BAYES-MCMC RECONSTRUCTION FROM 3D EIT DATA USING A COMBINED LINEAR AND NON-LINEAR FORWARD PROBLEM SOLUTION STRATEGY

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ABSTRACT: Extracting meaningful information from EIT data is a challenging task due to highly correlated data and substantial noise. Domain discretisation further complicates the situation by also making it an ill-posed problem requiring substantial regularisation. The Bayesian approach not only provides a natural setting in which to specify and interpret regularisation, but also leads to distributional results allowing probabilistic statements to be made, such as confidence intervals. When combined with MCMC methods great flexibility in modelling and output summary is possible. A major issue, however, when using MCMC algorithms is the number of times the forward problem must be solved. In many cases this means that the computational time is prohibitive - this is particularly true for 3D problems. This paper proposes the mixing of full non-linear solution of the forward problem with linearised solution. The linear approximation works well for local changes in the solution and so is ideally suited to use in MCMC algorithms. Over multiple iterations, however, repeated linearisation accumulates errors and so strategic full non-linear solutions are used to correct the path of the algorithm. The combined strategy provides a reliable approach with computational times that make the MCMC method feasible in practical situations.

Keywords: EIT, Image reconstruction, Inverse problem, MCMC, Linearisation

1. INTRODUCTION

This paper looks at the use of Bayesian approaches linked with Markov chain Monte Carlo (MCMC) algorithms for reconstruction of a conductivity distribution from electrical impedance tomography (EIT) data. The inverse problem in EIT is both non-linear and ill posed, and a Bayesian approach provides a natural setting in which to specify and interpret regularisation in terms of prior information. MCMC provides a flexible tool for conductivity reconstruction, as for similar inverse problems, allowing full investigation of the posterior distribution not just mode estimation. In MCMC however the forward problem must be solved several thousands times, which is very time consuming, especially for 3D EIT, and, in general, for cases where the number of pixels is large. In this paper a combined strategy is proposed for the MCMC updates in which linearised forward solution for local conductivity changes is mixed with full non-linear solution. This leads to acceptable computation time without substantial loss of accuracy making Bayes-MCMC a practical approach for 3D non-linear problems.

2. MODELLING

The solution of an inverse problem requires regularization in order to ensure stability and reliability, and regularization can be viewed as including prior information. Thus a Bayesian approach is not only desirable but is essential for such a problem. Indeed Bayesian methods encompass much more than simply reporting a posterior mode and can be regarded as more general than regularization. Tomographic techniques, where a section through an object is imaged using measurements taken outside or on the boundary of the object, are well known, especially for medical applications.

Commonly domain discretization (pixellization) is employed and image reconstruction will become an ill-posed inverse problem. These difficulties are emphasised for soft-field modalities such as electrical impedance tomography (EIT), which is considered here. In general an image comprises a vector of impedivities, but most often resistivities only are imaged. Electrodes are attached to the boundary of the

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object and whilst currents are injected, measurements of potential differences (voltages) are recorded at numerous electrodes. The relationship between resistivity and voltage is nonlinear (whenever the domain dimension exceeds 1).

If the contents of the domain are known, then boundary voltages can be calculated through the solution of Maxwell's equations and the corresponding boundary conditions (Somersalo 1992) for electromagnetism. In practice this is done numerically using the finite element method. This is the direct problem or forward solution. It is well posed and voltages can be obtained at least to the accuracy of measurements. Inverse solution is the focus here, especially the use of prior information. Modelling of prior information is the key to gaining good information about the solution of the problem, and is necessarily dependent upon the application being considered. Hence the objectives are very practical: to produce an approach which makes efficient use of the data and outputs useful solutions. The combination of realistic physical models and efficient statistical models requires close interaction between engineers and mathematicians.

3. BAYESIAN APPROACH AND MCMC ALGORITHM

The recorded voltages, $V$, are subject to measurement errors, which are often substantial. If these errors are assumed to be independent and normally distributed with equal variance, $\tau^2$, then this leads to the likelihood: the conditional distribution of the voltages for a given conductivity distribution, $V|\sigma \sim N(V(\sigma),\tau^2)$, with probability density function

$$f(V | \sigma) = (2\pi\tau^2)^{-\nu/2} \exp\{-||V - V(\sigma)||^2/2\tau^2\}$$  \hspace{1cm} (1)

where $V(\sigma)$ are voltages calculated from the given conductivity distribution. For the application to EIT considered here the true spatial conductivity distribution is expected to be relatively smooth and will be modelled by a Gibbs prior distribution, or homogeneous Markov random field (see for example Winkler 2003), with probability density function

$$\pi(\sigma) = Z(\beta) \exp\{-\beta||\sigma - \sigma^M||^2\}$$  \hspace{1cm} (2)

where $\sigma^M$ is a vector of nearest neighbour means and $\beta$ is a smoothing (regularisation) parameter. The use of a discrete approximation to the gradient of the conductivity distribution makes this a second-order prior with a mode when the gradient is constant. Many other choices are possible including the use of zero and first-order priors with modes corresponding to zero conductivity and constant conductivity. Also other norms can be used, for example the $L_1$-norm corresponds to the total variation prior.

The posterior distribution of any image given the data is then simply the result of combining the prior and likelihood distributions using Bayes' theorem:

$$\pi(\sigma | V) \propto f(V | \sigma) \pi(\sigma).$$ \hspace{1cm} (3)

Inference about $\sigma$ is then based on the posterior distribution $\pi(\sigma | V)$.

A standard Metropolis-Hastings algorithm is used to produce approximate samples from the posterior distribution by simulating an ergodic Markov chain, which has the required distribution as its limiting distribution. The use of such methods for parameter estimation, and more general density exploration, is now quite widespread and so only brief details will be given here. The reader is directed to works such as Besag et al. (1995), Liu (2001) and Winkler (2003) for general information and to Aykroyd et al. (2003) and West et al. (2004) for examples applied to EIT.

The general approach is as follows. At each stage a candidate state is proposed from some, almost arbitrary, distribution. This candidate is then accepted or rejected stochastically according to a probability that maintains ergodicity. It is common to use a proposal distribution, which is symmetric, in which case
the acceptance probability calculation is simplified. Also it is usual to consider only single parameter changes, but larger subsets or even the whole parameter set can be considered simultaneously. Particularly for nonlinear inverse problems even a single parameter change requires a forward solution and hence it is more efficient to propose simultaneous changes to all parameters.

Let the proposed set of parameters be \( \sigma' = \sigma + \Delta \sigma \), where the perturbation in the \( i \)-th component is from \( \mathcal{N}(0, \nu) \) with the proposal variance, \( \nu^2 \), used to tune the algorithm. To achieve reversibility, the proposal is accepted, and the parameter vector updated accordingly with probability

\[
\alpha(\sigma', \sigma) = \min\{1, \frac{\pi(\sigma' | V)}{\pi(\sigma | V)} \}
\]

otherwise it is rejected and the previous value retained. Here the parameters are non-negative, hence a negative proposal is immediately rejected and no change is made. This whole process is repeated very many times producing a sequence, \( \sigma', \sigma', ..., \sigma' \). Initially the values in this sequence depend on the initialisation of the algorithm, but eventually behave like a sample from the required posterior distribution. The initial, burn-in, phase can be long for poor starting values and inefficient algorithms. A practical approach to identifying the required length of the burn-in is to run the algorithm from various distant starting points and monitoring the variability between the chains relative to the variability within the chains. Only after convincing evidence should equilibrium be declared. A second issue is how long a chain is required. Autocorrelation within the chain can lead to a substantially longer chain being required than with independence. Time series analysis using autocorrelation time can be used to perform appropriate sample size calculations. It is also common to thin the sequence, by taking only every \( 1-in-n \), to produce an approximately uncorrelated final sample. Details of these issues can be found in Besag et al. (1995), Liu (2001) and Winkler (2003).

Once the pseudo-sample has been generated from the posterior distribution a number of possible estimators are available. The most common choice is the posterior mean, which can be estimated by the sample mean of the posterior sample collected after equilibrium has been declared. Further, other summaries can be considered, such as posterior variance, using the sample variance, and credible intervals using the percentiles of the sample distribution. This paper deals with generating the pseudo-sample and in particular possible reduction in the computation time by applying some linearisations.

4. LINEARISED FORWARD SOLUTION

In EIT the electrical conductivity distribution \( \sigma \) within the domain of interest \( \Omega \), and the voltage measurements collected at its boundary, \( V \), according to the applied current patterns, are related by

\[
F(\sigma) = V = V^* + e
\]

where the function \( F \) is nonlinear in \( \sigma \), and \( V \) is the array of noise or error (\( e \)) contaminated voltage data, and \( V^* \) is the error-free measurement for conductivity \( \sigma \). Let \( F \) be twice differentiable in \( \Omega \) so that linearising at a feasible point \( \sigma_k \) and using its Taylor expansion we obtain the model

\[
V_k (\Delta \sigma) = F(\sigma_k) + F'(\sigma_k)^T \Delta \sigma + O(\|\Delta \sigma\|^2).
\]

This linearisation model is known to be valid within a limited neighbourhood of the point \( \sigma_k \) defined by an appropriate constraint

\[
\|\Delta \sigma\| \leq \Delta_k
\]

which invariably limits the size of the linear step, from \( \sigma_k \) to \( \sigma_{k+1} \), while the exact value of \( \Delta_k \) is adjusted based on the agreement between \( F(\sigma_k + \Delta \sigma) \) and \( V_k(\Delta \sigma) \). Our aim is to exploit the above model so as to limit the computational costs. In particular for MCMC updates that satisfy the limitation in \( \|\Delta \sigma\| \), i.e.
within the trust region of the linear model. Having computed $F'$ (the Jacobian) we can estimate the next updates using the model in (7), saving the time to assemble $F$. The advantage of this shortcut is of course relevant to the computational demand for assembling the operator $F$ and solving the arising linear system of equations, here based on the finite element implementation of the complete electrode model, against the cost of performing the matrix-vector multiplication in equation (7) and computation of the Jacobian matrix.

5. CONCLUSION

In this paper we study the theoretical background for the proposed use of linearisation in MCMC algorithms. Our numerical experience, which will be shown during the presentation in conference, shows that instead of solving the full forward problem for each voxel, under certain conditions, we can use the Jacobian matrix and recalculate the forward problem in some minimum number of voxels. For small conductivity perturbations and for the voxels in the centre of the object, the linear approximation of the forward problem can be used. An error assessment of the linear approximation compared with the full forward solution given the knowledge of the noise error in the measurement instrumentation provides a guide to when and where to use the linearisation. It is worth mentioning, that the linearisation scheme is already widely used for deterministic inverse solutions.

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