Bounds for (generalised) Lyapunov exponents for deterministic and random products of shears

Rob Sturman
School of Mathematics
University of Leeds

Applied Analysis & Computation Seminar, 18 May 2017
UMassAmherst
Joint work with James Springham (Leeds)
Jean-Luc Thiffeault (University of Wisconsin-Madison)
Motivation — fluid mixing

Periodic, deterministic, random

Rob Sturman (University of Leeds)
Motivation — fluid mixing

Basic model is a combination of shears:

\[ G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \& \quad F = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]
The Arnold Cat Map — compose $F$ and $G$

$$H = G \circ F \text{ (a toral automorphism)}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ is hyperbolic}$$
The Arnold Cat Map — compose $F$ and $G$

$H = G \circ F$ (a toral automorphism)

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

is hyperbolic

1. Lyapunov exponents

$$\lim_{n \to \infty} \frac{1}{n} \log \|DH^n_x v\| = \log(3 + \sqrt{5})/2$$
The Arnold Cat Map — compose $F$ and $G$

$$H = G \circ F \text{ (a toral automorphism)}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ is hyperbolic}$$

1. Lyapunov exponents

$$\lim_{n \to \infty} \frac{1}{n} \log \|DH^n_x v\| = \log(3 + \sqrt{5})/2$$

2. The Cat Map is strong mixing: $\forall A, B,$

$$\lim_{n \to \infty} \mu(H^n(A) \cap B) = \mu(A)\mu(B)$$
The Arnold Cat Map — compose $F$ and $G$

$H = G \circ F$ (a toral automorphism)

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is hyperbolic

1. Lyapunov exponents

$$\lim_{n \to \infty} \frac{1}{n} \log \| DH^n_x v \| = \log\left(3 + \sqrt{5}\right)/2$$

2. The Cat Map is strong mixing: $\forall A, B,$

$$\lim_{n \to \infty} \mu(H^n(A) \cap B) = \mu(A) \mu(B)$$

3. Exponential decay of correlations

$$\int (\phi \circ H^n) \psi d\mu - \int \phi d\mu \int \psi d\mu = O(\theta^n)$$
Periodic, deterministic, random

Linked twist maps on the torus

\[ P = \{(x, y) | y \leq 1/2\}, \quad Q = \{(x, y) | x \leq 1/2\} \]

and define

\[ F(x, y) = \begin{cases} 
(x + 2y, y) & \text{if } (x, y) \in P \\
(x, y) & \text{if } x \notin P 
\end{cases} \]

\[ G(x, y) = \begin{cases} 
(x, y + 2x) & \text{if } (x, y) \in Q \\
(x, y) & \text{if } x \notin Q 
\end{cases} \]

And finally

\[ H(x, y) = G \circ F(x, y) \]
LTM

- A trajectory always landing in overlap $S$ is uniformly hyperbolic
- Almost all trajectories land outside $S$
- The closer to a boundary, the longer the return to $S$
- Arbitrarily long sequences of $F$s and $G$s
- But these are rare enough that Lyapunov exponent $\lambda$ is non-zero
- No uniform growth $\implies$ algebraic decay of correlations (polynomial, not exponential, mixing)
- Computing value of $\lambda$ is hard
LTM — return time partition
Random matrix products

\[ G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \& \quad F = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]

At each iterate, choose \( F \) or \( G \) with probability \( 1/2 \).
Random braiding of particle paths
Tangled magnetic fields

The "vision" for solar flux tubes

The magnetic field lines become "braided" due to MHD frozen-in condition + turbulence
Definition of Lyapunov exponents

Diffeomorphisms:

\[ \lambda(x, v) = \lim_{n \to \infty} \frac{1}{n} \log \| D_x H^n v \| \]
Definition of Lyapunov exponents

Diffeomorphisms:

$$\lambda(x, v) = \lim_{n \to \infty} \frac{1}{n} \log \| D_x H^n v \|$$

For products of random matrices $M_N = \prod_{k=1}^{N} A_{i_k}$:

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \| M_N \| = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \| X_N \|$$

almost surely, where $X_N = A_{i_N} X_{N-1}$. 
Definition of Lyapunov exponents

Diffeomorphisms:

\[ \lambda(x, v) = \lim_{n \to \infty} \frac{1}{n} \log \|D_x H^n v\| \]

For products of random matrices \( \mathcal{M}_N = \prod_{k=1}^{N} A_{i_k} \):

\[ \lambda = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\log \|\mathcal{M}_N\| = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\log \|X_N\| \]

almost surely, where \( X_N = A_{i_N} X_{N-1} \).

- Convergence is given by the celebrated Furstenberg-Kesten theorem (1960).
- Equivalence of the two definitions is given by the Oseledets multiplicative ergodic theorem (1968).
- \( \lambda \) is independent of choice of norm
Hard to compute analytically...

There is a paucity of exact results concerning Lyapunov exponents for random matrices, as famously lamented by Kingman (1973):

“Pride of place among the unsolved problems of subadditive ergodic theory must go to the calculation of [the Lyapunov exponent]”.... In none of the applications described here is there an obvious mechanism for obtaining an exact numerical value, and indeed this usually seems to be a problem of some depth.
... but easy computationally

The submultiplicativity of matrix norms provides “the most popular upper bound in the literature”:

$$E_k = 2^{-k} \mathbb{E} \left\{ \log \| M_{2^k} \| \right\},$$

(for products drawn from 2 possible matrices)

- The $E_k$ decrease monotonically to $\lambda$ (for any matrix norm) as $k \to \infty$
- This is ‘easy, if not efficient”
- the number of matrix product calculations required increases exponentially with $k$
Many (very good) algorithms and approximations...

Mannion, Products of $2 \times 2$ random matrices (1993)
Many (very good) algorithms and approximations...

Mannion, Products of $2 \times 2$ random matrices (1993)


**Theorem 1.** There is an algorithm giving approximations $\lambda_n$ to $\lambda$ defined in terms of the maximal eigenvectors and the eigenvalues of products of at most $n$ of the matrices. Moreover, there exist are $C > 0$ and $K > 0$ such that $|\lambda - \lambda_n| \leq K \exp\left(-Cn^{1+\frac{1}{d-1}}\right)$. 
Many (very good) algorithms and approximations...

Mannion, Products of $2 \times 2$ random matrices (1993)


**Theorem 1.** There is an algorithm giving approximations $\lambda_n$ to $\lambda$ defined in terms of the maximal eigenvectors and the eigenvalues of products of at most $n$ of the matrices. Moreover, there exist are $C > 0$ and $K > 0$ such that $|\lambda - \lambda_n| \leq K \exp\left(-Cn^{1+\frac{1}{d-1}}\right)$.

Why is this different?

- Rigorous, explicit (elementary) bounds
  (Not algorithmic, no complex functions, no Fourier decomposition)
- Positive and negative entries in matrices
  (but see also D. Viswanath, Random Fibonacci sequences and the number 1.13198824. . . Mathematics of Computation of the American Mathematical Society 69.231 (2000))
- Generalised Lyapunov exponents
Generalised Lyapunov exponents

- Often want more than knowledge of the infinite time limit of matrix product growth
- Study deviations from the long-time average
- Growth rate of the $q$th moment of the matrix product norm
Generalised Lyapunov exponents

- Often want more than knowledge of the infinite time limit of matrix product growth
- Study deviations from the long-time average
- Growth rate of the $q$th moment of the matrix product norm

Define the *generalized Lyapunov exponents* as

$$
\ell(q) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \| M_N \|^q,
$$

- Note that $\ell'(0)$ is the usual Lyapunov exponent
- $\ell(1)$ measures topological entropy (in the sense of growth of material lines)

“For large $|q|$, $\ell(q)$ is controlled by exceedingly rare realisations of the matrix products, and hence it is difficult to estimate reliably using Monte Carlo numerical methods, even with importance sampling.”
Group the matrices together

- W.l.o.g. assume that the first matrix in the product is \( F \)
- Group \( F \)'s and \( G \)'s together into \( J \) blocks

  e.g. \((FFGGG)(FG)(FGG)(FFFFGG)(FGG)\ldots\)

so the random product is

\[
\mathcal{M}_J = \prod_{j=1}^{J} F^{a_j} G^{b_j}, \quad a_j + b_j = n_j, \quad \sum_{j=1}^{J} n_j = N,
\]

with \( 1 \leq a_i, b_i < n_i \), so \( n_i \geq 2 \).

- Now the \( a_i \) and \( b_i \) are the i.i.d. random variables, with identical probability distribution \( \mathbb{P}(x) = 2^{-x}, \ x \geq 1 \).
- Hence, the joint distribution \( \mathbb{P}(a, b) = \mathbb{P}(a) \mathbb{P}(b) = 2^{-(a+b)} \).
- We have the expected values \( \mathbb{E}a = \mathbb{E}b = 2 \), so \( \mathbb{E}n = 4 \)
Group the matrices together

Recall

\[ \lambda = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \| X_N \| \]

almost surely, where \( X_N = F X_{N-1} \) or \( G X_{N-1} \). Then

\[ \lambda = \lim_{J \to \infty} \frac{1}{4J} \mathbb{E} \log \| X_J \| \]

where

\[ \| X_J \| = \frac{\| F^a J G^b J X_{J-1} \|}{\| X_{J-1} \|} \frac{\| F^{a_{J-1}} G^{b_{J-1}} X_{J-2} \|}{\| X_{J-2} \|} \ldots \frac{\| F^a_1 G^b_1 X_0 \|}{\| X_0 \|}. \]

We will compute bounds on each term in the RHS product, by bounding the orientation of the vectors \( X_i \), and recalling that the choice of norm is arbitrary.
Shears

We take the specific matrices

\[ F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]

and let

\[ M_{ab} = F^a G^b = \begin{pmatrix} 1 & b \\ a & 1 + ab \end{pmatrix}. \]

A more general problem takes the matrices

\[ F = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad M_{ab} = F^a G^b = \begin{pmatrix} 1 & b\beta \\ a\alpha & 1 + a\alpha b\beta \end{pmatrix}. \]
Simple case (single, positive shears)

Take

\[ F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]

Lemma

The cone \( C = \{0 \leq u/v \leq 1\} \) in tangent space is invariant under \( M_{ab} = F^a G^b \) for all \( a, b \geq 1 \). Moreover, this is the minimal such cone.
Norm bounds I.

Lemma ($L_\infty$-norm)

The norm $\|M_{ab}X\|_\infty$ for a vector $X \in C$ satisfies the bounds

$$1 + ab \leq \frac{\|M_{ab}X\|_\infty}{\|X\|_\infty} \leq 1 + a + ab$$
Norm bounds I.

Lemma \((L_\infty\text{-norm})\)

The norm \(\|M_{ab}X\|_\infty\) for a vector \(X \in C\) satisfies the bounds

\[
1 + ab \leq \frac{\|M_{ab}X\|_\infty}{\|X\|_\infty} \leq 1 + a + ab
\]

Lemma \((L_1\text{-norm})\)

The norm \(\|M_{ab}X\|_1\) for a vector \(X \in C\) satisfies the bounds

\[
1 + \frac{1}{2}(a + b + ab) \leq \frac{\|M_{ab}X\|_1}{\|X\|_1} \leq 1 + b + ab
\]
Lemma ($L_2$-norm)

The norm $\| M_{ab}X \|_2$ for a vector $X \in C$ satisfies the bounds

$$\frac{\| M_{ab}X \|_2^2}{\| X \|_2^2} \leq \frac{1}{2} \left( 2 + \mathcal{C}_{ab} + \sqrt{\mathcal{C}_{ab}(\mathcal{C}_{ab} + 4)} \right),$$

where

$$\mathcal{C}_{ab} = (a + b)^2 + a^2 b^2.$$ 

and

$$\frac{\| M_{ab}X \|_2^2}{\| X \|_2^2} \geq \min \left\{ \frac{1}{2} ((1 + a + ab)^2 + (1 + b)^2) \right\}$$
Bounding Lyapunov exponents

Putting it together

\[ \lambda = \lim_{J \to \infty} \frac{1}{4J} \mathbb{E} \log \|X_J\| \]

\[ = \lim_{J \to \infty} \frac{1}{4J} \mathbb{E} \log \frac{\|F^{a_J} G^{b_J} X_{J-1}\|}{\|X_{J-1}\|} \cdots \frac{\|F^{a_1} G^{b_1} X_0\|}{\|X_0\|} \]

Using the geometric probability distribution for the \(a\)'s and \(b\)'s, we then have, for the \(L_\infty\)-norm:

\[ \sum_{a,b=1}^{\infty} 2^{-a-b} \log(1 + ab) \leq 4\lambda \leq \sum_{a,b=1}^{\infty} 2^{-a-b} \log(1 + a + ab) \]

and similar expressions for the other two norms.
Accuracy in the simple case

The lowest upper bound \((U_2)\) and largest lower bound \((L_1)\) differ by about 8%. The true value (from explicit calculation via a standard algorithm) is 0.39625...

<table>
<thead>
<tr>
<th>Norm</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_1)</td>
<td>(L_1(1, 1) = 0.36886)</td>
<td>(U_1(1, 1) = 0.43835)</td>
</tr>
<tr>
<td>(L_2)</td>
<td>(L_2(1, 1) = 0.36347)</td>
<td>(U_2(1, 1) = 0.40277)</td>
</tr>
<tr>
<td>(L_\infty)</td>
<td>(L_\infty(1, 1) = 0.34613)</td>
<td>(U_\infty(1, 1) = 0.43835)</td>
</tr>
</tbody>
</table>

**Table:** Bounds for the maximal Lyapunov exponent for the matrix product in the case \(\alpha = \beta = 1\).
Positive shears of $\alpha$ and $\beta$

The invariant cone is now

$$C = \{ 0 \leq u/v \leq 1/\alpha \}$$
Positive shears of $\alpha$ and $\beta$

The invariant cone is now

$$C = \{0 \leq u/v \leq 1/\alpha\}$$

**Theorem**

*The Lyapunov exponent* $\lambda(\alpha, \beta)$ *for the product* $M_N$ *satisfies*

$$\max_{k \in \{1, 2, \infty\}} L_k(\alpha, \beta) \leq 4\lambda(\alpha, \beta) \leq \min_{k \in \{1, 2, \infty\}} U_k(\alpha, \beta)$$

where

$$L_k(\alpha, \beta) = \sum_{a,b=1}^{\infty} 2^{-a-b} \log \phi_k(a, b, \alpha, \beta)$$

$$U_k(\alpha, \beta) = \sum_{a,b=1}^{\infty} 2^{-a-b} \log \psi_k(a, b, \alpha, \beta),$$
\[
\phi_1(a, b, \alpha, \beta) = 1 + \frac{\alpha}{1 + \alpha} (a + b\beta + a\alpha b\beta)
\]

\[
\phi_2(a, b, \alpha, \beta) = \min \left\{ \left( (1 + a\alpha b\beta)^2 + b^2\beta^2 \right)^{1/2} \right\}
\]

\[
\phi_\infty(a, b, \alpha, \beta) = \begin{cases} 
1 + a\alpha b\beta & \alpha \geq 1 \\
\min \{ \max(1 + a\alpha b\beta, b\beta), \max(\alpha(1 + a + a\alpha b\beta), 1 + a\alpha b\beta) \} & \alpha < 1 
\end{cases}
\]

\[
\psi_1(a, b, \alpha, \beta) = 1 + b\beta + a\alpha b\beta
\]

\[
\psi_2(a, b, \alpha, \beta) = \left( 2 + C_{a\alpha b\beta} + \sqrt{C_{a\alpha b\beta}(C_{a\alpha b\beta} + 4)} \right)^{1/2}
\]

\[
\psi_\infty(a, b, \alpha, \beta) = \begin{cases} 
(1 + a + a\alpha b\beta) & \alpha \geq 1 \\
\max(1 + a\alpha + a\alpha b\beta, 1 + b\beta) & \alpha < 1 
\end{cases}
\]

where \(C_{a\alpha b\beta} = (a\alpha + b\beta)^2 + (a\alpha b\beta)^2\).
Accuracy of the bounds
Corollary

The Lyapunov exponent \( \lambda(\alpha, \beta) \) for the product \( M_N \) for \( \alpha, \beta \geq 1 \) satisfies

\[
K + \log \alpha \beta \leq 4 \lambda \leq K + \log(\sqrt{\alpha \beta} + \frac{1}{\sqrt{\alpha \beta}}) + \frac{1}{2} \log(1 + \alpha \beta),
\]

where

\[
K = \sum_{a,b=1}^{\infty} 2^{-a-b} \log ab \approx 1.0157 \ldots .
\]
Accuracy of the bounds
Accuracy of the bounds

Accuracy of the bounds in general shears.
Comparing two independent geometric distributions

Lemma

When $a$ and $b$ are both drawn from i.i.d. geometric distributions with parameter $1/2$, we have

$$P(a = b) = P(a > b) = P(b > a) = 1/3.$$  

Proof.

We have

$$P(a = b) = \sum_{i=1}^{\infty} P(a = i \cap b = i) = \sum_{i=1}^{\infty} 2^{-2i} = \frac{1/4}{1 - 1/4} = \frac{1}{3}.$$  

The remaining two equalities follow by symmetry.
The cone can be made smaller

Lemma

The cone $C = \{0 \leq \frac{u}{v} \leq \frac{1}{\alpha}\}$ is mapped into the following cones, in the following cases:

1. When $a = b$, $M_{ab}(C) = \{0 \leq \frac{u}{v} \leq \frac{1+\alpha\beta}{\alpha(2+\alpha\beta)}\}$;
2. when $a > b$, $M_{ab}(C) = \{0 \leq \frac{u}{v} \leq \frac{1+\alpha\beta}{\alpha(3+2\alpha\beta)}\}$;
3. when $a < b$, $M_{ab}(C) = C$.

and hence expressions like

$$4\lambda \leq \sum_{a,b=1}^{\infty} 2^{-a-b} \log \frac{1}{3} (\psi(a = b) + \psi(a > b) + \psi(a < b))$$
Accuracy of the bounds, improved

![Graph showing accuracy of bounds improved]
Theorem

We have

\[
4l(q, \alpha, \beta) \geq \max_{k \in \{1, 2, \infty\}} \left\{ \log \sum_{a,b=1}^{\infty} 2^{-a-b}(\phi_k(a, b, \alpha, \beta))^q \right\}
\]

\[
4l(q, \alpha, \beta) \leq \max_{k \in \{1, 2, \infty\}} \left\{ \log \sum_{a,b=1}^{\infty} 2^{-a-b}(\psi_k(a, b, \alpha, \beta))^q \right\}
\]

with \(\phi_k, \psi_k\) and defined as above.
Generalised Lyapunov exponent
Generalised Lyapunov exponents

When $q$ is a positive integer we can evaluate this easily by expanding the power $q$. Since

$$
\sum_{a,b=1}^{\infty} 2^{-a-b} a^n b^m = \left( \sum_{a=1}^{\infty} 2^{-a} a^n \right) \left( \sum_{b=1}^{\infty} 2^{-b} b^m \right)
$$

we require values of the polylogarithm $\text{Li}_{-q}(\frac{1}{2})$, defined by

$$
\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.
$$

For integer $q = -s$ we have special values

$$
\sum_{a=1}^{\infty} 2^{-a} a^n = 1, 2, 6, 26, 150, 1082, 9366, \ldots \text{ for } n = 0, 1, 2, 3, 4, 5, 6, \ldots
$$

and so the $L_\infty$ norm, for example, gives...
Generalised Lyapunov exponents

Corollary

Generalised Lyapunov exponents in the case $\alpha = \beta = 1$ are bounded by:

\[
\frac{1}{4} \log 5 \leq \ell(1) \leq \frac{1}{4} \log 7 \\
\frac{1}{4} \log 45 \leq \ell(2) \leq \frac{1}{4} \log 79 \\
\frac{1}{4} \log 797 \leq \ell(3) \leq \frac{1}{4} \log 1543 \\
\frac{1}{4} \log 25437 \leq \ell(4) \leq \frac{1}{4} \log 50531 \\
\frac{1}{4} \log 1290365 \leq \ell(5) \leq \frac{1}{4} \log 2578567.
\]
Switch one of the shears

Take

\[ F = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \]

Now

\[ M_{ab} = F^a G^b = \begin{pmatrix} 1 & b\beta \\ -a\alpha & 1 - a\alpha b\beta \end{pmatrix}. \]

is only hyperbolic \( \forall a, b \) when \( \alpha\beta > 4 \). We’ll assume \( \alpha, \beta > 2 \) for ease.

**Lemma**

*The cone \( C = \{-1 \leq u/v \leq 0\} \) in tangent space is invariant under \( M_{ab} = F^a G^b \) for all \( a, b \geq 1 \).*
Accuracy of the bounds

\[ \tilde{U}_1 - \tilde{L}_1 \]
\[ \tilde{U}_2 - \tilde{L}_2 \]
\[ \tilde{U}_\infty - \tilde{L}_\infty \]
\[ \tilde{U}_1 - \tilde{\hat{L}}_1 \]
\[ \tilde{U}_2 - \tilde{\hat{L}}_2 \]
\[ \tilde{U}_\infty - \tilde{\hat{L}}_\infty \]
Conclusions

- Alternating shears in (trivially) uniformly hyperbolic LTMs are more interesting — non-zero Lyapunov exponents
- We can derive pretty accurate, explicit, upper and lower bounds for Lyapunov exponents for random products of shears of varying strength
- Generalised Lyapunov exponents can also be bounded (which are also hard to compute)
- Our results, unusually, include matrices with negative entries
Linked twist maps also produce a sequence

$$FFGFFGFGGGGGFFGFGGGF \ldots = \prod_{j=1}^{J} F^{a_j} G^{b_j}$$

but now the $a_j, b_j$ are no longer random — we don’t have a probability distribution.

But a block $(F^a G^b)$ represents the return map to the hyperbolic overlap region $S$...

and such a block still possess the invariant cone and bounds on the growth of the vector norm...

and Kac’s lemma states that given a set $Q$ with measure $\mu(Q)$, the expected return time is $1/\mu(Q)$...

so this leads us back to the return time partition.
$x_1 = y_1 = 1/2$
Conclusions

Next?...

\[ x_1 = y_1 = \frac{1}{2} \]

\[ x_1 = y_1 = \frac{3}{4} \]