The Linked Twist Map Approach to Fluid Mixing
Ergodic Theory in Fluids

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Joint work with Steve Wiggins and Julio Ottino
Dynamical systems and fluids

**Fluids**
- incompressible fluid
- Poincaré section
- region of unmixed (stationary) fluid
- islands forming barriers to mixing
- “chaotic”

**Dynamical systems**
- invertible, area-preserving dynamical system
- Discrete time map, \( f : M \rightarrow M \)
- invariant (periodic) set \( f(A) = A \)
- KAM theory
- existence of a horseshoe
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Horseshoes in fluids

from [Chien, Rising, Ottino, JFM 170 355-77 (1986)]
### Topological
- topological space
- behaviour of individual trajectories
- dense orbit

### Measure-theoretic
- measure space
- need an invariant measure — Lebesgue measure $\mu$
- behaviour of sets of positive (or full) measure
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Ergodicity

**Definition**

\[
f \text{ is ergodic if } \mu(A) = 0 \text{ or } 1 \text{ whenever } f(A) = A.
\]

Birkhoff ergodic thm \(\implies\) “time averages = spatial averages”

Central notion is *indecomposability*

\[\text{ergodicity } \implies \text{“no islands of unmixed fluid”}\]
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Mixing

\[ \lim_{n \to \infty} \frac{\mu(f^n(A) \cap B)}{\mu(B)} = \mu(A) \]

Intuitive definition is that upon iteration, sets become asymptotically independent of each other.
The Bernoulli property

Bernoulli means “statistically indistinguishable from coin tosses”

The Ergodic Hierarchy
Bernoulli $\implies$ Mixing $\implies$ Ergodicity

... plus lots more!
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... plus lots more!
Define annuli $P$ and $Q$ on the torus $\mathbb{T}^2$ which intersect in region $S$. 

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LTM Approach to Fluid Mixing
Linked Twist Maps on the torus

The horizontal annulus $P$ has a horizontal twist map....
Linked Twist Maps on the torus

\[ F(x, y) = (x + f(y), y) \]

for points in \( P \)

\( f(y) \) could be linear...
Linked Twist Maps on the torus

...or not, but must be monotonic
After $F$, apply a vertical twist

$$G(x, y) = (x, y + g(x))$$

to points in $Q$. Again $g$ must be monotonic.

The combined map $H(x, y) = G \circ F$ is the linked twist map.
Mixing properties of LTMs on the torus

Domain is two intersecting annuli with two distinct regions of intersection.
The action of a twist map is to take a line...
Linked Twist Maps on the plane

... and twist it around the annulus.

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LTM Approach to Fluid Mixing
An egg-beater can be viewed as either linked twist map on the plane, or on the torus. from [Ottino, J, *Sci. Am.*, **260**, 56–67 (1989)]
The Blinking Vortex

Streamlines in the first half of the advection cycle

Streamlines in the second half of the advection cycle
The Blinking Vortex

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Streamlines in the first half of the advection cycle

Streamlines in the second half of the advection cycle
The Partitioned Pipe Mixer


Cross-sectional streamlines
The Rotated Arc Mixer

Microfluidics — electroosmotic flow

Fill hybridization chamber with "target" solution of mRNA
Attach "probes" (DNA strands) to silicon surface in hybridization chamber
Introduce syringes to form a source-sink pair
Streamlines from source to sink
DNA Hybridization

Second set of streamlines
DNA Hybridization

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LTM Approach to Fluid Mixing
DNA Hybridization

\[ f(y) = ry(1 - y) \]
\[ g(x) = rx(1 - x) \]
DNA Hybridization

Short pumping time $\rightarrow$ Long pumping time

from [J.M. Hertzsch, R. Sturman & S. Wiggins, 2006]
DNA Hybridization

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LTM Approach to Fluid Mixing
Off-centre sources and sinks
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Future Directions

- Monotonicity
- Transversality
- Speed of mixing
- Diffusion
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Future Directions

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Duct flows

- Schematic view of a duct flow with concatenated mixing elements
- Red and blue blobs of fluid mix well under a small number of applications
- Changing only the position of the centres of rotation can have a marked effect on the quality of mixing
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Schematic view of a duct flow with concatenated mixing elements

Red and blue blobs of fluid mix well under a small number of applications

Changing only the position of the centres of rotation can have a marked effect on the quality of mixing