Gauge Invariance and Implementability of the S-Operator
for Spin-0 and Spin-$\frac{1}{2}$ Particles
in Time-Dependent External Fields

S. N. M. Ruijsenaars*

Department of Physics, Princeton University, Princeton, New Jersey 08540
Communicated by Leonard Gross
Received February 21, 1978

It is proved that the classical $S$-operator for relativistic spin-0 and spin-$\frac{1}{2}$
particles in time-dependent external fields is gauge invariant, and that $S_{-\nu}$ and
$S_{+\nu}$ are entire functions of the coupling constant in the Hilbert-Schmidt norm.
As a result the Fock space $S$-operator exists for any real value of the coupling
constant, and is gauge invariant. The external fields and the gauge function are
assumed to be real-valued resp. complex-valued functions in $S(R^4)$.

1. INTRODUCTION

This paper deals with the $S$-operator for massive spin-0 and spin-$\frac{1}{2}$ particles
in time-dependent external fields, described by the Klein-Gordon resp. Dirac
equation. The external fields are described by infinitely differentiable, fast
decreasing functions on space-time. (The results could be easily extended to more
general functions, but we do not consider this.) In the spin-0 case we admit
scalar and vector fields, in the spin-$\frac{1}{2}$ case any linear combination of the five
well-known tensor types.

In Section 2 it is shown that the classical (i.e. single particle) $S$-operator is
gauge invariant in the sense that it remains the same if the vector field $A_a$ is
replaced by $A_a + \partial_a A$ with $A$ a (complex valued) function in $S(R^4)$. This
immediately results from the relation between the time evolution operators in
the presence or absence of the gauge term that we establish below (Eq. (2.4)).

The main result of Section 3 is that the transformation of the Fock space field
operators, generated by the classical $S$-operator, can be unitarily implemented
in Fock space for the fields mentioned above, i.e. that the Fock space $S$-operator
exists for these fields. This is a consequence of the more general result that
$S_{-\nu}$ and $S_{+\nu}$ are Hilbert-Schmidt entire functions of the coupling constant and
well-known implementability criteria.

* Work supported in part by NSF Contract MPS 74-22844.
In the spin-$\frac{1}{2}$ case the implementability result is not new. In fact, this has been recently proved by Palmer [1] for the Dirac equation in any space-time dimension with a larger class of time-dependent fields than we consider. Unfortunately, his proof is very long and complicated. Our method of proof is inspired by Palmer’s, but it is much shorter and simpler, and it can easily be used for the spin-0 case as well. It might also be useful for higher spin theories.

In the spin-0 case the result extends results of [2, 3]. In [2] Bellissard proves that implementability holds for the above-mentioned fields, provided the coupling constant is small enough (he also shows this in the spin-$\frac{1}{2}$ case), while in [3] it is shown that implementability holds for scalar and electric fields (in the spin-$\frac{1}{2}$ case electric and “pseudo-electric” fields). The present result lifts the restrictions on the coupling constant and the type of field.

The results of this paper have been announced in [4], to which the reader is also referred for general information on the external field problem.

2. Gauge Invariance

A. Spin-$\frac{1}{2}$

We consider the Dirac equation

\[ (-i\slashed{\partial} + m - B(x))\psi(x) = 0 \]  

(2.1)

and the same equation with an extra gauge term:

\[ (i\slashed{\partial} + m - B(x))\psi(x) = 0. \]  

(2.2)

Here,

\[ V(x) = \gamma^\rho B(x) \]  

(2.3)

is a function from $\mathbb{R}^4$ to the Hermitean $4 \times 4$ matrices, whose entries are in $S(\mathbb{R}^4)$. The complex-valued function $\Lambda$ in (2.2) is assumed to be in $S(\mathbb{R}^4)$ as well. We claim that the interaction picture time evolution operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, dx)^4$, associated with (2.2) resp. (2.1), are related by

\[ U(t, -\infty; V, \Lambda) = \exp(iHt) \exp(i\Lambda(t, \cdot)) \exp(-iHt)U(t, -\infty; V) \quad \forall t \in \mathbb{R}, \]  

(2.4)

where $H_0$ is the free Hamiltonian

\[ H_0 = -i\mathbf{u} \cdot \mathbf{v} + \beta m, \]  

(2.5)
which is self-adjoint on the Sobolev space $W^1_0(R^4)$ (cf. [3, Sect. 3A, B]). Since

$$U(t, -\infty; V, A)$$

is the unique solution of the (strong) integral equation

$$U(t, -\infty) = 1 + i \int_{-\infty}^{t} ds \exp(iH_0 s)[\mathbf{a} \cdot (\nabla A)(s, \cdot) + \dot{A}(s, \cdot) + V(s, \cdot)]$$

$$\cdot \exp(-iH_0 s) \ U(s, -\infty)$$

(2.6)

that is norm continuous and uniformly bounded on $R \cup \{-\infty, \infty\}$ (cf. [3, esp. Theorem 2.4]), it suffices to show that on a dense subspace $M$ of $\mathcal{H}$ one has in the strong topology

$$\frac{d}{dt} A(t, -\infty) = i \exp(iH_0 t)[\mathbf{a} \cdot (\nabla A)(t, \cdot) + \dot{A}(t, \cdot) + V(t, \cdot)]$$

$$\cdot \exp(-iH_0 t) A(t, -\infty).$$

(2.7)

where $A(t, -\infty)$ denotes the r.h.s. of (2.4). To prove this, we observe that as a consequence of [3, (2.29)] we have

$$U(t, 0; V)D(H_0) = D(H_0) \quad \forall t \in R$$

(2.8)

and therefore

$$U(t, -\infty; V)M = D(H_0) \quad \forall t \in R$$

(2.9)

where

$$M \equiv U(-\infty, 0; V)D(H_0)$$

(2.10)

(cf. [3, (2.21)]). Since $\exp(iA(t, \mathbf{x})) - 1 \in S(R^4)$ and Sobolev spaces are invariant under multiplication by test functions,

$$\exp(iA(t, \cdot))D(H_0) \subset D(H_0) \quad \forall t \in R.$$  

(2.11)

From (2.9) and (2.11) we conclude, using also [3, (2.22)], that on $M$ we have

$$\frac{d}{dt} A(t, -\infty) = i \exp(iH_0 t)[[H_0, \exp(iA(t, \cdot))]_+ \exp(-i\dot{A}(t, \cdot)) + \dot{A}(t, \cdot)$$

$$+ [\exp(i\dot{A}(t, \cdot)), V(t, \cdot)]_+ \exp(-i\dot{A}(t, \cdot)) + V(t, \cdot)]$$

$$\cdot \exp(-iH_0 t) A(t, -\infty).$$

(2.12)

Using (2.5) it follows that (2.7) holds. Hence, (2.4) holds. By taking the limit $t \to \infty$ in (2.4) we conclude

$$S(V, A) = S(V').$$

(2.13)
We have proved:

**Theorem 2.1.** The classical S-operator for spin-1/2 particles is gauge invariant in the sense described above.

In view of Corollary 3.2 below the Fock space S-operator exists for the fields considered above. Therefore (2.13) implies:

**Corollary 2.2.** The Fock space S-operator for charged spin-1/2 particles is gauge invariant.

B. Spin-0

We proceed in the same fashion as in A. We now have the Klein–Gordon equations

\[
(\partial_{\mu} - iA_{\mu}(x))[\partial_{\nu} - iA^{\nu}(x)] + m^2 - A_{\nu}(x) \phi(x) = 0
\]

resp.

\[
(\partial_{\mu} - iA_{\mu}(x) + (\partial_{\nu}A(x)))[\partial_{\nu} - i(A^{\nu}(x) + (\partial^{\nu}A)(x))] + m^2 - A_{\nu}(x) \phi(x) = 0,
\]

where \(A, \ldots, A_{4}\) resp. \(A\) are real-valued resp. complex-valued functions in \(S(R^4)\). We assert that (2.4) again holds true, where the operators \(U\) now denote the interaction picture evolution operators on \(\mathcal{H} = W_{\eta,2}(R^4) \oplus W_{\eta,4}(R^4)\) corresponding to (2.15) resp. (2.14), and where \(H_0\) is the free Hamiltonian

\[
H_0 = \begin{pmatrix}
0 & 1 \\
-\Delta + m^2 & 0
\end{pmatrix},
\]

which is self-adjoint on \(W_{\eta,2}(R^4) \oplus W_{\eta,4}(R^4)\) (cf. [3, Sect. 4A, B]). Reasoning as in A we infer that it suffices to show that on a dense subspace

\[
\frac{d}{dt} A(t, -\infty) = i \exp(iH_0t)
\]

\[
\begin{pmatrix}
(i\nabla \cdot [\mathbf{A} - (\nabla A)] + i[\mathbf{A} \cdot (\nabla A)] \cdot \nabla - i \mathbf{A} \cdot (\nabla A)^2 + A_{4} \cdot \mathbf{A} \cdot (\nabla A) + \mathbf{A}_{4} \cdot \mathbf{A}) (t, \cdot) \\
\exp(-iH_0t) A(t, -\infty)
\end{pmatrix}.
\]

However, (2.8–12) also hold in this case, but now in (2.12)

\[
V(t, \cdot) \equiv \begin{pmatrix}
i\nabla \cdot \mathbf{A} + i \mathbf{A} \cdot \nabla - i \mathbf{A} \cdot \mathbf{A}^2 + A_{4} \cdot \mathbf{A} \cdot (\nabla A) + \mathbf{A}_{4} \cdot \mathbf{A}
\end{pmatrix}(t, \cdot).
\]

(2.17)
Using this and (2.16) it is straightforward to verify (2.17), and thus (2.4) follows. As a result we obtain (2.13). We have proved:

**Theorem 2.3.** The classical S-operator for spin-0 particles is gauge invariant in the sense described above.

Since the Fock space S-operator exists by virtue of Corollary 3.4 below, we also have:

**Corollary 2.4.** The Fock space S-operator for charged spin-0 particles is gauge invariant.

### 3. Implementability

#### A. A Prototypical Case

The crucial idea of the proof that the off-diagonal parts of the S-operator (see below) are Hilbert–Schmidt (H.S.) is quite simple. For clarity we first treat a special case. We consider integral operators $T_{\epsilon n}^{(m)} (n \geq 1, \epsilon, \epsilon' = +, -)$ on $L^2(\mathbb{R}^3, dp)$ with kernels

$$T_{n}^{(m)}(p, q) = \int d\mathbf{k}_1 \cdots d\mathbf{k}_{n-1} \int_{-\infty}^{\infty} dt_1 \exp(iE_p\mathbf{t}_1) \hat{F}(t_1, \epsilon p - k_1) \int_{-\infty}^{t_1} dt_2 \times \exp(iE_{k_1}(t_2 - t_1)) \hat{F}(t_2, k_1 - k_2) \int_{-\infty}^{t_2} dt_3 \cdots \int_{-\infty}^{t_{n-1}} dt_n \times \exp(iE_{k_{n-1}}(t_n - t_{n-1})) \hat{F}(t_n, k_{n-1} - \epsilon' q) \exp(-iE_q t_n).$$

Here,

$$E_k = (k^2 + m^2)^{1/2}$$

and $\hat{F}(t, \mathbf{k})$ is the partial Fourier transform of a function $F(t, \mathbf{x})$ in $S(\mathbb{R}^3)$. One easily verifies that the integral is absolutely convergent and that $T_{\epsilon n}^{(m)}(p, q)$ is jointly continuous and $C^\infty$ in $p$ and $q$. As will be seen, the kernels $T_{\epsilon n}^{(m)}(p, q)$ are prototypes for the kernels that occur in the classical S-operator.

We claim that for any $l \in \mathbb{N}$ there is a $C$ such that

$$| T_{\epsilon n}^{(m)}(p, q) | \leq C^n (\epsilon! + 1)^{-l} (1 + |\epsilon p - \epsilon' q|^2)^{-l}.$$  

(Here and from now on $C$ denotes some positive number not depending on $n$.) Indeed,
Thus, changing variables,

\[ |T_{xx'}(p, q)| \leq C^* \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \prod_{i=1}^{n} \left( (1 + t_i^2)^{-1} \right) \int d^2 k_1 \cdots d^2 k_{n-1} \]

\[ \times (1 + |\epsilon p - k_1|^2)^{-1} \prod_{i=1}^{n-2} (1 + |k_i - k_{i+1}|^2)^{-1-2} \]

\[ \times (1 + |k_{n-1} - \epsilon' q|^2)^{-1-2}. \tag{3.4} \]

By using the estimates

\[ (1 + |t - s|^2)^{-1} \leq 2(1 + t^2)^{-1}(1 + s^2) \tag{3.6} \]

and

\[ 1 + \left| \sum_{i=1}^{n-1} s_i \right|^2 \leq (n - 1) \prod_{i=1}^{n-1} (1 + s_i^2) \tag{3.7} \]

(3.3) follows. Now it is well known that a measurable function \( T(p, q) \) satisfying

\[ \left( \sup_{p} \int dq |T(p, q)| \right) \left( \sup_{q} \int dp |T(p, q)| \right) = C^* < \infty \tag{3.8} \]

defines a bounded integral operator \( T \) (with \( ||T|| \leq C \)) on \( L^2(\mathbb{R}^3) \). (To see this, note that (3.8) implies that \( T \) is bounded from \( L^1 \) to \( L^1 \) and from \( L^\infty \) to \( L^\infty \), and apply the Riesz-Thorin theorem (5).) In view of (3.3) this implies that \( T_{xx'}^{(n)} \) is a bounded operator on \( L^2(\mathbb{R}^3) \) and that \( \sum_{n=1}^{N} T_{xx'}^{(n)} \) converges in norm to a bounded operator \( T_{xx'} \). (In the spin-0 and spin-\( \frac{1}{2} \) cases the corresponding assertions for the classical \( S \)-operator are a direct consequence of its structure (cf. [3, Sect. 2]), i.e. one does not need the above argument. We have given it nevertheless, since it might be useful for higher spin theories, and since we have occasion to use estimates like (3.3)).

We now assert that \( T_{xx'}^{(n)} \) are actually H.S. and that \( \sum_{n=1}^{N} T_{xx'}^{(n)} \) converge in \( || \cdot ||_2 \) to \( T_{xx'} \). This clearly follows if we can prove that for any \( l, m \in \mathbb{N} \) there is a \( C \) such that

\[ |(E_\rho + E_\varphi) T_{xx'}^{(n)}(p, q)| \leq C^*(n!)^{-1}(1 + |p + q|^2)^{-1}. \tag{3.9} \]
THE \( S \)-OPERATOR FOR EXTERNAL FIELDS

53

(Of course, (3.9) with \( l = m = 2 \) already suffices.) We prove (3.9) for \( T_{_{\pm \pm}}^{(n)} \). The proof for \( T_{_{\pm \pm}}^{(n)} \) is similar. We first observe that

\[
\exp(iE_p t + iE_q t) \, T_{_{\pm \pm}}^{(n)}(p, q) = \int dk_1 \cdots dk_{n-1} \int_{-\infty}^\infty dt_1 \exp(iE_{p_1} t) \hat{F}(t_1 - t, p - k_1) \int_{-\infty}^{t_1} dt_2 \times \exp(iE_{p_2} (t_1 - t_2)) \hat{F}(t_2 - t, k_1 - k_2) \int_{-\infty}^{t_2} dt_3 \cdots \int_{-\infty}^{t_n} dt_n \times \exp(iE_{p_n} (t_{n-1} - t_n)) \hat{F}(t_n - t, k_{n-1} + q) \exp(iE_{p_n} t) .
\]

Indeed, this follows by making the change of variables \( t_i \to t_i + t \) (\( i = 1, \ldots, n \)) on the right-hand side. Differentiating this identity \( m \) times with respect to \( t \), and setting \( t = 0 \) and

\[
F^{(i)}(t) = \partial^i_t F,
\]

we obtain

\[
(iE_p + iE_q)^m T_{_{\pm \pm}}^{(n)}(p, q) = \sum_{i_1, \ldots, i_m = 0}^m \frac{m!}{i_1! \cdots i_m!} \int_{-\infty}^{t_1} dt_1 \cdots \int_{-\infty}^{t_n} dt_n \times \exp(iE_{p_1} t_1) \hat{F}^{(i_1)}(t_1, p - k_1) \int_{-\infty}^{t_2} dt_2 \times \exp(iE_{p_2} (t_1 - t_2)) \hat{F}^{(i_2)}(t_2, k_1 - k_2) \int_{-\infty}^{t_3} dt_3 \cdots \int_{-\infty}^{t_n} dt_n \times \exp(iE_{p_n} (t_{n-1} - t_n)) \hat{F}^{(i_n)}(t_n, k_{n-1} + q) \exp(iE_{p_n} t).
\]

Now by the same estimates we used to prove (3.3) it follows that there is a \( C \) such that each term in the sum satisfies the estimate (3.3) with the same \( C \) for each term. Since the sum has less than \( (m + 1)^n \) terms (3.9) results, and thus our assertion is proved. Note that if one considers the more general integral operators

\[
T_{_{\alpha \alpha}}^{(n)}(p, q) = f_{_{\alpha \alpha}}(p, q) \, T_{_{\alpha \alpha}}^{(n)}(p, q),
\]

where \( f_{_{\alpha \alpha}}(p, q) \) are measurable functions satisfying

\[
|f_{_{\alpha \alpha}}(p, q)| \leq C(1 + \rho^p)(1 + \rho^q)^s, \quad r, s > 0,
\]

one can no longer conclude that \( T_{_{\pm \pm}}^{(n)} \) define bounded operators on \( L^2(R^3) \).
However, in view of (3.9) it again follows that $T_{\pm}^{(n)}$ are H.S., and that the operators $\sum_{n=1}^{N} T^{(n)}_{\pm}$ converge in $\| \cdot \|_{k}$ to H.S. operators.

In closing this subsection we should like to point out that the results obtained above can also be derived if one regards $T^{(n)}_{\pm}$ as operators on $L^2(\mathbb{R}^d)$ for any $k \geq 1$, i.e. if in (3.1) $p, k, q \in \mathbb{R}^d$ and $F \in S(\mathbb{R}^{k+1})$. Indeed, this easily follows by inspection of the estimates. Moreover, one could consider more general functions $F$. We leave this to the interested reader, however (cf. also (1)).

B. Spin-$\frac{1}{2}$

On the Hilbert space $\mathcal{H} = \mathcal{H}_{+} \otimes \mathcal{H}_{-}$, where $\mathcal{H}_{+}, \mathcal{H}_{-} = L^2(\mathbb{R}, dp)^{\otimes}$, the classical $S$-operator for external fields $V(x)$ as defined in Section 2A with coupling constant $\lambda$ is given by

$$(S(V))_{ij}(p) = f_{ij}(p) + \sum_{n=1}^{\infty} \sum_{i' = 1, 2} \int d\mathbf{q} R^{(n)}_{ij}(p, \mathbf{q}) f_{ij}^{(n)}(\mathbf{q}),$$

where

$$R^{(n)}_{ij}(p, \mathbf{q}) = (i\lambda)^n \sum_{e_1, \ldots, e_{n-1}} \int dk_1 \cdots dk_{n-1} \int_{-\infty}^{\infty} dt_1 (m|E_p|^{1/2} \tilde{v}_i(p)) \exp(iE_0 t_1)$$

$$\times \tilde{v}(t_1, \mathbf{p} - \mathbf{k}_1) \int_{-\infty}^{t_1} dt_2 [(E_{k_1} |\bar{\gamma}^0 - e_1 k_1 \cdot \gamma + \epsilon_1 m)](2E_{k_1})^{-1} \bar{\gamma}^0$$

$$\times \exp(-i\epsilon_1 E_{k_1}(t_1 - t_2)) \tilde{v}(t_2, \mathbf{k}_1 - \mathbf{k}_2) \int_{-\infty}^{t_2} dt_3 \cdots \int_{-\infty}^{t_{n-1}}$$

$$\times \exp(-i\epsilon_{n-1} E_{k_{n-1}}(t_{n-1} - t_n)) \tilde{v}(t_n, \mathbf{k}_{n-1} - \epsilon q)$$

$$\times \exp(-i\epsilon_{n} E_{\bar{a}}(t_n)) w_j^{(n)}(q)(m|E_{\bar{a}}|^{1/2})$$

(cf. [3, Sect. 3, esp. (3.35)]). We now have the following result.

**Theorem 3.1.** The operators $S(V)_{+}$ and $S(V)_{-}$ on $\mathcal{H}$ are $\| \cdot \|_{k}$-entire functions of $\lambda$.

**Proof.** We consider only $S_{+}$. The proof for $S_{-}$ is similar. It is clearly sufficient to show that $R^{(n)}_{+}$, considered as integral operator on $L^2(\mathbb{R}^d)$, is H.S., and that $\sum_{n=1}^{N} R^{(n)}_{+}$ converges in $\| \cdot \|_{k}$, uniformly in $\lambda$ for $\lambda$ in any compact subset of $C$. To prove this we use the results of the preceding subsection. By comparing (3.1) and (3.16) one sees that $R^{(n)}_{+}$ is equal to a sum of $2^{n-1} 4^{2n} \mathbf{a}$ terms that have the same structure as $T^{(n)}_{+}$, the only differences being that the...
$E_{\phi}$ in (3.1) can also have a minus sign, that additional bounded functions of $p, k_i$ and $q$ occur in the integrand, and that instead of one test function $F$ sixteen different test functions $V_{p,q}$ can occur. However, with these differences the analogue of (3.12) still holds true for all these terms, as follows from inspection of (3.10). Since only a finite number of derivatives of a finite number of test functions is involved, there is no difficulty in verifying that the analogue of (3.9) holds for each term, and thus that

$$[E_p + E_q] \circ R_{p,q}^{(a)}(p, q) | \leq | \lambda |^{-\frac{1}{2} C''(\eta)} (1 + | p + q |)^{\alpha}. \quad (3.17)$$

Consequently, our assertion follows, so the theorem is proved.

**Corollary 3.2.** The Fock space $S$-operator for charged spin-$\frac{1}{2}$ particles interacting with the external fields described above exists for any real value of the coupling constant.

**Proof.** Cf., e.g., [6].

**C. Spin-0**

In this case one has $\Psi = \Psi_{-} \oplus \Psi_{+}$ with $\Psi_{-} = L^2(R^3, dp)$, and the $S$-operator on $\Psi$ satisfies

$$(S(\lambda V)f)(p) = f(p) + \sum_{n=1}^{\infty} \sum_{\epsilon = \pm 1} \int \int \int_{-\infty}^{\infty} dt_1 dt_2 dt_3 (2E_p)^{-1/2} \exp(iE_p t_1)$$

$$\times [\lambda \hat{A}(t_1, \epsilon p - k_1) (\epsilon p + \epsilon E_{\phi}) - \lambda \hat{A}(t_1, \epsilon p - k_1) \cdot (\epsilon p + k_1)]$$

$$\times \exp(-i\epsilon \lambda E_{\phi}(t_1 - t_2))(\lambda \hat{A}(t_2, k_1 - k_2) (\epsilon_1 E_{\phi} + \epsilon_2 E_{\phi}) - \lambda \hat{A}(t_2, k_1 - k_2) \cdot (k_1 + k_2) - \lambda \hat{A}(t_2, k_1 - k_2)]$$

$$\times [\lambda \hat{A}(t_2, k_1 - k_2) (\epsilon p + \epsilon E_{\phi}) - \lambda \hat{A}(t_2, k_1 - k_2) \cdot (\epsilon p + k_1)]$$

$$\times \exp(-i\epsilon \lambda E_{\phi}(t_1 - t_2))(\lambda \hat{A}(t_1, \epsilon p - k_1) (\epsilon_1 E_{\phi} + \epsilon_2 E_{\phi}) - \lambda \hat{A}(t_1, \epsilon p - k_1) \cdot (\epsilon p - k_1)$$

$$\times \lambda \hat{A}(t_1, \epsilon p - k_1) \cdot (\epsilon p - k_1)]$$

$$\times \exp(-i\epsilon \lambda E_{\phi}(t_1 - t_2))(\lambda \hat{A}(t_1, \epsilon p - k_1) (\epsilon_1 E_{\phi} + \epsilon_2 E_{\phi}) - \lambda \hat{A}(t_1, \epsilon p - k_1) \cdot (\epsilon p - k_1)$$

$$\times \lambda \hat{A}(t_1, \epsilon p - k_1) \cdot (\epsilon p - k_1)]$$

$$(3.19)$$

(cf. [3, Sect. 4, esp. (4.30)]).
Theorem 3.3. The operators $S(\lambda^2\gamma_\pm)$ and $S(\lambda\gamma_\pm)$ are $\|\cdot\|$-entire functions of $\lambda$.

Proof. Assume first that $A_\mu = 0$. We can then argue in the same fashion as in the spin-$\frac{1}{2}$ case to conclude that

$$\|(E_p + E_q) R_{\pm}^{(m)}(p, q)\| \leq (1 + |\lambda|^2)^m C^m(p) (-1 + p^2 q^2)^{-\frac{1}{2}}, \quad (3.20)$$

from which the statement follows. Let now $A_\mu \neq 0$. Then the additional functions $p^\mu E_{-\mu}^{-1/2}$, $q^\nu E_{-\nu}^{-1/2}$ and $k_i^\mu k_i^{-\nu} E_{-\nu}^{-1}$ are unbounded, and so we cannot immediately conclude that (3.20) holds in this case as well. However, we have ($\epsilon = +, -$)

$$|k_\nu^\mu + e k_{i+1}^\nu| \leq |k_i - k_{i-1}| + 2E_{i+1}, \quad \mu = 0, \ldots, 3, \quad i = 0, \ldots, n - 2,$$

where $k_0 = p$, and

$$|k_{n-1}^\nu + e q^n| \leq |k_{n-1} + q| + 2E_q, \quad \mu = 0, \ldots, 3. \quad (3.21)$$

We can now write down the analogue of (3.12) for $R_{\pm}^{(m)}(p, q)$ (using (3.19)) and estimate as before, taking $m = 3$ and using (3.21) and (3.22). The first terms on the right-hand sides of (3.21–22) are taken care of by the test functions, while the second term in (3.21) is canceled by the factors $(2E_{i+1})^{-1}$ in (3.19). Using the extra factor $(E_p + E_q)$ to get rid of the second term in (3.22) one again obtains (3.20). The theorem is proved.

Corollary 3.4. The Fock space $S$-operator for charged spin-0 particles interacting with scalar and vector fields described by real-valued functions in $S(R^4)$ exists for any real value of the coupling constant.

Proof. Cf., e.g., [6].

In [7, Theorems 2.4, 3.4] analyticity properties of the Fock space $S$-operator have been derived. Theorems 3.1 and 3.3 imply that the number $l_\rho$ introduced there equals $\infty$, and thus that $l_\rho = l_{\rho}^*$ resp. $l_\rho = l_{\rho}^*$ (cf. [7, (2.78) resp. (3.64)]).

Acknowledgment

The author would like to thank the referee for suggesting several improvements.

References