Rates of mixing in models of fluid flow

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Joint work with James Springham, now in Perth
Chaotic mixing

Decay of correlations

Achieving the upper bound

Future directions

Introduction

Chaotic motion leads to exponential behaviour...

- Topological entropy is computable (Thurston-Nielsen classification theorem)
- Lower bound on the complexity.

[P. L. Boyland, H. Aref, & M. A. Stremler, JFM. 403, 277 (2000)]
Introduction

"...but walls seem to slow things down"

“regions of low stretching which slow down mixing and contaminate the whole mixing pattern...”

[Gouillart et al., PRE, 78, 026211 (2008)]

Also:

[Chernykh & Lebedev, JETP, 87(12), 682 (2008)]

[Lebedev & Turitsyn, PRE, 69, 036301, (2004)]
Dynamical systems approach

Dynamical systems modelling fluids

**Fluids**
- incompressible fluid
- spatial/temporal periodicity
- region of unmixed (stationary) fluid
- ‘chaotic advection’

**Dynamical systems**
- invertible, area-preserving dynamical system
- Discrete time map, \( f : M \to M \)
- invariant (periodic) set \( f(A) = A \)
- stretching & folding
  - horseshoe (topological)
  - (non)uniform hyperbolicity (smooth ergodic theory)
Mixing

$f$ is (strong) mixing if

$$\lim_{n \to \infty} \mu(f^n(A) \cap B) = \mu(A) \mu(B)$$
Mixing

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$$\lim_{n \to \infty} \mu(f^n(A) \cap B) = \mu(A)\mu(B)$$

Equivalently in functional form:

$$C_n(\varphi, \psi) = \left| \int (\varphi \circ f^n) \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \to 0$$

as $n \to \infty$ for scalar observables $\varphi$ and $\psi$.

At what rate does $C_n$ decay to zero?
Simple models

Stretching in alternating directions

Egg-beater flows

Blinking vortex

Pulsed source-sink mixers

Partitioned pipe mixer
A simple model of alternating shears giving chaotic dynamics on the 2-torus.

\[ F(x, y) = (x + y, y), \quad G(x, y) = (x, y + x) \]

\[ H(x, y) = G \circ F = (x + y, x + 2y) \]
Arnold Cat Map

A simple model of alternating shears giving chaotic dynamics on the 2-torus.

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\[ DH = A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \]

is a hyperbolic matrix and the Cat Map is *uniformly hyperbolic*. The Cat Map is strong mixing (not difficult to show).
Define annuli $P$ and $Q$ on the torus $\mathbb{T}^2$ which intersect in region $S$. 

**Linked Twist Maps on the torus**

Simple models
Simple models

Linked Twist Maps on the torus

The horizontal annulus $P$ has a horizontal twist map....
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Simple models

Linked Twist Maps on the torus

\[ F(x, y) = (x + f(y), y) \]

for points in \( P \).

\( F \) is the identity outside \( P \).

\( f(y) \) could be linear...
Linked Twist Maps on the torus

...or not, but must be monotonic
After $F$, apply a vertical twist

$$G(x, y) = (x, y + g(x))$$

to points in $Q$.

The combined map $H(x, y) = G \circ F$ is the linked twist map.
Or we can define on the plane (3-punctured disk), giving a model of the Aref blinking vortex.
A simple LTM

Linear twists on annuli half the width of the torus.
\[ F(x, y) = (x + 2y, y), \quad G(x, y) = (x, y + 2x) \]

\[ H(x, y) = G \circ F(x, y) \]

LTMs as defined here are strong mixing (Pesin theory).
Decay of correlations for the Cat Map

Assume w.l.o.g. that $\int \psi d\mu = 0$ and compute

$$C_n(\varphi, \psi) = \int (\varphi \circ H^n) \psi d\mu$$

Expanding (analytic) $\varphi$ and $\psi$ as Fourier series,

$$\varphi(x) = \sum_{k \in \mathbb{Z}^2} a_k e^{i k \cdot x}, \quad \psi(x) = \sum_{j \in \mathbb{Z}^2} b_j e^{i j \cdot x},$$

Linearity and orthogonality means that

$$C_n(\varphi, \psi) = \int \sum_{k \in \mathbb{Z}^2} a_k e^{i k \cdot A^n x} \sum_{j \in \mathbb{Z}^2} b_j e^{i j \cdot x} dx = \sum_{k \in \mathbb{Z}^2} a_k b_{-k A^n}.$$

Since $A$ is a hyperbolic matrix, the exponential growth of $|kA^n|$ together with the exponential decay of Fourier coefficients yields (super)exponential decay of $C_n$. 
Young Towers (Lai-Sang Young, 1999)

Young Towers give a procedure for computing decay of correlations for more general systems (with some hyperbolicity).

- Locate a set $\Lambda$ with ‘good hyperbolic properties’ (actually hyperbolic product structure)
- Run the system forwards and check when iterates of $\Lambda$ intersect $\Lambda$ (in the right way)
- Measure and record the return times for $\Lambda$
- Obtain the statistical behaviour for the return time system
- Pass the findings back to the original system
Construct return time function $R : \Lambda \to \mathbb{Z}^+$ and measure $\mu(x \in \Lambda | R(x) > n)$:

- If $\int R \, d\mu < \infty$, then $H$ is ergodic.
- If $\int R \, d\mu < \infty$ and gcd of $R$ is 1, then $H$ is strong mixing.
- If

$$\mu(R > n) = \begin{cases} 
\mathcal{O}(n^{-\alpha}), & \alpha > 1 \text{ polynomial decay } \mathcal{O}(n^{-\alpha+1}) \\
C\theta^n, & \theta < 1 \text{ exponential decay } C\theta'^n \\
\mathcal{O}(n^{-\alpha}), & \alpha > 2 \text{ central limit theorem holds}
\end{cases}$$
Choose $\Lambda$ to be any rectangle with sides aligned along stable and unstable eigenvectors.

Iterate $\Lambda$ under the Cat Map until its image intersects $\Lambda$.  

\[ v^+ \]

\[ v^- \]

\[ \Lambda \]
Young Towers

Cat Map
Young Towers

Cat Map
Young Towers

Cat Map

\[ R(x) = n_1 \]

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Linked Twist Maps

Appears a perfect system to apply Young Towers.
Linked Twist Maps

Appears a perfect system to apply Young Towers. But...

- local stable and unstable manifolds only exist \( \mu \)-almost everywhere...
- ... and \( \Lambda \) is defined as the intersection of such manifolds...
- ... so must contain holes.
- In general for non-uniformly hyperbolic systems, constructing \( \Lambda \) is hard.
A related scheme

Chernov, Zhang, Markarian conditions

Study the return map to $S$ and check:
A related scheme

Chernov, Zhang, Markarian conditions

Study the return map to $S$ and check:

- Smoothness
- Hyperbolicity
- Return map is mixing
- Bounded distortion
- Bounded curvature
- Absolute continuity
- Admissible curves in the singularity set
- One-step growth of unstable manifolds
Chernov, Zhang, Markarian conditions

Study the return map to $S$ and check:

- Smoothness (Structure of the singularity set)
- Hyperbolicity (Easy)
- Return map is mixing (Surprisingly difficult)
- Bounded distortion (Trivial)
- Bounded curvature (Trivial)
- Absolute continuity (Easy)
- Admissible curves in the singularity set (Not too bad)
- One-step growth of unstable manifolds (Hard)
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- infrequently returning points (Nightmare)
A related scheme

**Decay of correlations for linear LTMs**

Theorem (Springham & Sturman, ETDS, (2012))

*For* $\alpha$-*Hölder observables and for a linear LTM* $H$,

$$C_n = \mathcal{O}(1/n)$$
A related scheme

Decay of correlations for linear LTMs

Theorem (Springham & Sturman, ETDS, (2012))

For $\alpha$-Hölder observables and for a linear LTM $H$,

$$C_n = O\left(\frac{1}{n}\right)$$

- This is an upper bound on the worst behaviour
- Compare with the topological result of a lower bound on the best behaviour
- As it’s an upper bound we haven’t yet shown polynomial decay rates actually happen. $C_n$ could still decay exponentially fast.
For LTMs we can compute explicitly the contribution to the decay integral of a particular region near the boundary.

Consider all points which take \( n \) iterates to enter the overlap \( S \). These form wedge-shaped regions \( B_n \). We will concentrate on \( W_n \).
A simple computation

\[ I_n = \int_{W_n} (\varphi \circ H^n) \psi d\mu \]

\[ = \frac{4}{3} \int_{\frac{1}{2}}^{1} \int_{0}^{(1-y)/2n} \varphi(x, y + 2nx) \psi(x, y) dx dy \]
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Consider the related double integral

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Substitute \( t = y + 2nx \):

\[ J_n = \frac{2}{3n} \int_{1/2}^{1} \psi(0, y) \int_{y}^{1} \varphi(0, t) \, dt \, dy \sim K/n \]
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Finally show \( \lim_{n \to \infty} n \left| I_n - J_n \right| = 0 \), that is, the contribution made to the correlation function by points near the boundary is asymptotically the same as the contribution made by points at the boundary.
Now we have

$$C_n \sim \frac{K}{n} + \int_{\mathbb{R} \setminus B_n} (\varphi \circ H^n)\psi d\mu.$$ 

Certainly the contribution from the remaining integral is no slower than $O(1/n)$ (from earlier).

[Sturman & Springham, PRE, (2012)]

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So we see polynomial decay at rate \( 1/n \) providing:

1. \( K \neq 0 \), i.e., the contributions from the wedges do not cancel each other out (do not choose non-generic symmetric observables)
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So we see polynomial decay at rate \( 1/n \) providing:

1. \( K \neq 0 \), i.e., the contributions from the wedges do not cancel each other out (do not choose non-generic symmetric observables)

2. The decay rate of the remaining integral is not exactly \( -K/n \) to leading order.
More general boundary behaviour

In a neighbourhood of the boundaries, replace linear twists with

\[ \tilde{F}(x, y) = (x + 2y^p, y) \quad \text{and} \quad \tilde{G}(x, y) = (x, y + 2x^p). \]

Then

**Lemma**

\[ \int_{B_n} (\varphi \circ \tilde{H}^n) \psi \, d\mu \sim \frac{K}{n^{1/p}} \]

and so

**Conjecture**

\[ \int_{LTM} (\varphi \circ \tilde{H}^n) \psi \, d\mu \sim \frac{K'}{n^{1/p}} \]
Numerics — linear case
Different boundary conditions

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Different boundary conditions

Numerics — quadratic case

$$\tilde{f}(y) = 1 - \frac{\cos^{-1}(4y - 1)}{\pi}$$

$$\tilde{g}(x) = 1 - \frac{\cos^{-1}(4x - 1)}{\pi}$$
Numerics — quadratic case
Different boundary conditions

Numerics — maximum width of striations

Red: linear, $O(1/n)$
Blue: quadratic, $O(1/n^2)$
Green: exponential decay of Cat Map
Rigorous extensions

- Planar linked twist maps

- Introduce curvature to the Young Tower argument

- Combine with topological ideas to get a more detailed picture of mixing
Non-rigorous ideas

- Young Tower method is practically tractable
  - Don’t work in Fourier space
  - Don’t need to think about observables
  - Can measure return times
  - Can ignore (maybe) fine details of the mathematics
- Residence-time distributions
- Compute return times for more realistic models
- Consequences of presence of islands
- Connection with lobe dynamics
Can the non-rigorous ideas be turned into a practical scheme for understanding data?

Fundamental ingredients:

- Periodicity (but what is the consequence of random forcing?)
- ‘Good’ hyperbolic region (product structure?)
- Trajectories which repeatedly return to the hyperbolic region ‘in the right way’ (dynamic renewal?)
- Method of recording return times
- Atmospheric or oceanographic?
Singularity set for $H_S$