Integrable Systems: An Overview

Preamble. The following pages present a bird's eye view on the field of integrable systems in the widest sense, including some historical perspective, a sketch of the mathematical problems and interconnections associated with the various systems, and of their applications in science. Expert readers are cautioned that this survey is somewhat Procrustean. Indeed, it is outside our scope to cover all aspects of this vast and kaleidoscopic area, and inevitably our selection is influenced by our previous work and limited knowledge. For brevity, we mention key ideas and methods via a small number of systems, typically those dating back to the pioneering stages of the class of systems under consideration. Furthermore, various models and concepts are referred to without explanation (e.g., Ising model, XYZ chain, moment map, Baker-Akhiezer functions, transfer matrix, Yang-Baxter equation,...). If need be, basic information on such items can be gleaned from web sources, for example Wikipedia and Google.

Contents

1. Classical integrable systems

The concept of 'completely integrable system' arose in the 19th century in the context of finite-dimensional classical mechanics. Hamilton had reformulated Newton's equations in a novel and—as it turned out—seminal form. He introduced so-called canonical coordinates $x_1, \dots, x_n$ (generalized positions) and $p_1, \dots, p_n$ (generalized momenta) to describe a mechanical system with n degrees of freedom. The temporal evolution of any initial state $(x_0, p_0) \in \mathbb{R}^{2n}$ is then governed by Hamilton's equations of motion $\dot{x} = \nabla_p H, \dot{p} = -\nabla_x H$ (where the dot denotes the time derivative), with the Hamiltonian $H(x,p)$ encoding the total energy of the system.

Using the Poisson bracket $\{A, B\} = \nabla_x A \cdot \nabla_p B - \nabla_p A \cdot \nabla_x B$ for functions A, B on the phase space Ω (the range of variation of the canonical coordinates (x,p)), Hamilton's equations can be rewritten as $\dot{x}_j = \{x_j, H\}, \dot{p}_j = \{p_j, H\}, j = 1, \dots, n$. Also, a conserved quantity I ('first integral') can be characterized by its Poisson commuting with H, i.e., $\{I, H\} = 0$. For example, H clearly Poisson commutes with itself, which expresses conservation of energy. Transformations $\Omega \to \Omega', (x,p) \mapsto (x', p')$ preserving Hamilton's equations are called canonical. The Hamilton formalism is invariant under such canonical maps. In particular, the canonical commutation relations $\{x_j, x_k\} = 0, \{p_j, p_k\} = 0, \{x_j, p_k\} = \delta_{jk}, j, k = 1, \dots, n$, and more generally Poisson brackets are invariant.

A dynamical system defined by a given Hamiltonian H on a $2n$-dimensional phase space

Ω is now called (completely) integrable if there exist additional functions $H_1, \dots, H_n$ on Ω (again referred to as 'Hamiltonians') such that $H_1, \dots, H_n$ are independent and in involution (i.e., all Poisson brackets $\{H_j, H_k\}$ vanish). Thus these Hamiltonians are conserved under the Hamiltonian evolution on Ω generated by each of them.

For integrable systems there is a key result (the Liouville-Arnold theorem) ensuring that there exists a canonical transformation to action-angle coordinates (x', p') such that the transformed Hamiltonians depend only on the action variables p'. As a consequence, the actions $p'_1, \dots, p'_n$ are conserved, whereas the angles $x'_1, \dots, x'_n$ evolve linearly in the evolution parameters $t_1, \dots, t_n$, as opposed to the dependence of x, p on $t_1, \dots, t_n$, which is highly nonlinear for the systems of interest. Hence one can explicitly solve ('integrate') Hamilton's equations, provided the action-angle map is explicitly known.

Although various integrable systems were discovered in the 19th century (e.g., the Euler and Kovalevskaya tops, Jacobi's geodesic flow on an ellipsoid), the subject lay dormant during the first seventy years of the 20th century. Results by Poincaré, to the effect that integrability is a highly exceptional property for the systems usually considered in classical mechanics, were an important factor contributing to this lack of interest.

This state of affairs changed dramatically after the discovery of the soliton phenomenon by Zabusky and Kruskal (1965). Before their work, soliton theory did not go far beyond some insights concerning the Korteweg–de Vries (KdV) equation $u_t + 6uu_x + u_{xxx} = 0$. This equation was introduced a century ago, to model the propagation of water waves in canals (t denotes time, x the position along the canal, and $u(t, x)$ the height of the wave). The empirical discovery by the Scottish engineer Russell of the solitary wave, a hump of water moving with constant speed and shape along the canal, dates back even further (1834). The corresponding solution $u = 2a^2/\cosh^2(a(x - x_0) - 4a^3 t)$ to the KdV equation also dates from the 19th century.

On the other hand, the extraordinary stability properties of these solitary waves were discovered much later in computer calculations by Zabusky and Kruskal. (Their research was partly inspired by similar unexpected stability results in a numerical simulation of energy flow in anharmonic lattices by Fermi, Pasta and Ulam.) They studied collisions of n solitary waves, and found that these waves emerge unscathed, with the same velocities and shapes as before the collision. A priori, this is not to be expected, since the KdV equation is nonlinear, so that solutions cannot be linearly superposed. In fact, once the pertinent solutions were found in explicit form, it became clear that the presence of a nonlinear interaction does show up: the positions of the solitary waves are shifted, compared to the positions arising from a linear superposition. Moreover, the shifts can be written as sums of pair shifts, leading to a physical picture of individual entities scattering independently in pairs. These particle-like properties led to the coining of the term 'soliton'.

The connection with the concept of 'completely integrable system' was first made by Zakharov and Faddeev (1971). Kruskal and coworkers had shown that the KdV equation has an infinite number of conservation laws, and that there exists a linearizing transformation, which maps the initial value $u(0, x)$ for the KdV Cauchy problem to spectral and scattering data of the Schrödinger operator $-d^2/dx^2 - u(0, x)$. The nonlinear evolution yielding $u(t, x)$ then transforms into an essentially linear time evolution of these data, so that $u(t, x)$ can be constructed via the inverse map, the so-called Inverse Scattering Transform (IST). Inspired by these findings, Zakharov and Faddeev showed that the KdV equation may be viewed as an infinite-dimensional classical integrable system, the spectral and scattering data being the action-angle variables, the IST the (inverse of the) action-angle map, and the infinity of conserved quantities the Poisson commuting Hamiltonians.

Ever since these pioneering works, the number of nonlinear partial differential equations in two space-time variables admitting n-soliton solutions has steadily increased, the most well-known examples being the KdV, modified KdV, sine-Gordon and (attractive) nonlinear Schrödinger equation. For many of these equations additional structural features have been shown to be present, including the existence of infinitely many conserved quantities, a Lax pair formulation, Bäcklund transformations, a prolongation algebra, and a linearizing map playing a role comparable to Fourier transformation for linear PDEs.

Nonlinear PDEs without solitons, but with several of the latter features, were also discovered, the repulsive nonlinear Schrödinger equation being a prime example. Furthermore, it was realized that two well-known relativistically invariant PDEs belong to this category. These are the massless Thirring and Federbush models, named after the authors who solved these PDEs through a linearizing map several years before the emergence of the soliton concept.

Important integrable equations of a different kind include higher-dimensional PDEs with soliton solutions (such as the Kadomtsev-Petiashvili and Davey-Stewartson equations) and soliton lattices (the infinite Toda lattice being a prime example). A further large class of equations consists of integrable discretizations of soliton equations, and discrete equations associated with Bäcklund transformations of soliton equations. The latter systems have been classified by Adler, Bobenko and Suris, yielding a rather short list that includes long-known lattice equations such as the discrete KdV and Nijhoff/Quispel/Capel equations. An infinity of explicit solutions is known for most equations on the ABS list. These solutions are also referred to as solitons, but in the pertinent lattice setting there is no clear notion of 'time' and 'scattering'.

At this point it should be mentioned that there exists no widely accepted definition of 'integrability' for classical systems with infinitely many degrees of freedom. Rather, the term is used whenever some of the above structural features are present. For example, the Euclidean self-dual Yang-Mills equations admit a Lax pair formulation and an infinity of (more or less) explicit solutions (the instantons), so that they are viewed as 'integrable', even though there is no notion of evolution. (The equations are elliptic rather than hyperbolic.) As a second example the Ernst equations from general relativity may be mentioned; here, the 'soliton solutions' are axially-symmetric space-times. As a final example, the discrete equations on the ABS list arise by imposing 'consistency around the cube' (an integrability condition closely related to the Yang-Baxter equation that will reappear several times in the sequel); on the other hand, a more general equation arises when one views integrability in terms of vanishing algebraic entropy (Viallet).

As concerns classical systems with finitely many degrees of freedom, the discovery of the soliton phenomenon led to a renewed interest in systems that are integrable in the aforementioned 19th century sense. As a result, a considerable number of new integrable systems were discovered in the seventies, the most prominent being the Calogero-Moser and Toda systems. These are n-particle systems whose defining Hamiltonian is of the form $H = \sum_{j=1}^{n} p_j^2/2m + U(x_1, \ldots, x_n)$. For the Toda systems the potential $U(x)$ is a certain sum of exponential pair potentials. For the Calogero-Moser systems one has $U(x) = \lambda \sum_{1\leq j<k\leq n} V(x_j - x_k)$. The simplest case is the rational potential $V(x) = 1/x^2$. This can be generalized to a hyperbolic or trigonometric potential (namely, $\nu^2/\sinh^2(\nu x)$ and

$\nu^2/\sin^2(\nu x)$, with $\nu > 0$), the most general integrable case being the elliptic pair potential $\wp(x;\omega_1,\omega_2)$ (with $\wp$ the Weierstrass function with periods $2\omega_1 \in (0,\infty), 2\omega_2 \in i(0,\infty)$).

For the latter systems the defining Hamiltonian H can be viewed as the time translation generator of a representation of the nonrelativistic space-time symmetry group (the Galilei group), the space translation generator being the total momentum $P = \sum_{j=1}^{n} p_j$ and the Galilei boost generator being given by $B = m\sum_{j=1}^{n} x_j$. The pair potential structure of H can be tied in with the Lie algebra A_{n-1}, and there exist integrable versions of these systems for all of the simple Lie algebras.

In the eighties it became clear that there exist integrable relativistic generalizations of the nonrelativistic Toda and Calogero-Moser systems. The defining Hamiltonian H for these systems is a sum of n terms, each of which is the product of the relativistic kinetic energy $mc^2\cosh(p_j/mc)$ (with c the speed of light) and a potential energy term involving the exponential function and the $\wp$-function, resp. The Lorentz boost generator coincides with the Galilei boost, whereas the space translation generator P (total momentum) differs from H only by the replacement of cosh by sinh. The requirement that H and P Poisson commute is very stringent and fixes the pair potential. As a bonus of this relativistic invariance requirement, integrability is preserved. Integrable versions of these systems (associated with A_{n-1}) are now known for all simple Lie algebras.

As it has turned out, the hyperbolic relativistic (A_{n-1}) Calogero-Moser systems are intimately related to the n-soliton solutions of a host of soliton equations. Indeed, the existence of these relativistic systems was first established (by Ruijsenaars and Schneider) with an eye on replacing the interaction of n sine-Gordon solitons by that of an equivalent interaction between n relativistic point particles, in the sense that the same scattering occurs (conservation of momenta and factorization in pair shifts). In fact, however, the relation to the sine-Gordon particle-like solutions (solitons, antisolitons, and their bound states, the so-called breathers) is much closer than can be readily explained. The solitons of other equations are linked to the particle systems by choosing different generators for the space-time dependence of the particles. In more detail, all of the Hamiltonians serving as space and time translation generators for the various soliton equations are obtained as suitable functions of the Lax matrix. The latter matrix is the key tool in constructing the action-angle map in great detail, and the soliton velocities correspond to suitable functions of the actions.

After this survey of classical integrable systems we turn to their role in the natural sciences. Especially for the various soliton PDEs the range of applications and the new insights spawned by them have been so fruitful and prolific, that it is often said that solitons have revolutionized nonlinear applied science. Besides modelling a wide variety of wave phenomena in physical contexts like hydrodynamics, acoustics, nonlinear optics, plasma physics and solid-state physics, they have found applications in quite different areas, such as molecular biology (energy transport along proteins), ecology (predator-prey equations), chemistry (charge density waves in organic conductors) and electronics (network equations). Especially striking is the ubiquity of the KdV, sine-Gordon and nonlinear Schrödinger equations. For example, the sine-Gordon equation $\phi_{xx} - \phi_{tt} = \sin\phi$ is used as a model for the propagation of dislocations in crystals, phase differences across Josephson junctions, torsion waves in strings and pendulas, and waves along lipid membranes. This equation arose already in the context of pseudo-spherical surfaces (which led Bäcklund to his now famous transformations), and is also used (in quantized form) as a toy model for elementary particles.

We proceed to discuss the mathematical aspects of classical integrable systems. From the perspective of dynamical systems they comprise a class of nonlinear PDEs and ODEs of a very special character, since minute changes suffice to destroy their explicit solvability. The existence of stable solitary wave solutions in the PDE context may be viewed as being due to a delicate balance between two competing effects: the nonlinearity of the equation tends to focus the waves, whereas the linear, dispersive part tends to pull them apart. The soliton therefore only appears in a nonlinear and dispersive context, and is even in this context non-generic. The special character of integrable PDEs also shows up in associated ODEs: the only movable singularities of the latter are poles (the Painlevé property).

The systems at issue are connected to many other subfields in mathematics. In particular, the construction and study of the above-mentioned linearizing map leads to questions in functional analysis (scattering and inverse scattering theory, spectral theory, integral equations,...), function theory and algebraic geometry (Riemann-Hilbert problem, elliptic, hyperelliptic, Baker-Akhiezer and theta functions, Jacobi varieties,...), differential geometry (geodesic flows on groups and symmetric spaces), symplectic geometry (Hamiltonian structures, moment map, Weinstein-Marsden reductions,...), and in Lie algebra and Lie group theory.

More generally, the latter area is intimately related to integrable system theory. Finite-dimensional systems connected with the finite-dimensional simple Lie algebras have already been mentioned above, but as it has turned out, there are also close links between affine Kac-Moody Lie algebras and infinite integrable systems. In particular, the Kyoto school (Date, Jimbo, Miwa, Sato,...) pioneered this connection. In their prolific work Hirota's τ-function, in terms of which soliton solutions and their theta-function generalizations can be written, has lost its enigmatic character and has risen to great prominence. Their framework furnishes a new view on the hierarchy of soliton PDEs (corresponding to the sequence of conservation laws) and enables a classification of various hierarchies by tying it to the representation theory of Kac-Moody algebras.

It is an intriguing circumstance that the arena of the Kyoto work concerning classical integrable systems is a well-known quantum-field theoretical construct, namely fermion Fock space. In fact, the Kyoto group was led to this arena by their previous work on certain quantum field models. These models are examples of quantum integrable systems, to whose discussion we now turn.

2. Quantum integrable systems

This section is concerned with certain 1-dimensional quantum systems and 2-dimensional lattice systems from classical statistical mechanics. In these settings, the adjective 'integrable' is used in various ways, depending on the model in hand. When the Poisson commuting Hamiltonians of a finite-dimensional integrable system admit a quantization such that the quantum versions commute, the resulting quantum system is universally called 'integrable'. For infinite-dimensional systems and finite lattice systems, however, there is less agreement on the notion of integrability. Often the term 'integrable' is used as a synonym for 'solved' or 'soluble'. However, what it means for a model to have been solved or to be solvable depends on the type of model and also on the views of individual workers in the area.

On the other hand, in cases where infinite quantum systems have an associated scattering behaviour, they are universally viewed as 'integrable' when the S-operator has a soliton structure. That is, the number of particles and the set of incoming momenta are conserved in the scattering process. In all cases where this is known to occur, the n-particle S-operator is also factorized as a product of S-operators for each particle pair. Just as at the classical level, this leads to a picture of independent pair collisions.

In any case, we restrict the discussion to systems for which the epithet 'integrable' is widely accepted. It is expedient to divide these systems in three classes. The first class consists of spin chains and two-dimensional lattice models. This class has proliferated considerably in the last twenty years, so we mainly discuss the prime examples, namely the XYZ chain and 8-vertex model, and their specializations. The second class consists of integrable quantum field theories. Again we only sketch the salient features of a representative sample. The third class consists of the integrable quantum versions of the aforementioned classical Calogero-Moser and Toda systems.

2.1 Spin chains and lattice systems

Bethe's seminal work on the XXX (isotropic) Heisenberg chain dates back to 1933. The key idea in his solution was his famous Ansatz, leading to explicit eigenfunctions of the chain Hamiltonian, thereby revealing the presence of particle-like excitations in the model (called magnons or spin waves in this case). Some twenty years ago Babbitt and Thomas rigourized and extended his results for the infinite chain in its ground state, proving orthogonality and completeness of the eigenfunctions, and showing that the magnons and their bound states have a solitonic scattering behaviour.

In equilibrium statistical mechanics it is customary to regard a model as 'exactly solved', once its infinite-volume partition function is known in closed form. Indeed, this leads to an explicit determination of the thermodynamics (vacuum energy density/free energy per site/pressure,..., depending on the model and its physical interpretation). The first 2-dimensional model that was solved in this sense is the Ising model on a square lattice (by Onsager in the forties). The next landmark work in this area was done some 20 years later by Lieb, who solved (special cases of) the 6-vertex model using techniques related to Bethe's Ansatz. Similar methods also played an important role in Baxter's solution of the more general 8-vertex model (early seventies).

After this pioneering period this area mushroomed considerably. A key ingredient for the solution of these 2-dimensional classical lattice systems and related ones consists in obtaining sufficient information on their transfer matrices. The latter also lead to an intimate relation of the systems to 1-dimensional quantum chain models. In particular, the Ising model corresponds to the XY spin chain, the 6-vertex model to the XXZ chain, and the 8-vertex model to the XYZ chain. In fact, the latter link enabled Baxter to calculate the vacuum energy density of the XYZ Hamiltonian.

For the 2-dimensional Ising model on a square lattice much more is known than the partition function. For example, the correlation functions are known explicitly for any temperature. Using this information it has been possible to control the continuum limit (temperature $\to$ critical temperature, lattice spacing $\to 0$, correlation length fixed). In this limit the correlation functions have been shown to converge to the Schwinger functions of an integrable relativistic quantum field theory, the so-called continuum Ising model.

For the far more challenging 6-vertex and 8-vertex models great strides forward have been made in the calculation of correlation functions, in particular via the algebraic Bethe Ansatz and the elegant solution of the quantum inverse problem by Maillet and Terras. Here one is nevertheless still far removed from controlling critical limits. In particular, it seems intractable to substantiate the conjectured convergence of 6-vertex model correlation functions to those of the quantum sine-Gordon field theory under a suitable continuum limit.

To be sure, there are several 'exactly solved' statistical mechanics models that have a quite different character, such as the strong coupling BCS model. These models seem not to be amenable to a form of Bethe Ansatz (which is closely related to quantum soliton behaviour), and are not likely to lead to integrable quantum field theories in suitable continuum limits.

In view of their low-dimensionality, the applicability of the above spin chains and lattice systems to natural phenomena would seem to be limited. Even so, they turn out to be remarkably accurate models for some physical systems that are approximately 1- or 2-dimensional, for example, thin films and crystals with highly anisotropic interactions. Furthermore, they are in fact playing a crucial role in equilibrium statistical mechanics, in particular in testing general approximation methods used to study critical phenomena.

A striking feature of the above models is their intimate relation to elliptic and allied functions. The latter arise in particular via the transfer matrices already mentioned. A pivotal feature of the solution methods is that there exists a 1-parameter family of commuting transfer matrices, which may be viewed as a quantum counterpart of the Poisson commuting Hamiltonians in classical integrable systems. Such a family arises via a nontrivial solution to an algebraic equation for the Boltzmann weights, called the star-triangle or Yang-Baxter equation. The crux is now that such solutions can be found by parametrizing the Boltzmann weights with elliptic theta functions.

From a mathematical viewpoint the developments in this area are connected to a wide range of issues in algebra and combinatorics. In particular, the representation theory of various algebraic structures (including quantum groups and quantum affine algebras) can be used to great advantage. Further relevant areas of mathematics include the theory of Toeplitz/Wiener-Hopf operators, and more generally functional analysis, especially in rigorous studies of the infinite-volume and continuum limit.

2.2. Quantum field theories

The physical picture associated with an integrable quantum field theory is drastically different from that of its classical counterpart. To explain this, we recall that there are classical integrable field theories without soliton solutions (for example, the Federbush model and the repulsive nonlinear Schrödinger equation). Also, even when pure soliton solutions do exist, the solution to a general Cauchy problem will also have a radiation (decaying) component. By contrast, integrable quantum field theories describe solely states of arbitrarily many quantum particles that exhibit soliton scattering. (It should however be noted that wave packets decay for asymptotic times, unless bound states are present.) Roughly speaking, the classical radiation turns into quantum solitons, whereas classical solitons with internal degrees of freedom ('breathers') show up as soliton bound states.

At the quantum level the nonlinear Schrödinger equation yields a field theory whose

Hilbert space arena is a boson Fock space. Its Hamiltonian leaves the n-particle subspace invariant. On this subspace it equals the n-dimensional Laplacean plus δ-function pair potentials. It is a striking historical fact that the soliton character of the nonlinear Schrödinger equation was revealed at the quantum level before the classical concept of soliton arose. Indeed, in the repulsive regime its Hamiltonian was diagonalized by a Bethe Ansatz in the early sixties, and the soliton scattering property can be read off from the eigenfunctions. (In the same sense, the quantum soliton arose in Bethe's work on the XXX model more than 30 years before the discovery of the classical soliton.) The attractive case can also be solved by a Bethe Ansatz. Here, n-particle bound states are present for all n, reflecting the state of affairs on the classical level.

If the nonlinear Schrödinger Hamiltonian is assumed to act on the Fock space for distinguishable particles, it can still be formally diagonalized by a Bethe Ansatz, as shown by Yang (1968). In this more general case the solitonic collision picture leads to a striking property of the scattering amplitudes. Specifically, their factorization in pair amplitudes is independent of the temporal ordering of collisions. This amounts to an algebraic equation that is nowadays known as the Yang-Baxter equation, and that we already encountered above in the very different setting of vertex models.

The relativistically invariant Federbush and Thirring models (and some related models, such as the Gross-Neveu model) have two vastly different quantum versions. First, the nonlinear PDEs at hand can be quantized without insisting that the resulting Hamiltonian be positive. This yields Fock space theories that may be viewed as relativistic analogues of the quantized nonlinear Schrödinger equation: On the n-particle subspace the Hamiltonian is formally equal to a sum of n free Dirac operators plus δ-function pair potentials. The Hamiltonian can then be diagonalized by a Bethe Ansatz, which reveals the soliton character of the quantum particles. The massive Thirring model exhibits soliton bound states, whereas the massless Thirring and Federbush models do not have bound states. This once more mirrors the classical situation: viewed as nonlinear PDEs, the massive Thirring model has solitons with internal degrees of freedom, whereas the two other models have only pure radiation solutions.

When on insists on a quantization that gets rid of the unphysical negative energies (by 'filling the Dirac sea'), then these models become extremely singular, just as any other positive energy interacting relativistic quantum field theory. Besides ultraviolet divergencies (which are renormalizable, but not superrenormalizable for the models at hand), the massless Thirring model has severe infrared divergencies. This gives rise to there being two very different 'exactly solved' versions on the positive energy quantum level. For the Federbush model Wightman pointed the way towards a mathematically rigorous solution in the sixties. In the eighties it was proved that his quantum fields indeed satisfy all of the Wightman axioms, and the soliton S-operator formally obtained by Federbush was rigorously reobtained via the Haag-Ruelle theory. Together with the closely related continuum Ising model, these are the only positive energy relativistic soliton quantum field theories whose nonperturbative solution is under analytic control.

To be sure, much more is known about the positive energy Federbush and continuum Ising models. The explicit expressions for the lattice correlation functions obtained by McCoy, Tracy and Wu, and their penetrating studies on the connection of their continuum limit 2-point functions with Painlevé transcendents triggered a far-reaching research program by Sato, Miwa and Jimbo. In a series of papers the latter authors obtained detailed information on the Wightman functions of the above models and related ones. They showed in particular that the n-point functions are linked to monodromy preserving deformation equations, generalizing the Painlevé ODEs pertaining to the $n = 2$ case.

We proceed to survey the remaining positive energy relativistic quantum field theories that are widely referred to as 'integrable' or even 'solved'. These field theories have a different analytical status, compared to the ones already mentioned. Indeed, it has not yet been proved that field theories with the explicit integrable features described below exist in the precise sense of axiomatic and constructive field theory. This is intimately related to more general problems concerning relativistically invariant Lagrangeans. In particular, even after Herculean efforts in constructive field theory, it is not known whether such a starting point can be promoted to a well-defined and unique positive energy quantum field theory.

The most prominent integrable field theories yet to be discussed are the sine-Gordon equation/massive Thirring model and nonlinear σ-models. The first two theories are vastly different as classical PDEs, but are now very widely viewed as being the same on the (positive energy) quantum level. The canonical picture is that the soliton and antisoliton of the classical sine-Gordon equation manifest themselves in the quantum version as the fermion and antifermion of the quantized massive Thirring model. Moreover, the breathers correspond to fermion-antifermion bound states. There is less agreement on the tenet that the radiation reveals itself at the quantum level as the lowest-energy bound state.

This boson-fermion equivalence was first argued to hold true by Coleman (1975), using perturbation theory and an already known special case of boson-fermion correspondence. At the same time, the famous DHN-formula for the bound-state energies was derived from a semi-classical quantization scheme. Not much later, it was shown that the infinitely many conservation laws of the massive Thirring model survive quantization (in the sense of perturbation theory—no anomalies), entailing conservation of particle number and solitonic scattering. This insight and other suggestive previous results came to a head with Zamolodchikov's discovery of the explicit S-operator for arbitrary values of the coupling constant (1977).

Soon thereafter, also the soliton-type S-operators of the nonlinear $O(N)$ σ-models were found. These S-matrices have all the features that are expected to hold (such as unitarity, crossing and factorization), and have been checked in perturbation theory. Factorizing S-matrices for other groups than $O(N)$ have also been found, but only few of these have been attributed to field theories.

A closely related development originated in the same period: the emergence of the quantum inverse scattering method. Roughly speaking, this is a quantum analogue of the classical inverse scattering method, with the Bethe Ansatz playing the role of the linearizing map. This method led in particular to a systematic derivation of the DHN-formula and S-matrix for the sine-Gordon equation/massive Thirring model. The starting point is a finite lattice approximation that can be treated via a Bethe Ansatz. The state of interest is not the vacuum state for the resulting pseudoparticles (the approximate pseudo-vacuum), but rather the state of lowest energy (the approximate true vacuum). When taking the cutoffs away one meets the usual divergence problems, which to date have not been rigorously handled.

The quantum inverse method, as applied to this class of models, evolved into a program to calculate form factors of the nonperturbative field operators, with the ultimate aim of determining correlation functions by summing over all relevant form factors. In the hands of Karowski, Smirnov, and a small army of further workers, this led to a lot of progress over the past decades. Even so, detailed information on correlation functions still seems out of reach.

At this point it should be mentioned that there do exist rigorous constructions starting from cutoff Langrangeans for the sine-Gordon, massive Thirring and Gross-Neveu models, yielding the existence of Wightman functions with the expected axiomatic properties. In this framework, however, it seems intractable to push through the Haag-Ruelle scattering theory so as to verify the soliton scattering picture. Worse yet, from this constructive perspective no distinction between these field theories and nonintegrable ones seems to be present.

Besides modelling some solid-state phenomena (including the Kondo effect), integrable quantum field theories are used as a laboratory to develop and test techniques and ideas that can be used in a nonperturbative study of more realistic quantum field theories. For example, the 2-dimensional nonlinear σ-models exhibit formal analogies with 4-dimensional non-abelian Yang-Mills theories, whose physical relevance is beyond doubt.

2.3. Quantum Calogero-Moser and Toda systems

The canonical quantization prescription $p_j \to -i\hbar\partial/\partial x_j, j = 1, \ldots, n$, (with $\hbar$ Planck's constant) yields an unambiguous defining quantum Hamiltonian for the nonrelativistic (A_{n-1}) Calogero-Moser and Toda cases: It is given by the Schrödinger operator $-\hbar^2 \sum_{j=1}^{n} \partial_{x_j}^2/2m+ U(x_1, , \ldots, x_n)$ on the Hilbert space $L^2(G, dx)$, with $G \subset \mathbb{R}^n$ the classical configuration space. (For example, G equals $\mathbb{R}^n$ in the Toda case, and the wedge $\{x_n < \cdots < x_1\}$ in the rational and hyperbolic Calogero-Moser cases.)

For a suitable choice of the Poisson commuting Hamiltonians (namely, those given by the symmetric functions of the Lax matrix), there are no ordering ambiguities, and canonical quantization yields n commuting partial differential operators (PDOs). More generally, for the remaining simple Lie algebras of rank n there exist n commuting PDOs whose classical versions are the Poisson commuting Calogero-Moser and Toda Hamiltonians.

For the simple Lie superalgebras of rank n there also exist n commuting PDOs, but it is a peculiar fact that these systems do not have an integrable classical limit. Special cases were first found and studied by Chalykh and Veselov, and the general case was treated by Sergeev and Veselov. More specifically, these systems can be viewed as deformed Calogero-Moser systems of trigonometric and hyperbolic type, the deformation being encoded in the occurrence of more than one particle mass and a corresponding reduction of symmetry.

Turning to the relativistic (A_{n-1}) Calogero-Moser and Toda systems, here one encounters ordering ambiguities already for the defining Hamiltonian. This is because $\cosh(\beta p_j)$ (with $\beta = 1/mc$) is multiplied by a function $f_j(x)$ that depends nontrivially on x_j. On the other hand, the quantization of the summands $\exp(\pm\beta p_j)$ is straightforward. Taking $\hbar = 1$ from now on, their quantum versions are the analytic difference operators (AΔOs) $T_j^{\mp} = \exp(\mp i\beta\partial/\partial x_j)$, which act on a wave function $\Psi(x)$ by shifting $x_j \to x_j \mp i\beta$. The function $f_j(x)$ should now be factored as $f_{j,+}(x)f_{j,-}(x)$ in a special way, and then $\cosh(\beta p_j)f_j(x)$ should be quantized as $(f_{j,+}(x)T_j^{+}f_{j,-}(x) + f_{j,-}(x)T_j^{-}f_{j,+}(x))/2$ to obtain an integrable quantization. More precisely, when this recipe is applied to the Poisson commuting symmetric functions of the Lax matrix, one winds up with n commuting AΔOs, as first shown by Ruijsenaars (1986).

It is plausible that this integrable quantization is essentially unique. In this connection we recall that there are no general results concerning the existence and/or uniqueness of an integrable quantization for a given classical integrable system.

The canonical quantization just sketched leads to commuting PDOs and AΔOs that are formally self-adjoint. Since already their 'kinetic energy' building blocks $i\partial_{x_j}$ and $T_j^{\pm}$ are unbounded, one is however led to some difficult functional analysis problems. To put these in a wider context, we recall that the spectral theorem, as applied to n commuting self-adjoint operators $H_1, \ldots, H_n$ on a Hilbert space $\mathcal{H}$, guarantees that there exists a unitary operator onto a spectral representation space $\hat{\mathcal{H}}$, diagonalizing the operators as real-valued multiplication operators $M_1(p), \ldots, M_n(p)$. However, just as its classical analogue (the Liouville-Arnold theorem yielding existence of action-angle maps), this structural result is of limited practical value. Indeed, it contains no information on the nature of this unitary operator and on the explicit definition of the Hilbert space $\hat{\mathcal{H}}$ and multiplication operators $M_j(p)$.

Returning to the above commuting operators, it is not feasible to directly define them as bona fide commuting self-adjoint operators. Rather the main problem is to obtain sufficiently explicit information on simultaneous eigenfunctions, and use suitable joint eigenfunctions for real eigenvalues to construct a unitary eigenfunction transform. Indeed, once this is achieved, this yields an explicit manifestation of the spectral theorem, self-adjointness and commutativity being a consequence of unitarity. Of course, the simplest example of this strategy is the case of no interaction. Then the joint eigenfunctions are just the plane waves $\exp(ix \cdot p)$, and the associated unitary transform is the Fourier transform.

The simplest interacting case is the trigonometric one. This is because the joint eigenfunctions are essentially multivariable orthogonal polynomials, whose Hilbert space aspects are easily understood. In the nonrelativistic A_{n-1} case the polynomials are the Gegenbauer polynomials for $n = 2$, and for $n > 2$ the Jack polynomials from statistics (independently introduced by Sutherland). In the relativistic A_{n-1} case the polynomials are the q-Gegenbauer polynomials for $n = 2$ (with q related to the speed of light c, Planck's constant $\hbar$, the inverse length scale parameter ν and the particle mass m by $q = \exp(-\nu\hbar/mc)$), and the Macdonald polynomials for $n > 2$. The BC_n generalizations involve additional external field couplings, physically speaking. In the PDO case the relevant polynomials are the Jacobi polynomials for $n = 1$, and for $n > 1$ the multivariable Jacobi polynomials introduced by Heckman and Opdam. In the AΔO case one arrives at the Askey-Wilson polynomials for $n = 1$, and for $n > 1$ at the Koornwinder polynomials.

In the PDO case the rational and hyperbolic transforms are known in considerable detail; for the rank-1 or (reduced) 1-particle case one obtains the Hankel transform and the so-called Jacobi transform, whose kernel (the eigenfunction of the defining PDO) is the Gauss hypergeometric function. In the AΔO case however, only the rational transforms are essentially known and understood. In particular, in the rank-1 case the kernel is again the Gauss hypergeometric function. This is a manifestation of a duality (or bispectrality) property that holds true more generally.

For the hyperbolic AΔO case there are to date only partial results. For the BC_1 specialization the unitary transform is known in great detail. Its self-dual kernel (the joint eigenfunction of four distinct 1-variable AΔOs of Askey-Wilson type) is a 'relativistic' generalization of the hypergeometric function, introduced by Ruijsenaars. For higher rank, however, there are only results (due to Chalykh) about self-dual joint eigenfunctions for the commuting A_{n-1} AΔOs when the coupling is restricted to a sequence of integer values. The hyperbolic results agree in particular with the conjecture that the relation of the relativistic n-particle systems with the sine-Gordon n-soliton solutions survives quantization. Indeed, for the 2-particle case the same scattering and bound-state spectrum occurs as for the sine-Gordon quantum field theory.

The results about elliptic eigenfunctions are even more fragmentary. We just mention that the second-order ordinary differential operators associated with the A_1 and BC_1 cases can be diagonalized by an orthonormal base involving Lamé and Heun functions, respectively.

For the commuting Toda PDOs and AΔOs the eigenfunction transform theory is also far from complete. For the A_1 differential operator case the pertinent eigenfunctions are of Whittaker and Mathieu type (corresponding to the nonperiodic and periodic 2-particle systems).

It is a highly remarkable circumstance that for the deformed trigonometric Calogero-Moser systems an infinity of joint eigenfunctions of multivariable polynomial type is explicitly known, but these polynomials are not orthogonal with respect to a suitable positive weight function. Therefore, they cannot serve to promote the commuting PDOs to self-adjoint commuting operators on a Hilbert space. In fact, no eigenfunctions without nonintegrable singularities are known at all.

It will be already clear from the above that this third group of integrable quantum systems is closely linked to special function theory. Indeed, an alternative view on the commuting quantum Hamiltonians arises from the theory of Lie groups and symmetric spaces (which leads to commuting PDOs) and from the theory of quantum groups and quantum symmetric spaces (which leads to commuting AΔOs). The link with special function theory is particularly well-known for the case of 1-variable functions and Lie group theory, where it was worked out in great detail by Vilenkin decades ago. Likewise, the work by Harish-Chandra, Helgason and others on the radial Fourier transform is quite relevant for the above commuting PDOs with special couplings. Indeed, this has been extended to arbitrary (real) couplings by Heckman and Opdam, both for the compact ('trigonometric') and noncompact ('hyperbolic') case.

We conclude by mentioning some further connections of this group of integrable quantum models with subfields in mathematics and physics. To begin, the polynomials that serve as joint eigenfunctions in the trigonometric regime also play an important role in enumerative combinatorics and random matrix theory. The search for suitable joint eigenfunctions for the commuting AΔOs of hyperbolic and elliptic type is connected to issues in the theory of linear analytic difference equations and Nevanlinna theory. The AΔOs are also related to Sklyanin algebras (in the elliptic case) and to affine Hecke algebras. Furthermore, the models are linked with Painlevé, Kniznik-Zamolodchikov, Yang-Baxter and WDVV equations, with Hitchin, Gaudin, WZNW and matrix models, with operators of Dunkl, Cherednik and Polychronakos type, with Huygens' principle, and with the bispectral problem.