_2(M^2)$ in the Källen-Lehmann representation of their two-point functions (cf. Ref. [29]) is already equal to a $\delta$-function at $m(s)^2$, and obviously $>0$ for $M > m(s) + 2m(-s)$. Worse still, there is no hope of obtaining canonical fields by a rescaling, since the time-zero field does not exist as an operator: given $f \in S(R)^2$ one has $\int dx f(x^1) \cdot \psi_{A,s}(0, x^1) \Omega \in \mathcal{F}$ only if $f = 0$. I like other technical results on two-point functions mentioned in this paper we shall prove this in Ref. [24], but in this case the reader may in fact easily verify the assertion himself: already the three-body component is not square-integrable by an almost trivial application of Fubini's theorem.

Let us finally consider the question whether the Hamiltonian $H_0$ may be expressed as an integral over space of the right-hand side of Eq. (4.6) at time zero. This is tantamount to asking if one can give a definition of the quadratic form products such that a quadratic form on, e.g., $\mathcal{D} \times \mathcal{D}$ results that on integration over space is the form of $H_0$. If one agrees with our proposal above for products $\psi_{A,s}(x) \psi_{B,s}(x)$, i.e., that they be understood as $\psi_{A,s}(x) \psi_{B,s}(x)$, the interaction term at the right-hand side of Eq. (4.6) clearly equals

$$-4\pi \lambda \sum_{s = \pm 1} \psi_{0,s,2}(0, x^1) \psi_{0,2,1}(0, x^1)$$

(4.38)

(In view of the remarks above we may replace $\lambda$ by any convenient function $g(\lambda)$, if need be. Note also that the two factors in (4.38) commute since they act on different factors of $\mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_{-1})$.) The other term equals

$$\sum_{s = \pm 1} \phi_{s,1}(0, x^1) \phi_{s,2}(0, x^1) \psi_{0,s}(0, x^1)$$

$$\sigma \sum_{s = \pm 1} \psi_{s,2}(0, x^1) \phi_{s,3}(0, x^1) \phi_{s,4}(0, x^1)$$

if one uses (4.35) in the first term, normal orders it and integrates over space one clearly obtains $H_0$. Thus the second term should cancel (4.38) on integration over space. However, we see no way of defining the term $\phi_{s,1} \phi_{s,2}$ so that this happens, except by decreeing that it equal $2\pi \lambda \sigma \psi_{s,1} \psi_{s,2}$, for which we fail to see any other reason than that this holds true at the classical level.

It seems likely to us then, that no cogent definition of the form products can be given such that $H_0$ results, and we see no reason why the situation for the higher conserved
charges should be better. On the unphysical sector we shall in fact prove that the charges cannot be expressed in the non-perturbative field operators in the way suggested by the classical polynomial expressions (cf. Section 7A). We feel that these results indicate that the corresponding assumption in the case of more complicated models like the positive energy massive Thirring and nonlinear $\sigma$-models is ill founded, in spite of its success in perturbation theory (cf. Refs. [30, 31] and references given therein).

5. The Massless Thirring Model

The classical massless Thirring model is described by the Lagrangean

$$L = \bar{\psi}(i\gamma^\mu \partial_\mu)\psi - \pi \lambda J^\mu J_\mu,$$

(5.1)

where

$$J^\mu = \bar{\psi} \gamma^\mu \psi, \quad \bar{\psi} = \bar{\psi} \gamma^0.$$

(5.2)

The resulting equations of motion can be written

$$i\left(\left(\begin{array}{c} \partial_0 + \partial_1 \\ \partial_0 - \partial_1 \end{array}\right) \begin{array}{c} \psi_1 \\ \psi_{-1} \end{array}\right) = 4\pi\lambda \left(\begin{array}{cc} |\psi_{-1}|^2 & \psi_{-1} \\ \psi_1 & |\psi_1|^2 \end{array}\right).$$

(5.3)

Comparing this with Eq. (4.3) one sees that the Federbush model reduces to two decoupled massless Thirring models with coupling constants $\lambda$ and $-\lambda$ if one lets $m(s) \downarrow 0$. Correspondingly, we shall see that the massless Thirring model admits a treatment that is completely analogous to that of the Federbush model. The general solution

$$\psi = \left(\begin{array}{c} \psi_0 \\ \psi_{-1} \end{array}\right)$$

can be written

$$\psi_0^{\lambda}(t, x^1) = \psi_0^{\lambda}(t, x^1) \exp\left[i\pi\lambda s \int dy^1 \epsilon(y^1 - x^1) |\psi_0^{\lambda}(t, y^1)|^2\right],$$

(5.4)

where

$$\left(\begin{array}{c} \psi_0 \\ \psi_{-1} \end{array}\right)$$

solves the free massless Dirac equation, and the interacting Hamiltonian density

$$\mathcal{H} = \bar{\psi}_0^{\lambda}(\gamma^5 \partial_1) \psi_0^{\lambda} + \pi \lambda J^\mu J_\mu,$$

(5.5)

equals the free Hamiltonian density

$$\mathcal{H} = \bar{\psi}_0^{\lambda}(\gamma^5 \partial_1) \psi_0^{\lambda}. $$

(5.6)
The remarks concerning conserved charges made in Chapter 4 apply here as well, and will not be repeated. We proceed instead to consider the quantized theories.

5A. The Unphysical Sector

As in Section 4A we first use the Fock space $\mathcal{H}(\mathcal{H}_1 \oplus \mathcal{H}_\lambda)$, but here $\mathcal{H}_\lambda$ is the space defined in Section 2D. We shall denote the second quantization of the operator $\hat{H}_0$ of Section 2D by $\hat{H}_0^\circ$. The analog of the classical solution $\psi_{\lambda,s}$ is the field

$$\psi^\circ_{\lambda,s}(t, x^1) = \exp(i\hat{H}_0^0 t) \psi^0_{\lambda,s}(0, x^1) \exp(-i\hat{H}_0^0 t), \quad (5.7)$$

where

$$\psi^0_{\lambda,s}(0, x^1) = c_s(x^1) \exp \left[ i\pi \lambda s \int_{-\infty}^{\infty} dy^1 \epsilon(y^1 - x^1) c^*_s(y^1) c_s(y^1) \right]. \quad (5.8)$$

Transforming to the rapidity Fock space $\mathcal{H}(\mathcal{H}_1 \oplus \mathcal{H}_\lambda)$ by means of the unitary operator

$$W^{-1} \psi^\circ_{\lambda,s}(t, x^1) W^{-1} = \psi^\circ_{\lambda,s}(t, x^1), \quad (5.9)$$

where $\psi^\circ_{\lambda,s}$ is defined in Section 3B. Proceeding as in Section 4A, it follows that $\psi^\circ_{\lambda,s}$ satisfies the equation of motion

$$i \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x^1} \right) \psi^\circ_{\lambda,s} = 4 \sin \pi \lambda \begin{pmatrix} \psi^0_{\lambda,1} \\ \psi^0_{\lambda,-1} \end{pmatrix}, \quad (5.11)$$

in the sense of quadratic forms on $\mathcal{Q} \times \mathcal{Q}$. The discussion following Eq. (4.20) can now be repeated with obvious changes, and will therefore be omitted.

Finally, let us consider in what sense, if any, the quantum Federbush model on the unphysical sector reduces to two decoupled Thirring models in the limit $m(s) \downarrow 0$. We first note that no sensible limit results if we remain on the rapidity Fock space. Instead, we should transform both theories to the momenta Fock space, and take the limit there. On this space it is obvious that the representation of the Poincaré group strongly converges to the tensor product of two massless representations that are equivalent to the massless Thirring representation, and that a similar result holds true for the free fields. However, it is not hard to see that the form limits of the Federbush field $\psi_{\lambda,s}$ on this space differ from the Thirring fields $\psi^0_{\lambda,s}$, $\psi^0_{\lambda,-s}$ by the occurrence of an additional $x$-dependent unitary operator; that this is bound to occur can already be seen at the classical level by taking the $m(s) \downarrow 0$ limit in the representation (4.4) of the general Federbush solution, and comparing the result with the representation (5.4) of the general Thirring solution. It is also obvious that the $S$-matrix $S_\lambda$ of the Federbush model (cf. Eq. (3.31)) does not reduce to $S^0_\lambda \otimes S^0_{-\lambda}$ in this limit, where $S^0_\lambda$ is the $S$-matrix (3.32) of the massless Thirring model on the unphysical sector. Taking, e.g.,
$m(1) = m(-1) = m \downarrow 0$, two unequal species positive energy Federbush particles with positive and different momenta always scatter, whereas two positive energy Thirring particles with positive momenta do not scatter. In Section 5B we shall consider the analogous question for the positive energy Federbush and massive Thirring models, in Section 7A for the massive Thirring model on the unphysical sector (remark (ii)).

5B. The Physical Sector

First, let us repeat the definition of the field $\psi_{\lambda, s}^0$ of Section 3C for $|\lambda| < \frac{1}{2}$:

$$\psi_{\lambda, s}^0 = \psi_{0, s}^0 \phi_{\lambda, -s}^F,$$

where

$$\psi_{0, s}(x) = \left(\frac{1}{2\pi}\right)^{1/2} \int d\theta \ e^{i\theta/2} [e_{s,1}(\theta) e^{-ix \cdot p_s} + ise_{s,-1}(\theta) e^{ix \cdot p_s}],$$

and where $\phi_{\lambda, s}^F$ is given by Eqs. (4.24)-(4.26), the sole difference being that now of course $p_s(\theta) = (e^{i\theta}, e^{i\theta})$. The field $\psi_{\lambda, s}^0$ is essentially equal to the field that Glaser [5] obtained in his study of the massless Thirring model after rearranging the right-hand side of (4.30) without triple dots, and after omitting the infinite wave function renormalization constant due to the absence of triple dots.

The situation as regards the equation of motion is analogous to Section 4B. Now one obtains

$$i \left( (\bar{\partial}_0 + \partial_1) \psi_{\lambda, 1}^0 \right) = 4 \sin \pi \lambda \left\{ \begin{array}{ccc} \psi_{\lambda, -1}^0 & \psi_{\lambda, 1} - 1 & \psi_{\lambda, 1}^0 \\ \psi_{\lambda, -1}^0 & \psi_{0, \lambda, 1} & \psi_{\lambda, 1} \\ \psi_{0, \lambda, 1} & \psi_{\lambda, 1} \\ \psi_{\lambda, 1} \\ \psi_{\lambda, 1} \\ \psi_{\lambda, 1} \end{array} \right\},$$

where

$$\psi_{0, \lambda, s}(x) = \left(\frac{1}{2\pi}\right)^{1/2} \int d\theta \ e^{i\theta/2} [e_{s,1}(\theta) e^{-ix \cdot p_s} + ise_{s,-1}(\theta) e^{ix \cdot p_s}].$$

Clearly, $\psi_{0, \lambda, s}^0$ is a non-local free field with anomalous dimension. If one settles for the product definition of Section 4B, this shows again that $\psi_{\lambda, s}^0$ is not a solution to the equation of motion. As before, the field $\xi_{\lambda, s}^0$ defined by omitting the factor $e^{i\theta}$ in Eq. (4.26) is a solution, and leads to the same $S$-matrix $S_{\lambda}^0$ (cf. Eq. (3.54)) as $\psi_{\lambda, s}^0$ [23]. Since neither $\psi^0$ nor $\xi^0$ give rise to an operator on smearing with any $F \in S(R^2)$ [24], the formal non-locality of $\xi^0$ is irrelevant in this case. However, just as in the Federbush case the second argument for our product definition is most likely invalid for $\xi^0$, and the reader may feel one should not base such definitions solely on a classical consideration.

The question of the feasibility of expressing the conserved charges in terms of the fields can be discussed along the same lines as in Section 4B, so that we shall not comment on this any further.

The remarks concerning the mass-zero limit of the Federbush model made in Section 5A also hold true on the physical sector for exactly the same reasons. It is of interest
that the same negative result can be shown to be true for the $S$-matrix of the positive energy massive Thirring model [32-33]; for those values of $\lambda$ for which a bound state just disappears (i.e., $\lambda = \frac{1}{2}(n - 1)$, $n = 1, 2, \ldots$) the $S$-matrix simplifies [34], and one can verify that the strong limit $m \downarrow 0$ of the (anti-)soliton $S$-matrix on momentum space exists. For (anti-)solitons moving in opposite directions the limit equals the massless Thirring $S$-matrix $S^0_{\pm}$ (cf. Eq. (3.54)), but for particles moving in the same direction the limit is not one, as predicted by $S^0_{\pm}$. In fact, in this case the limit is quite surprising: on transforming to massless rapidities it is equal to the massive Thirring model $S$-matrix.

Finally, let us add some comments concerning other approaches to the massless Thirring model (cf. Refs. [2, 20] and many references given there). These were inspired by the finding that the approach of Thirring [4] and Glaser [5] (cf. also Berezin [6]) does not lead to certain results that were considered desirable, and even leads to inconsistencies, as claimed by some authors. From our viewpoint these difficulties are to be expected; they simply reflect the fact that the field $\tilde{\psi}(x)$ is not an operator-valued distribution [24]. In particular, if one nevertheless tries to determine its $x$-space $n$-point functions, he is bound to encounter inconsistencies. We should like to emphasize that the above considerations show that Glaser’s field $\tilde{\psi}_{\lambda,\alpha}$ gives rise to a consistent relativistic quantum theory describing interacting massless particles. Furthermore, although it does not satisfy the equation of motion (at least as defined by us) it does come remarkably close to that, and behaves in this respect just like Wightman’s Federbush model field $\psi_{\lambda,\alpha}$. We have moreover proved these assertions in an explicit and rigorous way.

In our opinion, this cannot be claimed for the other fields that various authors associated with the massless Thirring model, culminating in the work of Klaiber [20]. Here, quadratic forms are freely exponentiated and multiplied, and the resulting field is claimed to be an operator-valued distribution (cf. Ref. [20, p. 154]). The sole argument for this claim is the fact that after a number of other formal manipulations the $n$-point functions turn out to be tempered distributions; the usual positivity condition is asserted to be obvious from the fact that the fields were defined in a Hilbert space with positive definite metric. We feel that the discussions of the equation of motion in the literature are not quite convincing as well. For instance, formula (4.13) is freely used (cf. Ref. [20, Eq. (IX.6)]), although the exponential is not normal ordered. However, in Chapter 4 we have seen that in this case this formula may be not only formal, but erroneous. Also, limit prescriptions for currents are given, but it is not shown that such limits exist in a suitable topology (cf. Ref. [20, pp. 171-175]), etc. The later paper by Dell’Antonio et al. [35] does pay attention to the issues raised above (in Section V and the appendices), but it does not seem conclusive to us. In particular, although the existence of their intertwining operator $U_\alpha$ is rigorously proved (cf. also the work by Streater and Wilde [36, 37]), its explicit form is left unspecified. We therefore do not see why the only explicit estimates in the paper, leading from Eq. (B5) to (B7), having a bearing on the existence of their $\tilde{\psi}_H(u, v)$ as an operator-valued distribution; basically, they only say that if $\phi(x)$ is the usual free massless scalar field, then the vector $\int dx F(x) \exp g \phi(x) \Omega$ (where $F \in S(R^3)$) does not
fail to be in Fock space for ultraviolet reasons. This fact was already pointed out by Wightman in his Cargèse lectures [2]; he had to introduce an indefinite inner product for infrared reasons.

In summary, then, it seems to us that the approach just discussed is of great interest, but that work remains to be done to render its results rigorous. If one succeeds in this, one has obtained a well-defined field theory somehow associated with the classical PDE (5.3), but this field theory does not describe particles, contains free parameters not present at the classical level, and has abnormal covariance properties for generic parameter values, leading to anomalous scaling dimensions and multi-valuedness in the extended tube, all in contrast to Glaser’s field $\varphi^0_{h,t}$.

6. THE CONTINUUM ISING MODEL

6A. General Features

To put the Ising field theory [7–15] in a proper perspective, we begin by enlarging on the general discussion in Sections 2A and 3A. We shall first summarize some formalism and results from Ref. [19] and then add an observation that establishes the connection between the “charged case” considered in Section 3A, and the “neutral case” needed for the Ising model.

As we have shown in Ref. [19], unitary/pseudounitary operators $U$ that commute with the charge conjugation operator $C$ defined by Eq. (2.5) can be used to generate Bogoliubov transformations on the “neutral particle” Fock spaces $F_0(\mathcal{H}_+)/F_0(\mathcal{H}_+)$ as follows: define the field operators

$$\psi(v) = c(P+v) + c^*(CP-v), \quad \forall v \in \mathcal{H},$$

where $c^{(*)}$ are the annihilation and creation operators on $F_0(\mathcal{H}_+)$. These operators satisfy the commutation relations

$$[\psi(u), \psi(v)^*]_+ = (u, v) \quad \text{fermions}, \quad (6.2)$$

$$[\psi(u), \psi(v)^*]_- = (u, qv) \quad \text{bosons}. \quad (6.3)$$

Provided $U$ commutes with $C$ and is (pseudo-)unitary the transformation

$$\psi(v) \rightarrow \psi(U^*v) \quad (6.4)$$

leaves (6.2)/(6.3) invariant, and amounts to a Bogoliubov transformation of the $c^{(*)}$ (and in contrast to the charged case any Bogoliubov transformation of the $c^{(*)}$ arises in such a fashion). The transformation is unitarily implementable if and only if $U_{+-}$ is Hilbert–Schmidt. The implementing operator $\mathbb{U}_n$ is then given by

$$\mathbb{U}_n = \text{det}(1 - Z_{+-}^*Z_{+-})^{1/4} \exp\left(\frac{1}{2} [Z_{+-}c^*c^* + (Z_{++} - 1) c^*c + (Z_{--} - 1) cc^* + Z_{+-}cc] \right). \quad (6.5)$$
in the boson case, and by
\[ U_n = \det(1 + Z_{++}^* Z_{++})^{-1/4} \exp(\frac{1}{2}[Z_{++} c^* c^* + (Z_{++} - 1) c^* c - (LZ_{+-} - 1) c c^* - Z_{+-} c^* c]) \] (6.6)
in the fermion case, provided \( \text{Ker } U_+ = 0 \); if the latter condition is not met, \( U_n \) is given by Eq. (6.11) in Ref. [19].

Of course, if the above conditions are satisfied, \( U \) also generates a Bogoliubov transformation on \( \mathcal{F}(\mathcal{H}) \), implemented by a unitary operator \( \hat{U}_{eh} \) on \( \mathcal{F}(\mathcal{H}) \), which is given by the right-hand side of Eqs. (3.7) and (3.8) resp. Let us now rewrite \( \hat{U}_{eh} \) in terms of new “neutral particle” annihilation and creation operators \( c_i^{(+)} \), defined by
\[ a = 2^{-1/2}(c_1 + ic_2), \]
\[ b = 2^{-1/2}(c_1 - ic_2). \] (6.7)
This amounts to considering the Fock space \( \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2) \simeq \mathcal{F}(\mathcal{H}) \) instead of \( \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2) \simeq \mathcal{F}(\mathcal{H}_1) \), where
\[ \mathcal{H}_1 \frac{1}{2} = \{ \psi \in \mathcal{H} \mid P_+ \psi = \pm P_- \psi \}. \] (6.8)
The extra condition \([U, C]_+ = 0\) is equivalent to certain (anti-)symmetry properties of the kernels (cf. Eq. (4.6) in Ref. [19]) that imply the relation
\[ U_{eh} \simeq \mathcal{H}_n \otimes \mathcal{H}_n, \] (6.9)
where \( \mathcal{H}_n \) is the operator at the right-hand side of Eq. (6.5)/(6.6), but with \( c^{(*)} \) replaced by \( c_i^{(*)} \). Since the spaces \( \mathcal{H}_1, \mathcal{H}_2 \) are naturally isomorphic to \( \mathcal{H}_1 \), one may regard the operator \( \mathcal{H}_n \) as a “square root” of the operator \( \mathcal{H}_{eh} \).

With these general considerations out of the way, let us now discuss the operators associated with the Ising field theory. As mentioned before, there is a fermionic and a bosonic version [14], corresponding to the classical field operators \( \phi_{el}^s(x) \) and \( \phi_{el}^A(x) \) of Sections 2B and 2C, which are unitary and pseudo-unitary resp. It is easy to see that these operators commute with the charge conjugation operator \( C \) by using Eq. (2.6) and the obvious fact that the operators \( \hat{C} \) of Eqs. (2.27)/(2.52) commute with multiplication by \( e(x^3) \). Hence, we can use these field operators to generate Bogoliubov transformations on \( \mathcal{F}(\mathcal{H}) \). Again, these transformations are not unitarily implementable, since the off-diagonal parts are not Hilbert–Schmidt. However, as in the charged case, the expression at the right-hand side of Eq. (6.5)/(6.6) may be interpreted as a quadratic form implementing the transformation in the sense of forms even if \( U_{eh} \) is not Hilbert–Schmidt, provided one omits the determinantal factor ("wave function renormalization"). In view of relation (6.9) the quantum fields on \( \mathcal{F}(\mathcal{H}_1) \) resulting in this way from \( \phi_{el}^s(x) \) (\( \epsilon = s, a; A = B, F \)) can be regarded as square roots of the quantum fields on \( \mathcal{F}(\mathcal{H}) \) obtained as detailed in Section 3A. That is, the Wightman functions of the latter field operators are squares of those of the
former. The latter fields correspond to a doubled version of the Ising model [8–11],
and will not further be considered here.

Before we explicitly define the various fields, we wish to point out that the pair
\( \phi^F_{cl}(x), \phi^B_{cl}(x) \) provides an explicit example of a pair \( U, Z \) as considered at the end of
Section 7 in Ref. [19]. That is, these operators are mutually conjugate, commute with
\( C \) and equal their own inverse, and they are not only unitary/pseudo-unitary, but also
self-adjoint/pseudo-self-adjoint. As a result, the implementing quantum fields will be
seen to be self-adjoint as quadratic forms.

6B. Definitions of Quantum Fields

The fermionic and bosonic Ising fields on \( \mathcal{F}_{\alpha}(H_+) \) and \( \mathcal{F}_{\alpha}(H_-) \), obtained from
\( \phi^F_{cl}(x) \) and \( \phi^B_{cl}(x) \) resp. in the way just explained, are explicitly given by

\[
\phi_-(x) = \exp(\frac{1}{2} [Z^A_{+-}c^*c + (Z^A_{++} - 1) c^*c \mp (Z^A_{--} - 1) cc^* \mp Z^A_{+-}cc]).
\]

where
\[
Z^A_{\theta_1, \theta_2} = \exp\{ix \cdot (\delta p(\theta_1) - \delta p(\theta_2))\} Z^A_{\theta_1}(\theta_1 - \theta_2).
\]

and

\[
Z^F(\theta) = \frac{1}{2\pi i} \left( \begin{array}{cc} P \cotanh \frac{\theta}{2} & -\tanh \frac{\theta}{2} \\ \tanh \frac{\theta}{2} & -P \cotanh \frac{\theta}{2} \end{array} \right),
\]

(cf. Eq. (2.37)), while

\[
Z^B(\theta) = \frac{1}{2\pi i} \left( \begin{array}{cc} P \sinh \frac{\theta}{2} & i \\ -i & P \cosh \frac{\theta}{2} \end{array} \right).
\]

(cf. Eq. (2.54)). Finally,

\[
\rho(\theta) = m(\cosh \theta, \sinh \theta), \quad m > 0.
\]

These fields are evidently covariant and satisfy

\[
\phi_-(x)^* = \phi_-(x)
\]
as forms on \( \mathcal{F} \times \mathcal{F} \). The field \( \phi^F_\text{cl} \) is the scaling limit of the single-site spin operator
of the Ising model for \( T \uparrow T_c \) [15], while \( \phi^B_\text{cl} \) is the scaling limit of a single-site "spin
operator" of a closely related lattice model, introduced by Sato \textit{et al.} [15].

Let us now consider the scaling limit of the Ising spin operator for \( T \downarrow T_c \), which
we shall denote by $\phi_{\pm r}(x)$. To define $\phi_{\pm r}$ and for later purposes it is convenient to introduce a field

$$\phi_{\pm}(x) = (4\pi)^{-1/2} \int d\theta [c(\theta) \exp(-ix \cdot p(\theta) - i\alpha) + c^*(\theta) \exp(ix \cdot p(\theta) + i\alpha)].$$

where $\epsilon = a$ or $s$ depending on whether $c^{(i)}(\theta)$ acts on $\mathcal{A}(\mathcal{H})$ or $\mathcal{A}(\mathcal{H})$. Note that $\phi_{\pm}(x)$ has the form of a free neutral Klein–Gordon field. We now define

$$\Phi_{\alpha}(x) = \tilde{\phi}_{\alpha}(x) \tilde{\phi}_{\alpha}(x)^*, \quad \alpha \in [0, \pi).$$

According to Sato et al. [15] one then has up to an irrelevant multiplicative constant

$$\phi_{\pm}(x) = \Phi_{\sigma/2}(x).$$

It is to be noted that the fields $\Phi_{\alpha}(x)$ are very likely not unitarily equivalent for different $\alpha \in [0, \pi)$, in view of the $\alpha$-dependence of their $n$-point functions for $n > 4$. The point is that although the $\alpha$-dependence in $\phi_{\alpha}(x)$ can of course be gauged away by picking $\tilde{c} = \tilde{c}(e^{\alpha}) c\tilde{c}(e^{-\alpha})$, this will make the new kernels $Z_{\alpha,\beta}^{(r)}$ depend on $\alpha$. (Clearly, $Z_{\alpha,\beta}^{(r)}$ does not change. Note in this connection that the expressions (8) and (10) in Ref. [12] for $\phi_{\alpha}(x)$ contain an error: a (non-trivial) factor $i$ is missing in the kernel $\alpha_1$.) We have introduced these fields, since it seems plausible to us that they will all turn out to be local, once it is proved that they are operator-valued distributions in the usual sense. Indeed, their formal locality can again be understood in terms of Bogoliubov transformations, as we shall explain in Section 6C.

Let us now introduce a similar field in the boson case. First, set

$$\psi_{\alpha}(x) = \left(\frac{m}{4\pi}\right)^{1/2} \int d\theta \left[c(\theta) \left(e^{i\theta/2} \right) \exp(-ix \cdot p(\theta) - i\alpha) + c^*(\theta) \left(-ie^{-i\theta/2}\right) \exp(ix \cdot p(\theta) + i\alpha)\right].$$

where as in Eq. (6.16) $\epsilon$ denotes action on $\mathcal{A}$ for $\epsilon = a$, on $\mathcal{F}$ for $\epsilon = s$. Thus, $\psi_{\alpha}$ has the form of a free Majorana field. Now define

$$\Psi_{\alpha}(x) = :\psi_{\alpha}(x) \phi_{\alpha}(x)^*: \quad \alpha \in [0, \pi).$$

This field is a generalization of the field $\phi_{\alpha}(x)$ of Sato et al. [14]. In Section 6C we shall make plausible that this field is non-local for any $\alpha$. However, it nevertheless gives rise to asymptotic free fields in the LSZ sense, connected by a unitary $S$-matrix, as we shall also detail in Section 6C.

6C. Locality and Scattering Properties

In Section 3A we have already presented an argument (based on the Källen–Lehmann representation of the two-point function) that implies formal locality of the
fields $\phi_{-A}(x)$, $A = F, B$. This argument can also be used for the fields $\Phi_{A}(x)$, provided we can prove that they also implement transformations generated by local classical field operators. We shall now show this, arguing formally at first.

For the implementing operator to be given by Eq. (6.6) one needs the condition $\text{Ker } U_{++} = 0$. If this condition is violated one needs extra annihilation and creation operators multiplying the right-hand side, and depending on orthonormal functions from $\text{Ker } U_{++}$ and $\text{Ker } U_{++}^{*}$ resp. (cf. Section 6 of Ref. [19]). In the case at hand, the operators $\hat{D}_{F}^{(x)}$ act as multiplication by $\tanh \pi y$ (cf. Eq. (2.33)), so that, formally, they have kernel $e^{i\pi \delta(y)}$. Hence, $U_{++}^{F(x)}$ acting on constant functions give zero. From this and a comparison of Eq. (6.17), and Eq. (6.11) in Ref. [19], we now conclude that $\Phi_{F}(x)$ implements the transformation generated by $-\partial(x)$.

Let us now briefly digress to show that the implementing property of $\phi_{-F}$ and $\Phi_{A}(x)$ (on which locality hinges) can actually be made rigorous. Since $Z_{F}$ is an unbounded operator, the quadratic forms $\phi_{-F}(0), \Phi_{A}(0)$ do not make sense on all algebraic tensors. However, they are well defined on algebraic tensors for which the constituent functions are in

$$D := \{ f \in S(R) \mid \int d\theta f(\theta) = 0 \}. \tag{6.21}$$

Indeed, for $v \in D$, where

$$D = \{ v \in H \mid P_{+}v, P_{-}v \in D \}, \tag{6.22}$$

the Fourier transform $\hat{\phi}$ is in the domain of $Z_{F}$, since it vanishes at the origin. Note this also implies $U_{F}D \subset D$. We claim that on this form domain, denoted $\mathcal{D}$, one has

$$\psi(v) \phi_{-F}(0) = \phi_{-F}(0)\psi(U_{F}v), \quad \forall v \in D \tag{6.23}$$

and

$$\psi(v) \Phi_{A}(0) = -\Phi_{A}(0)\psi(U_{F}v), \quad \forall v \in D. \tag{6.24}$$

To see this, note first that these formulas make rigorous sense, since $\psi(v)\mathcal{D} \subset \mathcal{D}$, $\psi(U_{F}v)\mathcal{D} \subset \mathcal{D}$. For $\phi_{-F}(0)$ one can now mimic the proof of Eq. (5.18) in Ref. [19] to obtain (6.23), since only the conjugacy relations (2.15) need to be used. To obtain (6.24), note that on $\mathcal{D}$ $e^{iA_{+}}(f)$ anticommute with the monomials $\int d\theta c^{(x)}(\theta)$ for any $f \in D$, and then proceed as in the first case. Of course, these relations imply more generally

$$\psi(v) \phi_{-F}(x) = \phi_{-F}(x)\psi(\phi_{0}(x)^{*}v), \quad \forall v \in U(x, 1)D, \tag{6.25}$$

which hold in the sense of forms on algebraic tensors built up from functions $f_{i} \in U(x, 1)_{++}D$. 

We proceed to consider the field $\Psi_{s}^{B}(x)$. It follows from the commutation relation (4.63) in Sato et al. [14] that for their value of $\alpha$,

$$\psi(v) \Psi_{s}^{B}(0) = \Psi_{s}^{B}(0) \psi(U^{B}v), \quad (6.26)$$

i.e., they claim that the form implementing the transformation generated by $U^{B}$ is not unique. We have been unable to explain this from the viewpoint of Bogoliubov transformations, since a "charged vacuum" cannot occur in the boson case (at least if the transformation is unitarily implementable). Indeed, we consider it likely that (6.26) only holds if $v = 0$ (if so, Eq. (4.63) in Ref. [14] is incorrect). For instance, if one takes the vacuum expectation of (6.26) one obtains the relation

$$-2\pi e^{2i\pi} \int d\theta \left( e^{\theta/2} \right) \left( P_{+}v(\theta) \right) = \int d\theta \left( e^{\theta/2} \right)$$

$$\cdot \left[ \int d\theta' \coth \left( \frac{\theta - \theta'}{2} \right) \left( P_{-}v(\theta') \right) + \int d\theta' \tanh \left( \frac{\theta - \theta'}{2} \right) \left( P_{+}v(\theta') \right) \right], \quad (6.27)$$

and it is far from clear that this can hold for all $v$ belonging to a dense subspace of $\mathcal{H}$, or even that the integrals at the left- and right-hand sides make simultaneous sense for a dense set of $v$. (Of course, other expectation values give rise to additional restrictions.) We conjecture therefore that if $\Psi_{s}^{B}(x)$ can be proved to be an operator-valued distribution, it will not be local.

Let us now study the scattering content of the various field theories introduced in Section 6B. Clearly the fields $\phi_{A}(x)$ do not couple the vacuum to the one-particle states, so that these field theories do not admit a particle interpretation in the usual sense (cf., however, Ref. [38], where an attempt is made to interpret the broken symmetry field theory $\phi_{A}(x)$ in terms of a highly unorthodox kind of particle). The situation is different for the fields $\Phi_{s}^{F}(x), \Psi_{s}^{B}(x)$, as we shall now show. For their values of $\alpha$, Sato et al. [13, 14] obtained in both cases the S-matrix

$$(S\psi)(\theta_{1}, ..., \theta_{N}) = (-)^{N-1/2} \psi(\theta_{1}, ..., \theta_{N}). \quad (6.28)$$

Their arguments for this are heuristic, and in particular several formal expressions occur in this connection whose precise meaning is obscure. However, a rigorous proof can be given that the operator $S$ is the S-matrix for $\Phi_{s}^{F}, \Psi_{s}^{B}, \alpha \in [0, \pi)$, and will now be sketched.

First, we observe that in one dimension symmetric and antisymmetric functions can be put in a natural 1–1 correspondence in two ways. Indeed, it is easy to see that the multiplication operators

$$(M_{\pm}\psi)(\theta_{1}, ..., \theta_{N}) = \prod_{1 \leq i < j \leq N} \varepsilon(\pm(\theta_{i} - \theta_{j})) \psi(\theta_{1}, ..., \theta_{N}) \quad (6.29)$$

have this property, and they clearly give rise to isometric maps (also denoted $M_{\pm}$) from $F_{s}(\mathcal{H})$ onto $F_{s}(\mathcal{H})$ and vice versa. Evidently,

$$M_{-} = M_{\alpha}S. \quad (6.30)$$
Note also that $M_\delta^2 = 1$ and that, e.g.,

$$M_+c_B^*(f_N) \cdots c_B^*(f_1) \Omega_B = c_F^*(f_N) \cdots c_F^*(f_1) \Omega_F,$$

(6.31)

provided that $\text{supp } f_{i+1}$ is to the left of $\text{supp } f_i, i = 1, \ldots, N - 1.$ (Here we have indexed $c^*$ and $\Omega$ to make clear to what Fock space they belong.) At first sight, this seems to imply that $M_\delta$ sets up a unitary equivalence between $c_B$ and $c_F$, since the above vectors are clearly total in $\mathcal{F}_B$. However, this is of course impossible, since $c_B^{i*}$ satisfies the CCR and $c_F^{i*}$ the CAR, and in fact a closer look reveals that the conclusion is false because of the restriction that the supports be disjoint. Nevertheless, the $M_\delta$ will enable us to define fields on one Fock space whose natural habitat is the other one, and will explain how it is possible to obtain a boson theory in a fermion Fock space and a fermion theory in a boson Fock space. Indeed, let us define the fermion Fock space fields

$$\phi_\alpha^\text{out}(x) = M_+ \phi_\alpha(x) M_+$$

(cf. Eq. (6.16)), and the boson Fock space fields

$$\psi_\alpha^\text{out}(x) = M_+ \psi_\alpha(x) M_+$$

(cf. Eq. (6.19)). Now let $f(F)$ be smooth solutions of the free Klein-Gordon (Dirac) equation with mass $m$. Then one has

$$\lim_{t \to \pm \infty} \int dx^1 \bar{f}(t, x^1) \bar{\Phi}_\alpha^F(t, x^1) = \int dx^1 \bar{f}(0, x^1) \bar{\phi}_\alpha^\text{out}(0, x^1),$$

(6.34)

$$\lim_{t \to \pm \infty} \int dx^1 \bar{F}(t, x^1) \cdot \Psi_\alpha^B(t, x^1) = \int dx^1 \bar{F}(0, x^1) \cdot \psi_\alpha^\text{out}(0, x^1)$$

(6.35)

in the sense of quadratic forms on $\mathcal{D} \times \mathcal{D}$, where $\mathcal{D}$ is the dense subspace of finite linear combinations of vectors of the form $\prod_{i=1}^N c^*(f_i) \Omega$, where $f_i(\theta) \in C_0^\infty(R)$. As a consequence, $S$ is the $S$-matrix of the boson fields $\Phi_\alpha^F$ and the fermion fields $\Psi_\alpha^B$; hence, these field theories are asymptotically complete. The proof of Eqs. (6.34)-(6.35) involves some non-trivial functional analysis, and will be presented elsewhere [23].

7. Bethe Transforms

Until now we have followed the usual approach to a relativistic field theory, in so far as we concentrated on the field operators and determined the scattering content of the theory by considering the asymptotic fields associated with the interacting field. On the unphysical sector one may try instead to make sense of the formal Hamiltonian $\hat{H}$, and determine whether it resembles for long times the free asymptotic dynamics of the field theory, in the sense that the wave operators

$$U_\pm = s \cdot \lim_{t \to \pm \infty} \exp(i\hat{H}t) J \exp(-iH_0t)$$

(7.1)
exist and are unitary. Here, $J$ is a comparison map from the Hilbert space $\mathcal{F}$ on which $H_0$ acts onto the space $\mathcal{F}$ on which $\tilde{H}$ acts. If the corresponding $S$-operator

$$S = U_+^{-1} U_-$$

equals the $S$-operator of the field theory one may set

$$\tilde{U}_i(a, A) U_\pm = U_\pm U_i(a, A),$$

where $U_i$ is the free representation of the Poincaré group, whose time translation generator is $H_0$. Then $\tilde{U}_i$ is a well-defined unitary interacting representation of the Poincaré group on $\mathcal{F}$, not depending on whether $U_+$ or $U_-$ is used in (7.3), since $S$ is Lorentz invariant. Moreover, the time translation generator of $\tilde{U}_i$ is just $\tilde{H}$ by virtue of the intertwining relations $U_\pm \exp(iH_0 t) = \exp(i\tilde{H} t) U_\pm$ following from (7.1). In this way one obtains a relativistic quantum theory that is physically equivalent to the field theory, since it is relativistically invariant and has the same particle spectrum and scattering by construction.

In Section 7A we shall construct such a theory for the unphysical Federbush and Thirring models, starting from the Hamiltonian densities (4.6) and (5.5) resp. We also establish the connection with the field approach, thereby obtaining the unitary equivalence of the time-zero interacting and free field operators, and a simple proof of Eqs. (3.35) and (3.36), as promised above. Another useful consequence of these considerations is that we shall be able to answer the question raised below Eq. (4.9).

In the physical case it is well known that the Hamiltonian cannot be given a sense on the Fock space on which it formally acts. Instead, one may try to construct unitary maps $U_\pm$ and a comparison operator $J$ such that (7.1) and (7.2) hold, if $\tilde{H}$ is defined through (7.3). In a previous paper [21] we have explained this approach in more detail, and described such maps for the Federbush and Ising models. In Section 7B we simplify these results, and extend them to the bosonic version of these models and to the fermionic and bosonic massless Thirring models. In all cases the kernels of $U_\pm$ have a Bethe Ansatz structure, reflecting the soliton structure of the field theory $S$-matrices (3.31), (3.32), (3.53), (3.54) and (6.28).

7A. The Unphysical Sector

The Hamiltonian approach to the Federbush and Thirring models takes as its starting point the formal expressions

$$\mathcal{H}_H = \sum_{s=\pm 1} \int dx^4 \{ c_s(x^4) (-i\gamma^\alpha \partial_\alpha + \gamma^0 m(s)) c_s(x^4) 

+ 4\pi\lambda(c_{1,s}(x^4) c_{1,1}(x^4) c_{1,2}(x^4) c_{-1,2}(x^4)

- c_{1,2}(x^4) c_{1,1}(x^4) c_{-1,1}(x^4)) \}$$

(7.4)
and
\[ \hat{H}_0^\omega = \int dx^1 \left[ \left( \frac{c_{s}^s(x^1)}{c_{s-1}^s(x^1)} \right) \left( -i \gamma^a \partial_1 \right) \left( \frac{c_1(x^1)}{c_{1-1}(x^1)} \right) + 4\pi \alpha c_1^s(x^1) c_1(x^1) c_{1-1}^s(x^1) \cdot c_{1-1}(x^1) \right]. \tag{7.5} \]
respectively. In Eq. (7.4),
\[ c_{s}^{(\ast)}(x^1) = \begin{pmatrix} c_{s,1}^{(\ast)}(x^1) \\ c_{s,2}^{(\ast)}(x^1) \end{pmatrix} \]
acts on the Fock space \( \tilde{\mathcal{F}}_s \) of Section 4A, while \( c_{s}^{(\ast)}(x^1) \) in Eq. (7.5) acts on the Fock space \( \mathcal{F}_s \) of Section 5A. The formal Hamiltonians \( \hat{H}_0^\omega \) are obtained by integrating the Hamiltonian densities (4.6) and (5.5) resp. at time zero over all of space, and substituting
\[ \psi_s(0, x^1) \rightarrow c_{s}(x^1), \quad \psi(0, x^1) \rightarrow \begin{pmatrix} c_1(x^1) \\ c_{1-1}(x^1) \end{pmatrix} \]
resp. The formal solution of the field equations is then given by
\[ \chi_{s}^{(\omega)}(t, x^1) = \exp(i\hat{H}_0^\omega t) c_{s}(x^1) \exp(-i\hat{H}_0^\omega t). \tag{7.6} \]
Thus, the initial data of the interacting field \( \chi_{s}^{(\omega)}(t, x^1) \) is the free field \( c_{s}(x^1) \), ensuring that the solution is canonical.

The expressions at the r.h.s. of Eqs. (7.4) and (7.5) are well defined as quadratic forms on, e.g., finite particle Schwartz space test functions. For \( \lambda = 0 \) these forms are the forms of the Hamiltonians \( \hat{H}_0 \) and \( \hat{H}_0^\omega \) of Sections 4A and 5A resp. However, for \( \lambda \neq 0 \) the forms are unfortunately not the forms of self-adjoint operators that would be the obvious candidates for \( \hat{H}_0 \) and \( \hat{H}_0^\omega \). One way to see this is as follows. From the quadratic forms one infers that only particles with different values of \( s \) interact, and that the interaction is concentrated on the hyperplanes of coinciding particle positions. Thus, the Hamiltonians \( \hat{H}_0^\omega \) should be self-adjoint extensions of the differential operators \( \hat{\gamma} \), restricted to \( S(R) \)-functions whose support stays away from the hyperplanes \( x_i = x_j \). In the two-particle case this leads (using permutation (anti)symmetry) to a self-adjoint extension of the operator \( \frac{1}{i}(d/dx) \) (where \( x = x_1 - x_2 \)) restricted to \( S(R) \)-functions with support away from the origin. Since the deficiency indices of this symmetric operator are \( (1, 1) \), there is a one-parameter family of self-adjoint extensions \( \hat{\theta}_\lambda \), characterized by the boundary condition
\[ \lim_{x \uparrow 0} f(x) = \exp(2\pi i/\lambda) \lim_{x \downarrow 0} f(x), \quad \lambda \in [\frac{1}{2}, \frac{3}{2}] \tag{7.7} \]
on the wave functions in the domain of \( \hat{\theta}_\lambda \). It is easy to see this implies
\[ \hat{\theta}_\lambda = \exp(i\pi \lambda \epsilon(x)) \hat{\theta}_0 \exp(-i\pi \lambda \epsilon(x)). \tag{7.8} \]
But this in turn shows that a function \( f \in S(R) \) satisfying \( f(0) \neq 0 \) is not in the form domain of \( \hat{\theta}_\lambda \), since functions with jump discontinuities are not in the form domain of
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In $\mathfrak{p}_0$, viz., the Sobolev space $H_{1/2}(\mathbb{R})$. To put the result of this argument in a physicist's language: the addition of $\delta$-function pair potentials to the free Hamiltonians $\hat{H}_0$ and $\hat{H}_0^{(0)}$ of Eqs. (7.4) and (7.5) resp. does not give rise to a self-adjoint operator.

However, Eq. (7.8) suggests a way of defining $\hat{H}_\lambda^{(0)}$ rigorously as a self-adjoint operator that does capture the features of the quadratic forms resulting from (7.4) and (7.5): set

$$\hat{H}_\lambda^{(0)} = M_\lambda \hat{H}_0^{(0)} M_{-\lambda},$$

(7.9)

where $M_\lambda$ is defined by

$$(M_\lambda \psi)(x_1, s_1; \ldots; x_N, s_N) = \exp \left[ -\frac{i}{\lambda} \sum_{i < j} \epsilon(x_i - x_j)(s_i - s_j) \right] \psi(x_1, s_1; \ldots; x_N, s_N).$$

(7.10)

(In (7.10) we have suppressed the indices $j_1, \ldots, j_N$ in the Federbush case.) Since $M_\lambda$ is unitary and $M_\lambda^* = M_{-\lambda}$, the operators $\hat{H}_\lambda^{(0)}$ are well-defined self-adjoint operators, reducing to $\hat{H}_0^{(0)}$ on wave functions with support away from the hyperplanes $x_i = s_j$, and on wave functions for which all $s_i$ are equal.

We shall now show that these definitions lead to the field theory $S$-operators $S_\lambda^{(0)}$ of Sections 4A and 5A, provided one defines suitable maps $U_\pm$ and $J$. As we have seen before, on $\mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_{-1})$ the Federbush Hamiltonian is given by

$$(H_0 \psi)(\theta_1, s_1; \theta_N, s_N) = \sum_{i=1}^N m(s_i) \delta_i \cosh \theta_i \psi(\theta_1, s_1; \theta_N, s_N),$$

(7.11)

while the Thirring Hamiltonian reads

$$(H_\theta^{(0)} \psi)(\theta_1, s_1; \theta_N, s_N) = \sum_{i=1}^N \delta_i e^{\sqrt{\theta_i}} \psi(\theta_1, s_1; \theta_N, s_N).$$

(7.12)

Since $H_0^{(0)} = \mathcal{W}^{(0)} - H_0^{(0)} \mathcal{W}^{(0)}$, the obvious definition for the comparison map $J$ is

$$J = \mathcal{W}^{(0)}.$$  

(7.13)

We now claim that the operators

$$U_\pm^{(0)} = M_\lambda \mathcal{W}^{(0)} S_\lambda^{(0)} + \lambda/2$$

(7.14)

have all the desired properties. Indeed, it is obvious that they are unitary and satisfy Eq. (7.2). The proof that they are the wave operators for the pair $\hat{H}_\lambda^{(0)}, H_0^{(0)}$ with comparison map $J = \mathcal{W}^{(0)}$ proceeds along the same lines as the proof of Theorem 3.2 in Ref. [21] and will therefore be omitted. It also follows as in Ref. [21] that $U_\pm^{(0)}$ are actually the wave operators for a large class of dynamics, containing in particular the conserved charges corresponding to $H_\lambda^{(0)} \psi_\theta^{(0)}$ with $n$ odd (cf. (4.9)).

Let us now compare the present approach to the field theory approach of Sections 4A and 5A. In the latter case we defined an interacting Hamiltonian $\hat{H}_0^{(0)}$ and a time-
zero field $\tilde{\psi}_{\lambda,s}^{(0)}(0, x^1)$ differing from the free time-zero field $\tilde{\psi}_0^{(0)}(0, x^1)$. Here, the interacting Hamiltonian is $\tilde{H}_\lambda^{(0)}$, defined by Eq. (7.9), while the time-zero field $\tilde{\psi}_0^{(0)}(0, x^1) = c_s(x^1)$ coincides with the free time-zero field $\psi_0^{(0)}(0, x^1)$. The connection between the two points of view is simple: they are unitarily equivalent, and may be regarded as different pictures of the same theory. That is, $M_\lambda$ not only interwines $\tilde{H}_0^{(0)}$ and $\tilde{H}_\lambda^{(0)}$ according to Eq. (7.9), but also satisfies

$$\tilde{\psi}_{\lambda,s}^{(0)}(t, x^1) = M_\lambda \tilde{\psi}_s^{(0)}(t, x^1) M_{-\lambda},$$

This follows from the relation

$$\tilde{\psi}_{\lambda,s}^{(0)}(0, x^1) = M_{-\lambda} c_s(x^1) M_\lambda,$$

which can be verified by a straightforward calculation using Eqs. (4.10), (5.8) and (7.10).

This relation has several other useful consequences. Firstly, it establishes the already announced unitary equivalence of $\psi_{\lambda,s}^{(0)}(0, x^1)$ and $\psi_0^{(0)}(0, x^1)$. This implies that $\psi_{\lambda,s}^{(0)}(t, x^1)$ satisfies equal time (anti) commutation relations. Thus, by covariance one formally obtains the locality relations (3.30). (A rigorous proof could presumably be constructed by exploiting the finite propagation speed of $\exp(-i\tilde{H}_\lambda^{(0)}t)$.)

Secondly, it leads to a proof of Eqs. (3.35)-(3.36) that is not only simple, but also illuminates the connection between the "LSZ" point of view and the Hamiltonian point of view. Indeed, using the above and the fact that $F_s$ is a free solution we have for $\psi_1, \psi_2 \in \mathcal{D}$,

$$\int dx^1 \tilde{F}_s(t, x^1) \cdot \langle \psi_1, \psi_{\lambda,s}(t, x^1) \psi_2 \rangle = \int dx^1 \tilde{F}_s(0, x^1) \cdot \langle \psi_1, \psi_{\lambda,s}(t, x^1) \psi_2 \rangle = \int dx^1 \tilde{F}_s(0, x^1) \cdot \langle \psi_1, \psi_{\lambda,s}(t, x^1) \psi_2 \rangle,$$

But in view of Eq. (7.1) and the unitarity of $U_\pm$ the factors in brackets have strong limits $U_\pm^{-1}$ for $t \to \pm \infty$. Thus, using Eq. (7.14) we get

$$\lim_{t \to \pm \infty} \int dx^1 \tilde{F}_s(t, x^1) \cdot \langle \psi_1, \psi_{\lambda,s}(t, x^1) \psi_2 \rangle = \int dx^1 \tilde{F}_s(0, x^1) \cdot \langle \psi_{\pm_{\frac{1}{2}}} \psi_1, S_{\pm_{\frac{1}{2}}}^{-1} \psi_2 \rangle,$$

which is just Eq. (3.35). The proof of Eq. (3.36) is similar.

A third consequence is that we can now easily answer the question raised below Eq. (4.9) for the case that the conserved charge is the Hamiltonian. Indeed, the question whether $\tilde{H}_0^{(0)}$ can be expressed in terms of the non-perturbative time-zero field $\tilde{\psi}_0^{(0)}(0, x^1)$ in the same way as at the classical level is equivalent to the question whether $\tilde{H}_\lambda^{(0)}$ can be expressed in terms of $c_s(x^1)$ as formally suggested by Eqs. (7.4)-(7.5). We have already answered this question negatively above, and it is important to note that this is not due to our choice of "coupling constant" dependence in Eq. (7.10)
(which was of course inspired by the wish to obtain a theory essentially the same as the one of Sections 4A and 5A resp.). It is not hard to see that the change in domain effected by $M_s$ is also too strong for the higher order Hamiltonians $\hat{H}_s(\psi_s)$ and their Thirring analogs to be expressible as quadratic forms in the time-zero field operators, as suggested by the polynomial expressions at the classical level. (For the nonlinear Schrödinger equation on the boson sector the same is true for the polynomial charges $H_n$ obtained via the inverse scattering formalism if $n \geq 6$, as shown by Oxford [39].)

A fourth interesting consequence is that it implies Haag's theorem (cf., e.g., Ref. [27]) is false for a relativistic field theory in which the Hamiltonian is unbounded from below. This is, however, perhaps not surprising, since the proof of this theorem makes use of the assumption $\hat{H} \geq 0$. The examples just given show that this assumption is essential.

We conclude this section with some remarks concerning: (i) The connection with Thirring's work; (ii) The massive Thirring model on the unphysical sector; (iii) Coupling constant dependence; (iv) The chiral Gross-Neveu model on the unphysical sector.

(i) In his original paper [4] Thirring solved his model on the unphysical sector in the traditional fashion of theoretical physics, in the sense that he obtained formal incoming and outgoing eigenfunctions of Bethe Ansatz type, and read of the S-matrix from these functions. Since the kernels of the above unitary wave operators $U_+^0$ are essentially his eigenfunctions and since $S_+^0$ is essentially his S-matrix, the above considerations solve the usual mathematical problem that is raised by results of this nature, viz.: To define a self-adjoint Hamiltonian $\hat{H}$ for which the wave operators (7.1) exist if a suitable free comparison dynamics $H_0$ and comparison map $J$ is introduced; these wave operators $U_{\pm}$ should be asymptotically complete and should have the formal incoming and outgoing "eigenfunctions" as kernels, and correspondingly be related by $U_+ S = U_- $, where $S$ is the S-matrix found by the "time-independent" approach.

To solve the corresponding problem for the massive Thirring model on the unphysical sector, starting from the results of Berezin and Sushko [40], is considerably harder. We intend to come back to this elsewhere, and only mention some aspects relevant to the present paper in the next remark.

(ii) If one adds the term $m[c^+_s(x^3) c_{-s}(x^3) + c^+_{-s}(x^3) c_{s}(x^3)]$ to the integrand of Eq. (7.5) one obtains the formal Hamiltonian of the massive Thirring model on the unphysical sector. The considerations following Eq. (7.5) are applicable to this case as well, leading to the rigorous definition given by (7.9), where now of course $\hat{H}_0$ is the second quantization of the massive Dirac operator $-i\gamma^0 \partial_x + \gamma^0 m$. In particular, if $\hat{H}_s$ is defined in this way, the field operator $\hat{\chi}_s(t, x^3)$ defined by (7.6) is the rigorous solution to the massive Thirring model on the unphysical sector.

For the same reasons as in the massless case, the classical polynomial conserved charges obtained via inverse scattering do not give rise to operators if $\tilde{\varphi}(0, x^3)$ and $\tilde{\psi}(0, x^3)$ are replaced by $c^+(x^3)$ and $c(x^3)$ resp. However, the discussion of the scattering
is considerably more complicated than for the massless case, since in this case $\tilde{H}_\lambda$ has a rich bound state spectrum [40] (cf. also the recent work by Korepin [41]). Thus,

$$\int dx^1 F(t, x^1) \cdot \tilde{\chi}_\lambda(t, x^1) = \int dx^1 \bar{F}(0, x^1) \cdot e^{i\tilde{H}_\lambda t} e^{-i\tilde{H}_0 t} e^{i\tilde{H}_1 t}$$

(7.19)

(where $F$ is a smooth solution to the Dirac equation of mass $m$) can only converge on the sector in which all particles are unbound. For the other sectors one would have to construct polynomial field operators along the lines of Ref. [42].

Finally, let us see if the massive Thirring model reduces to the massless Thirring model in the limit $m \downarrow 0$. First we note that the above definition of $\tilde{H}_\lambda$ implies that $\lim_{m \downarrow 0} \tilde{H}_\lambda = \tilde{H}_\lambda^0$ in the strong resolvent sense. It readily follows from this that the operator $\int dx F(x) \cdot \tilde{\chi}_\lambda(x)$ strongly converges to the operator $\int dx F(x) \cdot \tilde{\chi}_\lambda^0(x)$ on $\mathcal{H}_\lambda$ and on the dense subspace of finite particle vectors in $\mathcal{H}_\lambda$ for any $F \in S(\mathbb{R}^2)$, in contrast to the situation in the Federbush case (cf. Section 5A). However, as in the case of the positive energy massive Thirring model, the $S$-matrix on the no bound state sector only converges to $S_\lambda^0$ for particles moving in opposite directions, while it stays the same (and therefore disagrees with $S_\lambda^0$) for particles moving in the same direction, if it is reexpressed in terms of massless rapidities.

(iii) Recently, Bergknoff and Thacker [43] reobtained part of the results of Berezin and Sushko [40] concerning the unphysical sector of the massive Thirring model in their study of the physical sector. Their unphysical $S$-matrix has a different coupling constant dependence than the one of Ref. [40]. The discrepancy is the same as the difference in the results of Thirring [4] and Glaser [5]. In both cases the reason is easily understood from the considerations above: the coupling constant dependence of the rigorous Hamiltonian $\tilde{H}_\lambda$, expressed through the multiplication operator $M_\lambda$, is not dictated by the formal Hamiltonian $\tilde{H}_\lambda$ of Eq. (7.4) (used by these authors), since it has no immediate connection with $\tilde{H}_\lambda$. It is therefore largely immaterial how one parametrizes the torus of different self-adjoint extensions. However, if one insists on having an equation of motion that resembles Eq. (5.3) as closely as possible, one should replace $\lambda$ throughout by $f(\lambda)$, defined by Eq. (4.21) (cf. the discussion there). This coupling constant dependence is different from that of all references mentioned above.

(iv) The generalization of the massless Thirring model to particles with an internal degree of freedom described by $SU(N)$ is usually called the chiral Gross-Neveu model. To study the physical sector of this model Andrei and Lowenstein [44, 45] recently solved this model on the unphysical sector by a Bethe Ansatz. Their solution does not amount to the obvious extension of Thirring's solution, but only reduces to it if all particles have the same internal "spin" quantum numbers. In particular, two colliding particles with unequal spin values can either transmit or reflect. This is not in conflict with the one-parameter family of self-adjoint extensions discussed above (cf. Eqs. (7.8)-(7.9)), since this was derived under the assumption of permutation (anti-) symmetry. For two particles of different spin values this need not
hold, and one can define a non-equivalent extension of the operator $i \partial_1 - i \partial_2$, as was essentially done in Refs. [44, 45]. It is interesting that this other definition, which is in no way dictated by the formal Hamiltonian, was essential to obtain the bound state spectrum on the physical sector (as we learned from N. Andrei).

7B. The Physical Sector

In Ref. [21] we found Bethe transforms for the Federbush and Ising models by making an educated guess at what the kernels of $U_\pm$ (i.e., the incoming and outgoing eigenfunctions of the perturbed Hamiltonian $\tilde{H}$) should look like for the known $S$-matrices to result. From the expressions for $U_\pm$ unitarity was not obvious, so that we had to do a bit of work to prove this. However, we have meanwhile realized that one can write the transforms $U_\pm$ as a product of three obviously unitary operators, much like in Section 7A. By some trial and error we then found Bethe transforms of this structure for the positive energy massless Thirring model as well. We shall now present these results. We restrict ourselves to the case where $H_0$ acts on the rapidity Fock space, and correspondingly get a point form relativistic dynamics [21]. However, if one interprets the $\theta$'s below as momenta, and correspondingly changes $\mathcal{H}(a, A)$, one obtains relativistic dynamics of instant form (provided one takes $m(1) = m(-1)$ in the Federbush case). In all cases the transforms are unitary on the respective Fock spaces for distinguishable particles, and satisfy there the relations (7.1) by a stationary phase argument (which is easiest on this large space). One then gets dynamics for the various field theories by restriction to the boson and fermion sectors.

First, let us consider the Ising case. Here, the Fock space is $\mathcal{F}(\mathcal{H}_+)$, where $\mathcal{H}_+ = L^2(d\theta)$, and the representation of the Poincaré group is given by

$$
(H(a, A) \psi)(\theta_1, ..., \theta_N) = \exp \left[ ia \cdot \sum_{j=1}^{N} p(\theta_j) \right] \psi(\theta_1 - \alpha, ..., \theta_N - \alpha), \quad (7.20)
$$

where

$$
p(\theta) = m(\cosh \theta, \sinh \theta). \quad (7.21)
$$

We take $J$ to be the second quantization of Fourier transformation, defined by

$$
(Jf)(\gamma) = (2\pi)^{-1/2} \int d\theta \exp(i\theta \gamma) f(\theta), \quad f \in \mathcal{H}_+. \quad (7.22)
$$

Thus, $J$ maps $\mathcal{F}$ onto another Fock space, denoted $\hat{\mathcal{F}}$. We now set

$$
U_+ = M_-JM_+. \quad (7.23)
$$

where $M_\pm$ are the multiplication operators defined by Eq. (6.29). Obviously, (7.2) holds, and if the interacting representation is defined by (7.3), it follows that (7.1)
holds by stationary phase. Indeed, these transforms are simply another representation of the transforms of Section 4A in Ref. [21].

Consider now the Federbush and massless Thirring cases. The Fock space is $\mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_{-1})$, where $\mathcal{H}_s$ is a copy of $L^2(d\theta)^2$, and $\mathcal{U}(a, A)$ is given by Eqs. (3.11)–(3.13), except that the factor $\delta_s$ is omitted in the exponential. Let us now specialize to the Federbush case. Here, we define $J$ as the second quantization of the unitary one-body operator

$$
(Jf)(y, s, \delta) = \left(\frac{m(s)}{2\pi}\right)^{1/2} \int d\theta \exp(im(s)\theta) f(\theta, s, \delta), \quad f \in \mathcal{H}_1 \oplus \mathcal{H}_{-1}, \quad (7.24)
$$

and then we set

$$
U_\pm = S_\pm J^* S_{\mp A/2}^\dagger, \quad (7.25)
$$

where $S_\pm$ is the Federbush $S$-matrix (3.53). Again, (7.2) is obvious, and (7.1) follows by stationary phase, once $\mathcal{H}$ is defined through (7.3). These operators are equal to the Bethe transforms of Section 4B in Ref. [21], as can be easily verified.

Finally, consider the massless Thirring case. Define $J$ as the second quantization of (7.24), with $m(s)$ replaced by 1. As before, we denote the range of $J$ by $\mathcal{H}$ and we denote the multiplication operator on $\mathcal{H}$ given by (3.53) with $\theta_i \rightarrow y_i$ by $S_0$. Now we set

$$
U_\pm = S_{\pm A/2} J^* S_{\mp A/2}^\dagger, \quad (7.26)
$$

where $S_\pm$ is the massless Thirring $S$-matrix (3.54). Then (7.2) is clear, and (7.1) follows from a stationary phase argument as in Ref. [21]. More generally, as in the Ising and Federbush cases, $U_\pm$ and $S$ are the wave and scattering operators for all of the conserved charges and their "pull-backs" to $\mathcal{H}$; the dynamics $H_0$ is singled out only by its physical property of being the time translation generator of the representa tion of the Poincaré group.

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