presented in this paper with the marginal stability criteria and scale selection for such an unbounded flow is left for future work. If the hydrostatic approximation is not made and horizontal diffusion is not ignored, instead of a Schrödinger-like equation \( \ddot{\psi}(y) - F(\cdots)\dot{\psi}(y) = 0 \), equations follow containing the sixth-order derivative (when \( P = 1 \)), or the eighth-order derivative (when \( P \neq 1 \)). In view of the discussion in §4.2, we expect little to be gained from a study of these far more complicated eigenvalue problems, unless \( N^2 \ll |f Q| \) (when \( P = 1 \)) or \( P N^2 \ll |f Q| \) (when \( P \neq 1 \)).

R. C. K. acknowledges support from the National Science Foundation (grants OCE 05-26033 and 07-26686), G. F. C. acknowledges support from the National Science Foundation (grants OCE 05-25776 and 07-26482) and the Ministero Istruzione universita e Ricerca (MIUR D. M. 26.01.01 n. 13).

Appendix A. Bounds on growth rates

The cubic (3.1) can be written as

\[
\left( s + \frac{a}{P} \right) [(s + a)^2 - |f Q|] + \left( s + \frac{a}{P} \right) \frac{\pi^2 l^2 E}{a} |f Q| + (s + a) \frac{\pi^2 l^2 E}{a} N^2 = 0. \tag{A 1}
\]

If there is a positive real root \( s > 0 \), then \((s + a)^2 - |f Q|\) has to be negative, since \( E, P, l^2, a \) and \( |f Q| \) are all positive and \( N^2 \geq 0 \). Hence we must have \( s < |f Q|^{1/2} \).

There can also be complex-conjugate pairs of roots. Setting \( s = b \pm i\omega \) in (A 1), the real part is

\[
(b + \frac{a}{P}) [(b + a)^2 - \omega^2 - |f Q|] + \left( b + \frac{a}{P} \right) \frac{l^2 \pi^2 E}{a} |f Q| + (b + a) \left( \frac{l^2 \pi^2 E}{a} N^2 - 2\omega^2 \right) = 0,
\]

(A 2)

and the imaginary part is

\[
\pm i\omega \left[ (b + a)^2 - \omega^2 - |f Q| + \frac{l^2 \pi^2 E}{a} |f Q| + 2 \left( b + \frac{a}{P} \right) (b + a) + \frac{l^2 \pi^2 E}{a} N^2 \right] = 0.
\]

(A 3)

Since by assumption \( \omega \neq 0 \), an equation \( \omega^2 = \cdots \) follows. Substitution in (A 2) yields

\[
(b + a) [(b + a)^2 - |f Q|] + (b + a) \left[ \frac{|f Q|}{a} + \frac{N^2}{2a} \right] l^2 \pi^2 E
\]

\[
+ \left( b + \frac{a}{P} \right) \left[ (b + a) + 2(b + a)^2 + \frac{l^2 \pi^2 E}{2a} N^2 \right] = 0. \tag{A 4}
\]

If \( b > 0 \), we see that \((b + a)^2 - |f Q|\) has to be negative. Therefore, if there are complex-conjugate roots with real part \( \text{Re}(s) = b > 0 \), then \( b < |f Q|^{1/2} \).

Appendix B. Weakly diffusive scale selection for the inertial instability of an arbitrary shear flow

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We start from equations (2.1)–(2.5), where we now allow \( U \) to vary arbitrarily with \( y \). Taking \( N^2 \) to be constant, the homogeneity in \( z \) allows us to consider disturbances
of the form
\[(u', \psi, \sigma) = \text{Re}[\bar{u}(y), \tilde{\psi}(y), \tilde{\sigma}(y)] \exp(st) \exp(imz)],\]
where \(\psi\) is the streamfunction (2.9), and \(\sigma = -g\rho'/\rho_0\) is a fluctuating buoyancy acceleration. Differentiating (2.7) with respect to \(t\), substituting from (2.1) and (2.4), and dividing by \(m^2\) gives
\[
\frac{(N^2 + s^2)}{m^2} \frac{d^2}{{dy}^2} \tilde{\psi} - (s^2 + f Q)\tilde{\psi} = \frac{v}{m^2} \left( \frac{d^2}{{dy}^2} - m^2 \right) \left[ s \left( \frac{d^2}{{dy}^2} - m^2 \right) \tilde{\psi} + im\tilde{\sigma} + \frac{\kappa}{v} \frac{d\tilde{\sigma}}{dy} \right],
\]
where \(Q(y) = f - dU/dy\) is the absolute vorticity.

B.1. The non-diffusive system

When \(v = \kappa = 0\), it is well-known that instability is only possible when \(f Q < 0\) somewhere in the flow. Denoting the cross-stream location of the global minimum of \(f Q\) by \(\bar{y}\), and its value by \(\bar{f}Q < 0\), the maximum growth rate of disturbances is given by \(|\bar{f}Q|^{1/2}\). This can be established by multiplying (B 1) by \(\tilde{\psi}^*\) and integrating across the domain, after setting the right-hand side of (B 1) to zero. Furthermore, this growth rate is approached as \(|m| \to \infty\), by disturbances which become highly localized in the cross-stream direction around \(y = \bar{y}\). The cross-stream localization occurs on a length scale \(\sim |m|^{-1/2}\), for both unstratified flows (e.g. Bayly 1988) and stratified flows (e.g. Griffiths 2007). Thus, the lateral boundary conditions – whether they be for periodic, bounded or unbounded flow – become irrelevant in this limit.

To study these most unstable modes, it is necessary to introduce a rescaled cross-stream coordinate \(Y\), the optimal choice being
\[
Y = (a|m|)^{1/2}(y - \bar{y}), \quad \text{where} \quad a = \left( \frac{(f Q)^\nu}{2(N^2 + |f Q|)} \right)^{1/2}, \quad (B 2)
\]
and where \((f Q)^\nu = d^2(f Q)/dy^2\) evaluated at \(y = \bar{y}\) is assumed to be non-zero. The limiting dynamics are obtained by making a Taylor expansion for \(f Q\) around \(y = \bar{y}\) (justifiable because of the cross-stream localization of the modes), and by looking for small deviations of \(s^2\) from its limiting value:
\[
f Q \sim \bar{f}Q + \frac{(f Q)^\nu}{2a|m|} Y^2 + \frac{(f Q)^\nu}{6(a|m|)^{3/2}} Y^3 + \cdots, \quad s^2 \sim |\bar{f}Q| - \left( \frac{a}{|m|} \right) s_1^2 + \cdots, \quad |m| \to \infty.
\]
Substituting in (B 1), and writing \(d^2/dy^2 = a|m|d^2/dY^2\), the \(O(1)\) terms cancel, and at \(O(|m|^{-1})\) we have
\[
(N^2 + |f Q|) a \frac{d^2\tilde{\psi}}{dY^2} + \left( \frac{s_1^2}{N^2 + |f Q|} - Y^2 \right) \tilde{\psi} = 0. \quad (B 3)
\]
This equation does indeed describe localized solutions, provided
\[
s_1^2 = (2n + 1)(N^2 + |f Q|), \quad n = 0, 1, 2, \ldots,
\]
with corresponding eigenfunctions \(\tilde{\psi} = H_n(Y) \exp(-Y^2/2)\), where \(H_n\) is the \(n\)th order Hermite polynomial. Thus, the limiting growth rate is given by
\[
s^2 \sim |\bar{f}Q| - \frac{(2n + 1)}{|m|} \left( \frac{(f Q)^\nu(N^2 + |f Q|)}{2} \right)^{1/2} \frac{1}{2} + \cdots, \quad |m| \to \infty. \quad (B 4)
\]
Although higher-order corrections to $s^2$ and $\tilde{\psi}$ may be derived, (B4) is sufficient for our purposes since it captures the inviscid monotonic growth rate increase as $|m| \to \infty$. From (2.1) and (2.4), the remainder of the leading-order solution is

$$\tilde{u} \sim -\frac{im\mathcal{Q}}{|f\mathcal{Q}|^{1/2}} \tilde{\psi}, \quad \tilde{\sigma} \sim -\frac{N^2(a|m|)^{1/2}}{|f\mathcal{Q}|^{1/2}} d\tilde{\psi}. \quad (B\,5\,a,\,b)$$

**B.2. The diffusive system**

We now evaluate the effect of diffusion on the inertial instability when the Prandtl number $P = \nu/\kappa$ is of order unity. We study the regime where $\nu$ and $\kappa$ are small, so that the inertial instability remains strong, with the large vertical wavenumber limit remaining appropriate. We anticipate that (B4) will be appropriate for moderately large $|m|$, describing an almost inviscid growth rate increase, whilst as $|m| \to \infty$ more appropriate would be a relationship describing an almost completely diffusive decay.

We need to assess for what scaling of $|m|$, in terms of $\nu$ and $\kappa$, the inviscid growth and diffusive decay terms balance, which will yield the most unstable modes.

Since the inviscid growth term of (B4) appears at $O(|m|^{-1})$ in (B1), we simply need to calculate when the terms on the right-hand side of (B1) scale like $|m|^{-1}\tilde{\psi}$, as $|m| \to \infty$. Using (B 5a, b), the largest terms on the right-hand side of (B1) as $|m| \to \infty$ are $\nu m^2 \tilde{\psi}$ and $-i\nu f m \tilde{\alpha}$, and are those originating from vertical diffusion in the two horizontal momentum equations. Using (B4) and (B 5a), to leading order these two terms sum to

$$2\nu m^2 |f\mathcal{Q}|^{1/2} \tilde{\psi}. \quad (B\,6)$$

Using (B 5b), the term $-\kappa d\tilde{\sigma}/dy$ (originating from the vertical diffusion of heat) scales like $\kappa m \tilde{\psi}$, and is much smaller than (B 6), as are all terms originating from horizontal diffusion ($\partial^2/\partial z^2 \gg \partial^2/\partial y^2$ under the scaling (B 2)). Thus the regime of interest occurs when (B 6) scales like $|m|^{-1} \tilde{\psi}$, i.e. when $|m| \sim \nu^{-1/3}$. With this understanding, (B 6) may be added to the right-hand side of (B 3), yielding a modified solvability condition

$$s^2 \sim |\mathcal{Q}| - \frac{(2n+1)}{|m|} \left(\frac{(\mathcal{Q})^2(N^2 + |\mathcal{Q}|)}{2}\right)^{1/2} - 2\nu m^2 |f\mathcal{Q}|^{1/2} + \cdots, \quad (B\,7)$$

where $|m| \to \infty$, $\nu \to 0$, and $|m| \sim \nu^{-1/3}$. This is a self-consistent procedure, since $\nu m^2 / \kappa m^2 \sim |m|^{-1} \ll 1$, so that the approximately non-diffusive balances leading to (B 5a, b) remain valid.

Equation (B7) describes the diffusive modification of the inertial instability. Although not valid for arbitrary $\nu$ and $|m|$, it should describe the most unstable inertial instabilities as $\nu \sim \kappa \to 0$. Maximizing $s^2$ with respect to $|m|$, the maximum growth rate $s_*$ satisfies

$$s_*^2 \sim |\mathcal{Q}| - \frac{3}{2} \{2\nu(2n+1)^2|\mathcal{Q}|^{1/2}(\mathcal{Q})^2(N^2 + |\mathcal{Q}|)^{1/3} + \cdots, \quad \nu \to 0, \quad (B\,8)$$

which is largest for the $n = 0$ mode. The most unstable vertical wavenumber $m_*$ satisfies

$$|m_*| \sim \left\{\frac{2n+1}{4\nu} \left(\frac{(\mathcal{Q})^2(N^2 + |\mathcal{Q}|)}{2|\mathcal{Q}|}\right)^{1/2}\right\}^{1/3} + \cdots, \quad \nu \to 0. \quad (B\,9)$$

These scalings, with $s_*$ decreasing from $|\mathcal{Q}|^{1/2}$ like $\nu^{1/3}$ and $m_* \sim \nu^{-1/3}$, are consistent with the particular cases reported by Griffiths (2003a), and Kloosterziel et al. (2007). There is disagreement with the results of §4.1 because it is a case for which $(\mathcal{Q})^p = 0.
The scalings apply when \( v \neq \kappa \) (unless \( v \ll \kappa \)), but the effects of \( \kappa \) on the inertial instability are rather weak in the large vertical wavenumber regime, as was noted in §4.3 for the uniform shear flow. The terms involving \( \kappa \) appear as an \( O(|m|^{-2}) \) correction to the growth rate in (B 7), and are not calculated here.

Less cumbersome scalings for \( m_\ast \) can be derived from (B 9) from case to case. We denote the length scale of \( fQ \) by \( L \), and take \( Q \sim f \), since for instability horizontal shears must be larger than \( f \). Thus:

(a) For unstratified flow on the \( f \)-plane, (B 9) implies \( m_\ast \sim (f/vL)^{1/3} \). Since \( L \) must also be the length scale for \( U \), and \( f \) is a characteristic shear, a characteristic velocity for the basic flow is just \( fL \), so that we can write the Reynolds number as \( Re = fL^2/v \), implying \( m_\ast \sim Re^{1/3}L^{-1} \).

(b) For hydrostatic flow (\( N \gg f \)) on the \( f \)-plane, (B 9) implies \( m_\ast \sim (N/vL)^{1/3} \sim (N/f)^{1/3}Re^{1/3}L^{-1} \). The characteristic vertical wavenumbers are much larger for this hydrostatic case than for the unstratified case, since stratification acts to reduce the vertical extent of the overturning circulations.

(c) For hydrostatic flow (\( N \gg f \)) on the equatorial \( \beta \)-plane, the length scale \( L \) of \( fQ \) is no longer that of \( U \). Rather, denoting a characteristic shear by \( \Lambda \), \( fQ \approx \beta y(\beta y - \Lambda) \) so that \( L \sim \Lambda/\beta \), and \( Q \sim f \sim \beta L \sim \Lambda \). Thus \( m_\ast \sim (N\beta/\Lambda v)^{1/3} \sim (N/\Lambda)^{1/3}Re^{1/3}L^{-1} \), consistent with the scaling given by Griffiths (2003a), as discussed in §4.2.

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