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Inertial waves in a spherical shell induced by librations of the inner sphere: experimental and numerical results

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Abstract

We begin with an experimental investigation of the flow induced in a rotating spherical shell. The shell globally rotates with angular velocity \( \Omega \). A further periodic oscillation with angular velocity \( 0 \leq \omega \leq 2\Omega \), a so-called longitudinal libration, is added on the inner sphere’s rotation. The primary response is inertial waves spawned at the critical latitudes on the inner sphere, and propagating throughout the shell along inclined characteristics. For sufficiently large libration amplitudes, the higher harmonics also become important. Those harmonics whose frequencies are still less than \( 2\Omega \) behave as inertial waves themselves, propagating along their own characteristics. The steady component of the flow consists of a prograde zonal jet on the cylinder tangent to the inner sphere and parallel to the axis of rotation, and increases with decreasing Ekman number. The jet becomes unstable for larger forcing amplitudes as can be deduced from the preliminary particle image velocimetry observations. Finally, a wave attractor is experimentally detected in the spherical shell as the pattern of largest variance. These findings are reproduced in a two-dimensional numerical investigation of the flow, and certain aspects can be studied numerically in greater detail. One aspect is the scaling of the width of the inertial shear layers and the width of the steady jet. Another is the partitioning of the kinetic energy between the forced wave, its harmonics and the mean flow. Finally, the numerical simulations allow for an investigation of instabilities, too local to be found experimentally. For strong libration amplitudes, the boundary layer on the inner sphere becomes unstable,
triggering localized Görtler vortices during the prograde phase of the forcing. This instability is important for the transition to turbulence of the spherical shell flow.

(Some figures may appear in colour only in the online journal)

1. Introduction

Inertial waves occur in rotating fluids in which the restoring force is supplied by the rotation itself, via the Coriolis force (Greenspan 1968). Of particular interest in planetary and stellar interiors are inertial waves in spheres and spherical shells. As similar as the two geometries may appear, there is also an important distinction between them: the well-defined global eigenmodes that exist in a full sphere in general do not carry over into a spherical shell (Stewartson et al 1970). Instead, inertial waves in spherical shells are concentrated on characteristic surfaces consisting of cones inclined to the rotation axis at an angle $\sin^{-1}(\omega/2\Omega)$, where $\Omega$ is the rotation rate and $\omega$ is the frequency of the inertial waves (Kerswell 1995). Different values of $\omega/\Omega$ can lead to ‘wave attractors’ of the characteristics and their—often very complicated—reflection patterns throughout the shell (Tilgner 1999, Rieutord et al 2000, Harlander et al 2007, Maas et al 2007).

One particularly convenient way not only to excite inertial waves (at least axisymmetric ones) but also to precisely select their frequencies $\omega$ is to impose an oscillatory component on the rotation rate of the outer and/or inner spheres. The first experiments of this type were done by Aldridge et al (1969) in a full sphere. The results in generally show good agreement with the expected global eigenmodes although the interpretation of these findings has been doubted by Zhang et al (2013). A limited number of spherical shell experiments were also reported by Aldridge (1975), but without enough detail to observe these characteristics (where we also note that at this time it was already known that the previous eigenmodes probably did not carry over to the shell, but not yet known what exactly to expect instead). The spherical-shell numerical calculations by Tilgner (1999) similarly triggered oscillations of a particular frequency in this way.

More recently, it has become increasingly evident that the so-called longitudinal librations, periodic variations in rotation rate about a fixed axis, exist in nature as well. See, for example, Comstock et al (2003) for ‘a solar system survey of forced librations in longitude’, as well as an analysis of the gravitational torques that cause such fluctuations in rotation rate. The realization that libration is an interesting and important phenomenon in planetary physics has led to further work in spheres (Busse 2010, Sauret et al 2010), spherical shells where in both cases the libration was imposed only on the outer sphere (Noir et al 2009, Calkins et al 2010), and even ellipsoids Chan et al 2011, Zhang et al 2011, Cébron et al 2012). Other geometries such as a librating rectangular prism (Maas 2001) and cylinders (Noir et al 2010, Swart et al 2010, Lopez and Marques 2011, Borcia and Harlander 2012, Sauret et al 2012) have also been studied. One central question addressed was the generation of a libration-driven zonal mean flow. Waves themselves can drive strong mean flows (Maas 2001, Tilgner 2007, Bordes et al 2012) but even without inertial waves, mean flows can be driven by libration (Busse 2010, Sauret et al 2010). In the latter case nonlinearities in the Ekman boundary layer are responsible for the flows, and not inertial waves or local instabilities.

Motivated both by these previous studies of inertial waves in spherical shells, as well as by the potential planetary applications, in this work we present an experimental and numerical
study in which the libration is imposed on the inner sphere alone. Compared to a global libration this is a more local source for wave and mean flow generation. It emphasizes aspects associated with the inner sphere, such as its tangent cylinder $C$, the cylinder parallel to the rotation axis and touching the inner sphere.

The remainder of this paper is organized as follows. Section 2.1 gives the experimental setup and section 2.2 discusses experimental results on inertial waves, the generation of harmonics and mean flows. Then in sections 3.1–3.5, these findings will be compared with numerical results and certain aspects are numerically investigated in greater detail. Section 3.1 outlines the governing equation and its numerical solution, section 3.2 presents the linear solutions, valid for infinitesimal libration amplitudes, and gives a scaling for the width of the shear layers, section 3.3 considers how higher harmonics arise for finite amplitudes, section 3.4 discusses the mean flow and how the kinetic energy is distributed between the forced wave, the harmonics and the mean flow. Moreover, a scaling for the width of the zonal jet on the cylinder tangent to the inner sphere is discussed. Section 3.5 demonstrates how instabilities may arise for even stronger forcings, and finally section 4 gives conclusions and possibilities for future work.

2. Laboratory experiments

2.1. Experimental setup

The experimental apparatus, shown in figure 1, previously used to study convection (Futterer et al 2007), consists of an anodized aluminum inner sphere suspended on a shaft of diameter 0.5 cm inside an acrylic outer sphere. The fluid filling the gap is ‘Wacker Ak 0.65’ silicone oil. Table 1 gives details on the geometry and the working fluid of the experiment.

The entire outer sphere is immersed inside a glass cube of sides 40 cm filled with water. Having a flat outer surface greatly reduces the optical distortions that would arise if the outer sphere were in contact with air. Having the entire apparatus surrounded by a water bath also helps to keep the temperature uniform. Both spheres are driven by a 122 W servo-motor RX130H, Mattke Inc. The motor for the steadily rotating component is visibly mounted outside of the apparatus (see figure 1). It drives the sphere together with the inner motor via a gear wheel. The inner motor, providing the libration of the inner sphere, is centrally installed over the rotation axes. Note that a precession of the inner sphere with respect to the outer sphere is impossible since both spheres are held in position by a single shaft. All the control signals and supply voltages of this rotating inner motor are performed by slip-ring assemblies. Hence, the inner and outer spheres can rotate independently, at rates up to 1.2 Hz, and are accurately controlled to within 0.03%.

<table>
<thead>
<tr>
<th>Table 1. Different parameters of the spherical shell experiment: (a) geometrical parameters of the apparatus and (b) experimental parameters.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Geometrical parameters</td>
</tr>
<tr>
<td>Radius—inner sphere $r_i$ (mm) 60</td>
</tr>
<tr>
<td>Radius—outer sphere $r_o$ (mm) 120</td>
</tr>
<tr>
<td>Gap width $d$ (mm) 60</td>
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<tr>
<td>Radius ratio $\eta = r_i/r_o$ 0.5</td>
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<tr>
<td>(b) Fluid properties</td>
</tr>
<tr>
<td>Kinematic viscosity $(T = 25^\circ C)$ $\nu$ (mm$^2$s$^{-1}$) 0.65</td>
</tr>
<tr>
<td>Density $\rho$ (g cm$^{-3}$) 0.76</td>
</tr>
</tbody>
</table>
Figure 1. Setup of the experiment. On the left half the laser for optical visualization is displayed. The spherical shell is immersed in a cube with glass walls to reduce optical distortion and to keep the temperature in the shell constant. The inner sphere is hardly visible due to the seeding of the fluid with aluminum flakes.

The entire system rotates at a rate $\Omega$. Within this rotating frame, the inner sphere undergoes an additional oscillation about the same axis, at a rate $\epsilon \cos(\omega t)$. The inner sphere’s libration frequency $\omega$ and amplitude $\epsilon$ can also be controlled to within 0.03%. We are able to achieve Ekman numbers $E = \nu / \Omega r_i^2$ as small as $2.5 \times 10^{-5}$, and libration frequencies $\hat{\omega} = \omega / \Omega$ and amplitudes $Ro = \epsilon / \Omega$ in the range $0.2 \leq \hat{\omega} \leq 2.0$ and $0.1 \leq Ro \leq 1.0$. In the following we will drop the hat, and refer to the frequency simply as $\omega$.

The flow was visualized by seeding the fluid with aluminum flakes, and illuminating it with a laser light sheet in a meridional plane. This method is less than ideal, not only in the sense that it offers at most qualitative information. It is also not particularly well suited to study rapidly oscillating flows, as the flakes then tend to simply ‘vibrate’, but without achieving the coherent alignment on which the visualization depends. On the other hand, this could also be viewed as an advantage, as it means this method is particularly sensitive to the steady part of the flow, which we will find to be among the most interesting aspects.

2.2. Results

Figure 2 shows meridional sections of the upper hemisphere of the spherical shell for an experiment with $E = 4 \times 10^{-5}$. The axis of rotation corresponds with the $y$-axis of the figures. The upper row shows visualizations for $\omega = 0.4$ and the lower one for $\omega = 0.8$. The non-dimensional amplitude is increasing from left to right $Ro = 0.2, 0.3, 0.4, 0.5$. The most prominent feature is a structure parallel to the axis of rotation touching the equator of the inner sphere. For inertial waves, the orientation of wave crests is related to the wave’s frequency. Thus, structures parallel to the axis of rotation are steady. The experimental and numerical results discussed later will confirm that the structure corresponds to a steady
Figure 2. Experimental results, at $\omega = 0.4$ (top row) and $\omega = 0.8$ (bottom row), $Ro = 0.2, 0.3, 0.4, 0.5$ (from left to right) and $E = 4 \times 10^{-5}$ for all. Inclined characteristics corresponding to the forced inertial wave and its harmonics are clearly visible, and in good agreement with the expected inclination angles. The structure on the tangent cylinder also shows up particularly well; as noted in the text, the visualization method is likely to be particularly sensitive to the steady flows obtained here. Note how, for larger $\omega$, a larger $Ro$ is needed to obtain a sharp jet.

prograde jet. It appears that the sharpness of the jet depends on $\omega$ and $Ro$. The second figure in the top row ($(\omega, Ro) = (0.4, 0.3)$) agrees well with the last figure from the bottom row ($(\omega, Ro) = (0.8, 0.5)$). For higher frequencies, the forcing amplitude needs to be stronger to obtain a prominent jet.

In the region to the right of the steady jet, we see inertial waves propagating upper-right. These waves are most visible in the two upper right figures ($(\omega, Ro) = (0.4, 0.4), (0.4, 0.5)$). As can be seen from the different angles of the structures, not only waves with the libration frequency have been excited. It can be expected that the additional frequencies correspond to higher harmonics of the forcing frequency.

To analyze this further we applied a proper orthogonal decomposition (see Holmes et al. 1996, chapter 3) to the time series of the gray-scale images shown in figure 2. This technique decomposes the field into a series of empirical orthogonal functions (EOFs), each explaining a certain amount of the total variance of the data. Figure 3 shows the decomposition for two experiments, one with $(\omega, Ro) = (0.4, 0.4)$ (see figure 2, third picture, upper row) and the other with $(\omega, Ro) = (0.8, 0.8)$. For both experiments we used $E = 4 \times 10^{-5}$.

For the experiment with $\omega = Ro = 0.4$, the forced wave explains 15.28% of the total variance. The harmonic waves with $\omega = 0.8, 1.2, 1.6$, barely visible in figure 2, are captured by EOF 7, 10 and 16, and explain just 1.57, 0.99 and 0.051% of the total variance. It is obvious from these values that the time series of the gray-scale images is noisy. Nevertheless, we believe that the numbers give a first guess on the distribution of the kinetic energy between the forced and harmonic parts. For the second experiment shown in figure 3 with libration frequency $\omega = 0.8$, there exists just one harmonic wave in the inertial wave frequency range $0 \leq \omega \leq 2$. The forced wave and its harmonics explain 5.26 and 2.52% of the total variance. Thus for higher forcing amplitudes, the first harmonic becomes stronger relative to the forced wave. Note that a large part of the total variance is related to imperfections in the glass of the outer sphere. Moreover, a significant part of the variance is captured within the inner part of the tangent cylinder, to the left of the steady jet visible in figure 2. There the motion seems to be less organized than in the outer part where we see the propagating waves.
Figure 3. (a) EOFs from the experiment with $E = 4 \times 10^{-5}$ at $\omega = 0.4$, $Ro = 0.4$ (top row) and $\omega = 0.8$, $Ro = 0.8$ (bottom row). (a) EOF 3, $1\omega = 0.4$, Var = 0.1528; (b) EOF 7, $2\omega$, Var = 0.0157; (c) EOF 10, $3\omega$, Var = 0.0099; (d) EOF 6, $1\omega = 0.8$, Var = 0.0526; and (e) EOF 8, $2\omega$, Var = 0.0252.

Figure 4. Results for $\omega = 0.71$. (a) Wave rays; (b) experimental EOF 1, variance is 0.1684, $E = 4 \times 10^{-5}$, $Ro = 0.6$; and (c) numerically obtained meridional flow for $E = 10^{-5}$, $10^{-6}$, $10^{-7}$, $Ro = 0$. The horizontal lines in (a) give the locations of the critical points at the inner sphere and the outer shell.

It is well known that in certain frequency windows, wave attractors exist, that is, limit cycles that accumulate the wave energy (Tilgner 1999, Rieutord et al 2000). In can be expected that for such frequencies, the attractor is the dominant pattern. The most simple attractor that is possible in a spherical shell consists of a single cell, touching the equator of the inner and outer spheres. In terms of wave rays, this attractor is shown for the geometry of our apparatus in figure 4(a). Its frequency is $\omega = 0.71$. Note that the attractor exists not just for this frequency, but for a frequency band whose limiting frequencies depend on the size of the shell.
Applying the proper orthogonal decomposition to the time series of the gray-scale images of an experiment with this frequency and $E = 4 \times 10^{-5}$, we find the single-cell attractor as EOF 1, the dominant pattern, accounting for 16.8% of the total variance (see figure 4(b)).

Note that the visualization technique using aluminum flakes is not able to orient the flakes properly when the forcing frequency becomes too large. This might explain why we obtain 15.28% of total variance for the forced wave with $\omega = 0.4$, but only 5.26% for $\omega = 0.8$. Only when the wave becomes focused on a simple attractor, the variance is large also for higher frequencies. Note further that a large part of the variance is captured by a background flickering of the pictures. This results from slight inhomogeneities of the aluminum flake distribution in the oil and the fact that we observe the motion in the laboratory and not the co-rotating frame. So far we have no evidence for randomly structured inertial waves that might form the background variance.

Finally, we discuss particle image velocimetry (PIV) measurements of the steady prograde jet that is visible in figure 2 as a bright streak parallel to the axis of rotation and touching the equator of the inner sphere. In contrast to figure 2, where the observations were done in the meridional plane, the PIV data have been taken from a horizontal plane, 60 mm below the equator, touching the inner sphere’s south pole. The sampling frequency was 15 Hz. Figure 5 shows time mean velocity vectors and velocity magnitude. Owing to the fact that the PIV system was mounted in the laboratory frame, the solid body rotation velocity $\bar{U} = \Omega r$ (that is 270 mm s$^{-1}$ at the inner and 540 mm s$^{-1}$ at the outer sphere) has been subtracted from each PIV snapshot prior to the averaging. For the experiment with $\omega = 0.6$ the averaging time was 9 forcing periods and for the one with $\omega = 0.8$ it was 11 periods. The maximum velocity of the jet in figure 5(a) is 22.7 mm s$^{-1}$ and in figure 5(b) 20.9 mm s$^{-1}$. This corresponds to non-dimensional velocities of 0.105 and 0.097, or, with the scaling used in figure 10, of 6.29 and 5.79. These values are somewhat smaller than the values shown in figure 10 for smaller $Ro$. Note that the local and temporal Rossby numbers, defined as $Ro_l = U/(2\Omega r_i)$ and $Ro_t = \omega/2\Omega$, are 0.042 and 0.038, and 0.3 and 0.4, respectively. It is obvious that for both experiments the steady jet shows some deviations from a perfect circular shape. The reason behind these structures might be slight imperfections of the experimental setup or Rossby wave-type features with very low frequencies. The latter might develop due to non-axisymmetric instabilities, similar to the familiar instabilities of the Stewartson layer (Hollerbach 2003, Hollerbach et al 2004, Schaeffer and Cardin 2005). Moreover, Aelbrecht et al (1999) found that Ekman layer instability occurred for $Ro_t < 1$, a criterion fulfilled in the cases presented.

Figure 5. PIV observation 60 mm below the equatorial plane, $E = 4 \times 10^{-5}$. (a) $Ro = 0.8$, $\omega = 0.6$ and (b) $Ro = 0.8$, $\omega = 0.8$. 

Applying the proper orthogonal decomposition to the time series of the gray-scale images of an experiment with this frequency and $E = 4 \times 10^{-5}$, we find the single-cell attractor as EOF 1, the dominant pattern, accounting for 16.8% of the total variance (see figure 4(b)).
3. Numerical simulations

3.1. Equations

Scaling length by \( r_i \), time by \( \Omega_1^{-1} \) and \( U \) by \( \epsilon r_i \), the dimensionless Navier–Stokes equation becomes

\[
\frac{\partial U}{\partial t} + Ro U \cdot \nabla U + 2 \hat{e}_z \times U = -\nabla p + E \nabla^2 U \tag{1}
\]

with associated boundary conditions

\[
U = \cos (\omega t) \sin \theta \hat{e}_\phi \quad \text{at} \quad r = 1, \quad U = 0 \quad \text{at} \quad r = 2. \tag{2}
\]

The three non-dimensional parameters are the Ekman number \( E \) measuring the overall rotation rate, and the scaled frequency \( \omega \) and amplitude \( Ro \) of the inner sphere’s libration. The range of interest is then \( E \ll 1 \) (to be in the rapidly rotating limit), \( \omega < 2 \) (to obtain inertial oscillations at all—recall the formula for the angle of the characteristics \( \sin^{-1}(\omega/2) \)) and the Rossby number \( Ro \) ranging from 0 (purely linear) up to \( O(1) \) (strongly nonlinear).

We numerically solve the system (1, 2), together with the incompressibility condition \( \nabla \cdot U = 0 \), using an axisymmetric version of the code described by Hollerbach (2000), in which \( (r, \theta) \) are expanded in Chebyshev and Legendre polynomials, respectively. The highest resolution used for the linear modes in section 3.2 was \( (r, \theta) = (400, 600) \), sufficient to obtain fully resolved results down to \( E = 10^{-7} \). For the nonlinear solutions in sections 3.3—3.5, \( (r, \theta) = (200, 600) \) were used, allowing \( E \) to be reduced down to \( 2 \times 10^{-5} \) at \( Ro = 0.4 \). The time stepping is a second-order Runge–Kutta method, modified to treat the diffusive terms implicitly. All solutions were time stepped until either an exactly periodic solution emerged (before the onset of the instabilities in section 3.5) or a statistically steady state (after the onset of instabilities).

3.2. Linear solutions

Figure 6 shows the linear solutions, that is the solution of (1) with \( Ro = 0 \), corresponding to an infinitesimal libration amplitude. The inclined characteristics, with angle \( \sin^{-1}(\omega/2) \), are clearly visible. They originate at the critical latitude on the inner sphere where they are tangent to the inner sphere (Hollerbach et al 1995). From this point they propagate outward throughout the shell, always reflecting at boundaries in such a way as to maintain the same characteristic angle. As \( E \) is reduced, more and more reflections can be tracked before the pattern eventually fades away. The first column of figure 6 should be compared with figures 3(a) and (e). In particular, figure 3(a) agrees well with the numerical result, and the structure of the shear layer and its reflections are nicely depicted by the EOF 3 of the time series of the gray-scale images. Note that in figure 3(a) we see a strong motion in the polar cap region which is less prominent in figure 6. However, it should be kept in mind that the simulations are two dimensional but the motion in the polar region has also a 3D (Rossby wave-like) component. Moreover, the shaft that keeps both spheres on their position is absent in the numerical simulation. This shaft can have some impact on the flow.

The two frequencies considered here yield reasonably simple reflection patterns and Tilgner (1999) and Rieutord et al (2000) consider some of the very complicated patterns and associated wave attractor dynamics one can obtain by scanning through the entire frequency range \( 0 < \omega < 2 \). Figure 4(c) shows the one cell wave attractor as a linear solution for \( \omega = 0.71 \). The solution corresponds well with the experimental result. However, note that the best qualitative match can be found for the Ekman number that is two orders of magnitude smaller
Figure 6. The linear modes with $Ro = 0$ in (1) for $\omega = 0.4$ (top row) and $\omega = 0.8$ (bottom row), and $E = 10^{-5}, 10^{-6}, 10^{-7}$ (from left to right). What is shown is the streamfunction of the meridional circulation at $\omega t = 0 \mod (2\pi)$, that is, after an integral number of periods. The pattern along the shear layers propagates with the inertial wave's phase speed perpendicular to the layers, but the characteristics are always located in the same place. White indicates clockwise circulation and gray counter-clockwise circulation. The associated azimuthal velocity $U_\phi$ is not shown. Finally, the line segments of length 0.2 (in red in the online version) in the $E = 10^{-5}$ plots indicate the particular slices through the layers shown in figure 7.

than the one of the experiment. The numerical solution for $E = 4 \times 10^{-5}$ looks too smooth and the attractor is hardly visible. This is surprising in view of the rather sharp shear layers that can be seen in figure 6 for the same Ekman number. The reason might be that wave forcing due to libration is less efficient when the shear layer is mapped onto the forcing region as is the case for wave attractor frequencies. Moreover, the single-cell diamond-shaped attractor actually consists of two counter-propagating waves. Thus, annihilation effects might not be excluded. It is instructive to note that the one cell attractor is, in general, not symmetric with respect to the equator. Annihilation should be strongest for the symmetric case. Increasing the forcing frequency to $\omega = 0.72$ and thus slightly enhancing the asymmetry, the attractor looks somewhat sharper compared to the numerical simulation shown in figure 4(c). This signals the relevance of annihilation effects.

Along the red lines in the figures 3(a) and (e), figure 7 shows slices through these layers of $U_t$, the component in the meridional plane tangential to the layer. As derived by Kerswell (1995), for the related problem of the precessionally driven spin-over mode, the thickness of the layers scales as $E^{1/3}$; the profiles in figure 2 are in excellent agreement with this scaling. Kerswell further demonstrated that the amplitude of the flow within these layers should scale as $E^{1/6}$. The results in figure 7 do decrease as $E$ is reduced, but at a rate slower than $E^{1/6}$. It is not known how much further $E$ would have to be reduced to see an $E^{1/6}$ scaling emerging, or whether perhaps the scaling is different in this problem after all.

3.3. Higher harmonics

For $Ro > 0$ the solutions will consist of more than just the linear modes. In the weakly nonlinear limit $0 < Ro \ll 1$, one would expect the interaction of the linear solutions with themselves, through the term $Ro U \cdot \nabla U$, to produce higher harmonics with frequencies $0$ and $2\omega$, then these will interact to produce still higher harmonics $3\omega$, $4\omega$, etc. The amplitudes of
Figure 7. Profiles of $U_t$, the tangential velocity along the layer, along the line segments indicated in figure 6. The first two panels are for $\omega = 0.4$ and the second two for $\omega = 0.8$. Within each pair, the first shows $U_t$ as a function of $d$, the actual distance along the segments in figure 6, with $d=0$ corresponding to the location of the characteristic. The broadest profiles correspond to $E = 10^{-5}$, the next to $10^{-6}$ and the narrowest to $10^{-7}$ (color-coded online as black, blue, red). The second panel in each pair shows the same profiles, but with $d$ replaced by $d/E^{1/3}$—that the widths then essentially collapse on top of one another indicates the agreement with the expected $E^{1/3}$ scaling.

Figure 8. The $\cos(\sigma t)$ part of the harmonics $\sigma = \omega$, $2\omega$, $3\omega$ (from left to right), and $\omega = 0.4$ (top row) and $\omega = 0.8$ (bottom row). $E = 4 \times 10^{-5}$ and $Ro = 0.1$ for both solutions. As in figure 6, what is shown is the streamfunction of the meridional circulation, with white indicating clockwise circulation, gray counter-clockwise.

these harmonics should scale as $O(1)$ for the fundamental mode $\omega$, $O(Ro)$ for the second harmonics 0 and $2\omega$, $O(Ro^2)$ for $3\omega$, and so on (where we recall also that due to the non-dimensionalization of $U$ as $\epsilon r_1$, the leading-order scaling with amplitude has already been accounted for, hence the expectation that the fundamental mode will be $O(1)$, independent of $Ro$).

The time stepping of (1) and (2) makes no assumptions about the scalings of the different harmonics, or indeed any separation into individual harmonics at all. Having computed $U(r, \theta, t)$ though, one can easily Fourier decompose the entire time series

$$U(r, \theta, t) = \sum_n \left[ U_n^c(r, \theta) \cos(\sigma_n t) + U_n^s(r, \theta) \sin(\sigma_n t) \right]$$

and examine the individual components $U_n$ separately.

Figure 8 shows the first three harmonics $\sigma = \omega$, $2\omega$ and $3\omega$, for the same fundamental frequencies $\omega = 0.4$ and $0.8$ as previously in figure 6. These results were computed at
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$Ro = 0.1$, but values as large as 0.4 still yielded similar spatial structures. Not surprisingly, the $\sigma = \omega$ fundamental modes are essentially the same as the previous linear modes in figure 6. With regard to the $2\omega$ and $3\omega$ harmonics, it is interesting to note that for $\omega = 0.4$ even $3\omega$ is still less than 2, whereas for $\omega = 0.8$ only $2\omega$ is less than 2, but $3\omega$ is already greater than 2. And sure enough, for $\omega = 0.4$ both the second and third harmonics exhibit the pattern of characteristics appropriate for their frequencies, whereas for $\omega = 0.8$ only the second harmonic does, with the third harmonic largely confined to the inner Ekman layer. That is, all higher harmonics $n$ that still satisfy $n\omega < 2$ are essentially inertial oscillations in their own right, simply triggered in a different way from the fundamental mode.

Figure 8 top row should be compared with figures 3(a)–(c) and figure 8 bottom row with figures 3(e) and (f). It is obvious that the EOFs nicely capture the dominant shear layers and their harmonics of the numerical simulations. Some differences remain: figure 3(b) looks somewhat more flared than the corresponding numerical results. From the second harmonics of the $\omega = 0.8$ case, we find in the experimental data only one of the three branches of the shear layer (figure 3(f)). Likely, the differences result from optical distortion and the large noise-to-signal ratio of the visualization.

From figure 8 it is obvious that the forced wave and its harmonics are excited at the critical points in the boundary layer of the inner sphere. A critical point is located where the slope of a shear layer matches the slope of the boundary. At the critical latitude on the inner boundary, anomalous Ekman pumping generates a shear layer tangent to the inner sphere’s surface (Kerswell 1995, Calkins et al 2010). Excited via this local process, inertial waves propagate in the form of wave beams throughout the bulk of the spherical shell. Note that for a boundary that curves convex inwards a similar mechanism drives motion away from the boundary layer. This is possible at the outer spherical shell (Kerswell 1995) but also for geometries mimicking a continental slope (Swart et al 2010). Note finally that we find fairly intense meridional circulation near the inner sphere in particular near the equator for the third harmonic at $\omega = 0.4$. In figure 3 the motion close to the inner sphere seems to be weaker. One reason for this might be due to the use of aluminum flakes for visualization. For higher frequencies or less well-organized motion, the flow cannot align the particles properly. Moreover, inhomogeneous light conditions have a significant influence on the quality of the pictures.

3.4. Mean flow

It is of particular interest also to consider the $\sigma = 0$ harmonic, as this corresponds to a steady rather than an oscillatory flow. Unlike the oscillatory components, where the meridional circulation (the part shown in figures 6 and 8) and the zonal flow are comparable in magnitude, this steady component is strongly dominated by the zonal flow $U_\phi$. Figure 9 shows this zonal flow, which is seen to consist of a prograde jet situated on the tangent cylinder $C$. It is clearly visible also in the experiment, e.g. in figure 2 and, as a top view, in figure 5. The zonal flow is reminiscent of the so-called Stewartson layer (Stewartson 1966) that is generated by a steady differential rotation of the inner sphere with respect to the outer one, but of course there are also important differences, not only with regard to how the two systems are forced, but also the fact that here we are dealing with a jet, with $U_\phi \approx 0$ on either side (see figure 5), whereas the Stewartson layer is a shear layer, with $U_\phi$ different on either side.

Figure 10 illustrates the structure of these zonal jets in more detail, presenting slices through them along the horizontal line $z = 1$, level with the top of the inner sphere. Both the distance along the line and $U_\phi$ have also been rescaled to reflect the expected (or conjectured) scalings: the distance from $C$, $s - 1$, has been rescaled by $E^{1/3}$, in accord with the
Figure 9. The steady ($\sigma = 0$) part of $U_\phi$, for the frequencies $\omega = 0.4$ and 0.8 and Ekman numbers $E = 4 \times 10^{-5}$ and $2 \times 10^{-5}$ as indicated. $Ro = 0.1$ for all four. The contour interval is 0.01, with positive contours only being shown, to avoid zero contours throughout the regions where $U_\phi \approx 0$.

Figure 10. Profiles of $U_\phi 1000E^{3/10}/Ro$, as functions of $(s-1)/E^{1/3}$, the stretched distance from the tangent cylinder. The top row is for $\omega = 0.4$ and the bottom row for $\omega = 0.8$. From left to right $E = 4 \times 10^{-5}$, $2 \times 10^{-5}$ and $10^{-5}$, as indicated. Dotted line (black) corresponds to $Ro = 0.1$, dashed line (blue) to $Ro = 0.2$ and solid line (red) to $Ro = 0.4$. That all six profiles for a given $\omega$ are essentially identical suggests that the indicated rescalings, in both $(s-1)$ and $U_\phi$, are correct. The further strong similarity between the $\omega = 0.4$ and 0.8 profiles suggests that these zonal jets are a robust feature, and do not depend on the particular choice of $\omega$.

previous scaling by Kerswell (1995), and one of the scalings (the innermost one) found in the Stewartson layer as well. The lower limit of $s$ in figure 10 is 0.5 that is half the radius of the inner sphere. The retrograde flow $U_\phi$ slowly converges to zero when $s$ approaches zero (the surface of the inner sphere). This corresponds with the PIV observations shown in figure 5. Note that the Stewartson layer also has outer scalings $E^{2/7}$ just inside $C$ and $E^{1/4}$ just outside $C$ though. Distinguishing whether such outer layers are present here as well would probably require several orders of magnitude variation in $E$. However, it can be conjectured that these outer layers are absent since, in contrast to the Stewartson layer there is no discontinuity in $U_\phi$. 

Turning next to the amplitude of $U_\phi$, we argued above that it ought to scale as $O(Ro)$. This is borne out by the numerical results; rescaling as $U_\phi/Ro$ produces virtually identical profiles for $Ro = 0.2$ and 0.4. Regarding its variation with $E$, this closely matches the $E^{-3/10}$ scaling derived by Noir et al (2001) for precessionally driven zonal flows. Tilgner (2007) similarly considered zonal flows induced by tidally driven inertial modes, and also obtained flows that show indications of divergence in the inviscid limit. That our zonal flows increase with decreasing $E$ is therefore consistent with these previous works, even if our range of variation in $E$ is far too small to make any definite statements regarding the precise exponent.

Tilgner's numerical results obtained for azimuthal wavenumber $m = 2$ show a stronger mean flow forcing in the bulk of the shell. Whether this difference results from differences in the wave forcing is an important issue. In contrast to Tilgner the libration is applied just at the inner sphere. Moreover, we apply an azimuthally symmetric forcing ($m = 0$). Analytical results on mean flow excitation in librating spheres have been obtained only for frequencies for which inertial waves are not important (Busse 2010). Thus only further numerical simulations could answer this open question.

Calkins et al (2010) found a different mean flow structure for their setup with an outer librating sphere: globally the flow was retrograde with a prograde jet along the outer equatorial region of the shell. Deviating from our result, the strength of the mean flow is independent of the Ekman number (and thus the viscosity). In their case the mean flow was due to nonlinear advection in the Ekman layer (see also Busse 2010, Sauret et al 2010). The discrepancy points to a stronger wave–mean flow interaction for our setup. Wave-driven mean flows depend essentially on viscosity. Recently, Sauret and Le Dizès (2013) studied mean flow generation in a spherical shell by asymptotic and numerical techniques in the limit of small amplitude and Ekman number. In addition to Busse (2010) they found that the mean flow also exhibits a discontinuity across the cylinder tangent to the inner sphere, very similar to the layer shown in figures 2 and 9. The mean flow is excited by nonlinear interactions in the Ekman layers. However, their analysis is valid for forcing frequencies larger than $2\Omega_1$ and wave–mean flow interactions are excluded.

Figure 11 shows how much energy is contained in the different Fourier modes in (3), and confirms many of our previous expectations about the amplitudes of the various harmonics. Specifically, the $\sigma = \omega$ primary mode is indeed essentially independent of $Ro$, whereas the energy in the higher harmonics consistently increases between $Ro = 0.2$ and 0.4. We note though that these values of $Ro$ are already sufficiently large that at least some modes are starting to deviate from the weakly nonlinear scalings. Another point to note is that for $E = 2 \times 10^{-5}$ and $Ro = 0.4$, the $\sigma = 0$ steady part of the flow contains almost as much energy as the $\sigma = \omega$ primary mode. For even smaller $E$, or larger $Ro$, the steady zonal flow would thus be the dominant flow structure and cannot be seen as harmonic of the forced wave.

3.5. Instabilities

The final feature to note in figure 11 is the presence of energy in the $Ro = 0.4$ solutions at frequencies $\sigma$ other than the various harmonics $n\omega$. That is, these solutions no longer follow the basic $2\pi/\omega$ periodicity of the imposed inner sphere libration. For these aperiodic solutions, a time series of 32 libration periods was used to do the Fourier decomposition (3).

To understand the nature of these aperiodic flow components, it is helpful to focus on $U' = U - U_\text{p}$, where $U_\text{p}$ is the periodic part, consisting of all the $n\omega$ harmonics. Figure 12 shows how the energy associated with $U'$ varies throughout the libration period. Each cycle is indeed different, confirming the aperiodic nature of this part of the flow, but the overall pattern is still strikingly similar between different periods, always consisting of a rapid increase
Figure 11. The kinetic energy, as it is distributed among the various Fourier modes in (3). The first two panels are for $\omega = 0.4$, the second two for $\omega = 0.8$; within each pair the Ekman numbers are as indicated. Pluses (blue) correspond to $Ro = 0.2$ and crosses (red) to $Ro = 0.4$. The small dots (red) indicate the instabilities of the $Ro = 0.4$ solutions.

Figure 12. The kinetic energy contained in the aperiodic part of the flow $U'$, as a function of the phase $\omega t$ throughout the libration cycle. $E = 2 \times 10^{-5}$, $Ro = 0.4$ and $\omega = 0.4$ and 0.8 in the two panels, as indicated. In each case six representative cycles are shown.

during the early part of the cycle, when the inner sphere is swinging through its equilibrium position in the prograde direction, followed by a gradual decrease throughout the rest of the cycle.

Figure 13 shows snapshots of $U'$ and $U$ throughout the first half of the cycle, where it is strongest. $U'$ is seen to consist of small vortices that form in the boundary layer on the inner sphere, gradually spreading outward as they fade away again. Comparing the two frequencies $\omega = 0.4$ and 0.8, the vortices are weaker, and peak somewhat later in the cycle for $\omega = 0.8$, consistent with the results in figure 12. These instabilities are evidently the equivalent of the Görtler vortices previously studied in considerable detail by Calkins et al (2010). Unlike ours, their vortices occurred at the outer sphere during the retrograde phase of the cycle. This is entirely consistent though: for a librating outer sphere the centrifugally unstable part of the cycle is the retrograde phase, whereas for a librating inner sphere it is the prograde phase. For higher frequencies, this local inertial instability has less time to grow. Therefore the vortices show up later and remain weaker for $\omega = 0.8$. In the experiments we observed that when $Ro$ is increased the flow becomes more irregular. This irregularity sets in earlier for low forcing frequencies. This observation can be explained by the Görtler instability that is more prominent for low forcing frequencies as is obvious from figures 12 and 13.

Finally, figure 14 shows snapshots of the full solutions $U$. The perturbations $U'$ are in fact completely overwhelmed by the periodic component $U_p$, consistent with the relative energies listed in figure 11. As expected, the general structure consists of superpositions of the individual harmonics in figures 8 and 3, respectively. The patterns again fluctuate throughout
Figure 13. Streamlines of the meridional part of $U'$ (black and white) and of $U$ (color), at the four points in the cycle $\omega t = k\pi/4$, for $k = 1, 2, 3, 4$, from left to right. The top row is for $\omega = 0.4$, the bottom row for $\omega = 0.8$. $E = 2 \times 10^{-5}$ and $Ro = 0.4$ for both solutions. Only part of the spherical shell, $r = [1, 1.2]$ and $\theta = [75^\circ, 90^\circ]$, is shown to concentrate on the vortices that form in the outer boundary layer. For the upper two rows, the contour interval is $4 \times 10^{-4}$, with white and gray indicating oppositely circulating flows. For the color figures the contour interval is $4 \times 10^{-3}$. 
the cycle, but are always concentrated on the characteristics of the various harmonics. The zonal flow $U_\phi$ is not shown; this is dominated by the steady component previously illustrated in figure 9.

4. Conclusion

We experimentally and numerically studied the flow induced in a rotating spherical shell. The shell globally rotates with angular velocity $\Omega$. A further periodic oscillation with angular velocity $0 \leq \omega \leq 2\Omega$, a so-called longitudinal libration, was added on the inner sphere’s rotation. This configuration has not been considered before. In previous works the sphere was either librating globally (Tilgner 2007) or just the outer sphere was librating (Noir et al 2009, Calkins et al 2010). In contrast to Busse (2010) and Sauret et al (2010) we focus on the excitation of inertial modes and do not take the limit of very low libration frequencies. The primary response of the experiments described was an inertial wave spawned at the critical latitudes on the inner sphere, and propagating throughout the shell along inclined characteristics. For sufficiently large libration amplitudes, the higher harmonics also become important. Seeding the fluid with aluminum flakes gives a surprisingly clear picture of the forced wave and its harmonics. In contrast, by using Kalliroscope, Noir et al (2001, 2009) did not find the inertial waves in their experiments with a librating outer sphere.

In our study, the steady component of the flow consists of a prograde zonal jet on the cylinder tangent to the inner sphere and parallel to the axis of rotation. The structure of the steady flow was surprisingly different from the previous results with global or outer-sphere libration. For a librating sphere, (Busse 2010) found a retrograde steady flow that becomes stronger near the outer boundary. The solution holds for a low libration to rotation ratio, and Busse argues that it should look essentially the same for a spherical shell. For the steady bulk flow, Sauret et al (2010) experimentally confirmed the analytical results by Busse (2010) but found an unpredicted prograde flow at the equator near the outer wall boundary layer. The steady flow in the bulk did not depend on either the Ekman number or the libration frequency. However, our experiments show that the steady jet on the tangent cylinder scales with the Ekman number. This is in agreement with Tilgner (2007) who reported on inertial wave-driven shear layers in his numerical simulations. The shear layers diverge for decreasing Ekman numbers.
The main result of this paper is that all features found experimentally and discussed above agree very well with numerical model simulations. However, the numerical model allows for a deeper analysis of certain facets of the flow in the shell.

The thickness of the shear layers found from the linear simulation scales with $E^{1/3}$, in excellent agreement with the scaling derived by Kerswell (1995). The thickness of the steady jet scales by $E^{1/3}$, also found for the innermost layer of the Stewartson layer. The outer layers of the Stewartson layer, scaling by $E^{2/7}$ and $E^{1/4}$, are probably absent since, in contrast to the Stewartson layer, there is no discontinuity in the steady flow. The amplitude of the steady jet seems to scale with $Ro E^{-3/10}$ comparable to the result by Noir et al (2001).

In full agreement with Noir et al (2009) and Calkins et al (2010), we found Görtler instability for a larger libration amplitude. Their instability occurred at the outer sphere during the retrograde phase of the cycle whereas we found the instability at the inner sphere during the prograde phase. However, this is consistent since for a librating outer sphere the centrifugally unstable part of the cycle is the retrograde phase, whereas for a librating inner sphere it is the prograde phase. In the experiments we observed that for large $Ro$ and small $\omega$ the flow becomes more irregular. This observation can be explained by the Görtler instability that has more time to grow for a low libration frequency.

Several questions regarding the experimental and numerical findings we presented are still open. Most important is to ask where the difference between the experiments with an librating outer and inner sphere come from. It is likely that, similar to the propagating waves, the steady jet is excited at the critical latitude on the inner librating sphere. For steady waves, wave rays are parallel to the axis of rotation and the critical latitude is the equator itself. It is therefore not too surprising that the jet forms a tangent cylinder touching the inner sphere’s equator. Why the jet depends on $E$ for the librating inner sphere but is independent of $E$ in Busse (2010) and Sauret et al (2010), where the outer sphere librates, is not clear yet. Also not answered is the question why Tilgner (2007) found a stronger mean flow forcing in the bulk of his globally librating shell. The difference might be caused by the different zonal wavenumbers of the forcing: we apply a zonally symmetric forcing but Tilgner (2007) used a zonal wavenumber two forcing. With respect to the linear solutions, the scaling of the shear layer amplitude is not clear yet. Kerswell (1995) demonstrated that the amplitude of the flow within the layers should scale as $E^{1/6}$. Our numerical results imply that the amplitude decreases slower than $E^{1/6}$.

One intriguing suggestion from the current visualization is that the zonal jet on the tangent cylinder seems to break down for sufficiently large Rossby numbers, especially at smaller frequencies. In addition to the axisymmetric Görtler instabilities considered here, this could also be due to a non-axisymmetric Kelvin–Helmholtz instability, similar to the familiar instabilities of the Stewartson layer (Hollerbach 2003, Hollerbach et al 2004, Schaeffer and Cardin 2005). Studying flow transitions such as this, both experimentally and numerically, will help to further improve our understanding of the full range of phenomena that exist in librationally driven flows.

The open question addressed might be answered by an extensive numerical parameter survey. However, this is beyond the scope of this paper where we focused on the comparison between experimental and numerical results. Notwithstanding, these issues will be picked up in a forthcoming study. It would also be of considerable interest to upgrade the experiment, doing more quantitative PIV measurement in the meridional and horizontal plane. Moreover, it would be instructive to add a differential rotation to the flow and compare the outcome with recent findings by Rieutord et al (2012).
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