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The GEOFLOW-EXPERIMENT ON ISS (PART III): BIFURCATION ANALYSIS

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ABSTRACT

Codimension 2 bifurcation of convective patterns in the nonrotating spherical Bénard Problem of Boussinesq fluid is studied using center manifold reduction. The aim is to determine, in the GEOFLOW-experiment framework, the physical parameters in order to obtain intermittent-like behaviour dynamics. We compute the critical aspect ratio and Rayleigh number corresponding to the $(\ell, \ell+1)$ mode interaction. In the case of the $(2,3)$ interaction, the existence of heteroclinic cycles are examined versus the Prandtl number.

INTRODUCTION

The bifurcation theory with symmetry is a powerful tool in the spherical Bénard problem, in which the near-onset convection in a self-gravity spherical shell of fluid is considered. In this problem, there are two natural bifurcation parameters: one is the Rayleigh number R_a which is proportional to the buoyancy force responsible for the onset of convection; the second one is the aspect ratio η which is the ratio of the inner to the outer radius of the shell. When the Rayleigh number is increasing the "trivial" solution loses its stability. Generically, a unique spherical mode ℓ is unstable (codimension 1, only R_a is allowed to vary). In this case, we expect only stationary or travelling waves solutions (Gaeta and Rossi, 1984). For specific aspect ratio numbers η_c, two modes $(\ell, \ell+1)$ are unstable (codimension 2, R_a and η vary). For this mode interaction the dynamics are more complex. It is already known since the article of Guckenheimer and Holmes (1988) that structurally stable heteroclinic cycles between group orbits of equilibria (i.e. steady states) can arise due to the symmetry of the problem. The $(1,2)$ interaction was studied from a numerical point of view by Friedrich and Haken (1986) and later by Armbruster and Chossat (1991) using group theoritic methods. A general study of the $(\ell, \ell+1)$ interaction was presented in Chossat and Guyard (1996). The authors show, under certain "generic" conditions, that heteroclinic cycles of various types exist and these connections are "robust" against small perturbation. Finally, Chossat (1999) has been proved that a heteroclinic cycle persists when the system is slowly rotating around an axis. In this case, not only steady-states but time periodic flows with the form of rotating waves are connected by heteroclinic orbits.

The GEOFLOW-experiment on ISS is an opportunity to find such intermittent-like behavior dynamics and thereby to corroborate the theory. But it has two specificities compared to the previous studies:

- The simulated central force field (dielectrophoretic force field) differs from the gravity field. Indeed, the gravity and buoyancy forces have the same mathematical form and then degeneracy occurs in the bifurcation equations. In particularly, the existence of heteroclinic cycles is proved only in the degeneracy case neighborhood (Chossat and Guyard, 1996).
- The presence of a thin conductor wire inside the shell (which is needed to control the potential of the inner sphere). This wire is breaking the symmetry of the problem and could play a relevant role on bifurcations.

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In this present work, the problem has the spherical $O(3)$-symmetry, i.e., there is not rotation and we neglect the influence of the wire. The aim of this paper is to determine if the degeneracy conditions could be satisfy with the experiment requirements. After a brief description of the model, we compute by a linear study the critical values of the both parameters R_a and η in order to obtain a mode interaction. Then, the bifurcation equations of the $(2,3)$ interaction are presented and the method of the bifurcation coefficients computation is described. Finally, we present the variations of these coefficients versus the Prandtl number.

GENERAL FRAMEWORK

The governing equations for perturbations $\vec{v}$ of the velocity field of the fluid and Θ of the temperature field are set in the Boussinesq approximation. They are given, after a suitable nondimensionalization, by

$$\begin{aligned} \frac{\partial \vec{u}}{dt} &= -\vec{\nabla} p + \Delta \vec{v} + \lambda \Theta g^*(r)\vec{r} - \vec{v}.\vec{\nabla}\vec{v} \\ \nabla.\vec{v} &= 0 \\ \frac{\partial \Theta}{dt} &= P_r^{-1}(\Delta\Theta + \lambda h^*(r)\vec{r}.\vec{v}) - \vec{v}.\vec{\nabla}\Theta \end{aligned} \tag{1}$$

where p is the pressure, P_r is the Prandtl number and λ is proportional to the square root of the Rayleigh number R_a:

$$\lambda = \sqrt{\frac{2R_a}{(1/\eta - 1)^3}}. \tag{2}$$

The functions $g^*(r) = \frac{1}{r^5}$ and $h^*(r) = \frac{1}{r^3}$ are respectively the dimensional less central force and the temperature gradient. The system of equations (1) is defined in the domain

$$\Omega = \{(r, \theta, \phi) \mid \eta < r < 1\},$$

where η is the aspect ratio. The fluid is viscous, then we consider "rigid boundaries": $\vec{v} = \vec{0}$ on $\partial\Omega$ and the temperature is imposed on boundaries: $\Theta = 0$ on $\partial\Omega$.

We develop each solution $(\vec{v}, \Theta)$ in the generalized spherical harmonics (Chossat and Giraud, 1983):

$$\mathbf{z}(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \alpha_m^\ell(r)\xi_m^\ell(\theta, \varphi), \tag{3}$$

where (r, θ, φ) are the spherical coordinates and :

$$\alpha_m^\ell = \begin{pmatrix} u_{0,m}^\ell & & & 0 \\ & u_{+,m}^\ell & & \\ & & u_{-,m}^\ell & \\ 0 & & & \Theta_m^\ell \end{pmatrix}, \quad \xi_m^\ell = \begin{pmatrix} T_{0,m}^\ell \\ T_{+,m}^\ell \\ T_{-,m}^\ell \\ T_{0,m}^\ell \end{pmatrix}.$$

The spherical functions $T_{k,m}^\ell$ are known by a recursive relation and they are determined by a formal computing.

Remark: We do not have use toroidal and poloidal decomposition, as in Friedrich and Haken (1986), because we need to differentiate the scalar potential to obtain the velocity field. This derivation is a source of numerical errors. Furthermore, this decomposition occurs only for an incompressible fluid.

LINEAR STABILITY

The pure conductive solution becomes unstable when the Rayleigh number increases (i.e. λ). The instability arises when the linearized system (1) has a zero eigenvalue. Due to the spherical symmetry, we obtain an ordinary differential system with the unique variable r for each mode ℓ. Furthermore, it does not depend on the m indice. The radial functions, i.e. $\alpha_m^\ell(r)$, are discretized thanks the Chebychev polynomials ("pseudo-spectral" method, Funaro, 1992). The eigenvalues problem is solved by an Arnoldi method (see e.g. Laure, 2000).

For each mode ℓ_c corresponds a critical value λ_c. This critical mode depends on the aspect ratio η. We can prove that ℓ_c tends to infinity when η tends to 1 (Chossat, 1979). Then, the $\ell_c(\eta)$ is an increasing function. The figure 1 shows the function $\ell_c(\eta)$ for $\ell_c \leq 6$. We remark that for some values of the ratio aspect, two modes co-exist. In particularly, for $\eta_c \cong 0.33$ the interaction (2,3) holds (Figure 1). The interaction (1,2) can be not reached experimentally, because it appears for $\eta < 0.3$.

Let us now consider the bifurcation situation for the critical values $\lambda_c = 19.8$ and $\eta_c = 0.333$ corresponding to the (2,3) mode interaction.

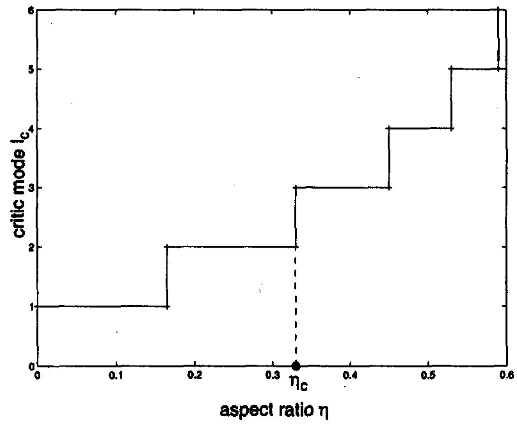

Fig. 1. Critic number ℓ value versus aspect ratio η

BIFURCATION PROBLEM

Center Manifold Reduction

The onset of convection can be examined by varying the two system parameters λ and η. Let us $Z(t) = (\vec{v}(t), \Theta(t))$, $\tilde{\lambda} = \lambda - \lambda_c$, $\tilde{\eta} = \eta - \eta_c$. The system (1) writes in a functional space which we do not precise (see Chossat, 1979):

$$\frac{\partial Z}{\partial t} = L_0 Z + \tilde{\lambda} L_1 Z + \tilde{\eta}(L_2 + 0(\tilde{\eta}^2))Z + M(Z, Z) = f(Z, \tilde{\lambda}, \tilde{\eta}), \tag{4}$$

where L_0, L_1, and L_2 are linear operators and M is a bilinear and symmetric operator. The eigenspace of L_0 is $V = V_2 \oplus V_3$ where V_ℓ is the subspace, which has dimension $2\ell + 1$, associated of the irreductible representation of ℓ degree of the $O(3)$ group. We shall denote by $(\zeta_m^\ell)_{m=-\ell,\dots,\ell}$ the corresponding base of V_ℓ. A vector $U(t) \in V$ van be expressed as

$$U = \sum_{m=-2}^{2} X_m \zeta_m^2 + \sum_{n=-3}^{3} Y_n \zeta_n^3, \tag{5}$$

where $X_{-m} = (-1)^m \overline{X}_m$ and $Y_{-n} = (-1)^n \overline{Y}_n$.

The dynamics and bifurcations from the basic state near the critical values η_c and λ_c of the parameters are governed by a system of ODE's which is the trace of the original system of PDE's (1) on its center manifolds. We define P_0 as the projection onto the kernel V of L_0 which commutes with L_0. If Z is the solution of Eq. (4), then Z writes (Chossat and Iooss, 1994):

$$Z(t) = U(t) + \Psi(U(t), \tilde{\lambda}, \tilde{\eta}) = U(t) + Y(t),$$

where $U(t) \in V$ and Ψ an application defined in Chossat and Iooss (1994). First, we decompose Eq. (4) as

$$\begin{aligned} \dot{U} &= P_0 f(U + \Psi(U, \tilde{\lambda}, \tilde{\eta}), \tilde{\lambda}, \tilde{\eta}) \\ \dot{Y} &= (I - P_0)(U + Y, \tilde{\lambda}, \tilde{\eta}). \end{aligned} \tag{6}$$

Then, we develop the first equation of the system (6) in terms of U, $\tilde{\lambda}$ and $\tilde{\eta}$, yielding the following series expansion:

$$\dot{U} = \sum_{pqr} \tilde{\lambda}^q \tilde{\eta}^r U^p_{qr}(U, ..., U),$$

where R^p_{qr} is p-linear and symmetric. This is the bifurcation equation, it consists of 5 equations for X_m and 7 equations for Y_n.

Equivariant Structure of the Bifurcation Equation

These equations keep the $O(3)$ symmetry of Eq. (1), and therefore have a certain equivariant structure. We give the explicit form of these polynomials equations for the order 3:

$$\left\{ \begin{aligned} \dot{X}_m &= \lambda_2 X_m + aQ_m(X, X) + bP_m(Y, Y) + X_m(e_{11} \parallel X \parallel^2 + e_{12} \parallel Y \parallel^2) \\ \dot{Y}_n &= \lambda_3 Y_n + cR_n(X, Y) + Y_n(e_{21} \parallel X \parallel^2 + e_{22} \parallel Y \parallel^2) + dS_n(Y, Y, Y). \end{aligned} \right. \tag{7}$$

In these equations, $Q = (Q_{-2}, \cdots, Q_2), P = (P_{-2}, \cdots, P_2)$ et $R = (R_{-3}, \cdots, R_3)$ are quadratic polynomials defined by:

$Q_2 = 2\ X_2X_0 - \sqrt{6}/2\ X_1^2$ $\quad P_2 = 3/2\ Y_1^2 - \sqrt{30}/2\ Y_0Y_2 + \sqrt{15}/2\ Y_{-1}Y_3$
$Q_1 = \sqrt{6}\ X_2X_{-1} - X_0X_1$ $\quad P_1 = \sqrt{12}\ Y_0Y_1 + 5\sqrt{10}\ Y_3Y_{-2} - 3\sqrt{10}\ Y_{-1}Y_2$
$Q_0 = 2\ X_{-2}X_2 + X_{-1}X_1 - X_0^2$ $\quad P_0 = 2\ Y_0^2 - 3\ Y_{-1}Y_1 + 5\ Y_{-3}Y_3$
$R_3 = X_2Y_1 - 1/2\sqrt{10}\ X_1Y_2 + \sqrt{10}/2\ X_0Y_3$
$R_2 = \sqrt{2}\ X_2Y_0 + \sqrt{10}/2\ Y_3X_{-1} - \sqrt{6}/2\ X_1Y_1$
$R_1 = -\sqrt{5}/5\ X_1Y_0 + 2/5\sqrt{15}\ X_2Y_{-1} - 3/10\sqrt{5}\ X_0Y_1 + X_{-1}Y_3 + \sqrt{6}/2\ X_{-1}Y_2$
$R_0 = \sqrt{5}/5\ X_1Y_{-1} + \sqrt{2}\ X_2Y_{-2} - 2/5\sqrt{5}\ Y_0X_0 + \sqrt{5}/5\ Y_1X_{-1} + \sqrt{2}\ X_{-2}Y_2$,
where $X_{-k} = (-1)^k\ \overline{X}_k, Y_{-k} = (-1)^k\ \overline{Y}_k$. The S_n are cubic polynomials in terms of Y which we do not explicit here (see e.g. Chossat and Lauterbach, 2000).

The numerical value of the coefficients in the Taylor expansion is determined by the physical conditions of the model.

Computation of the Bifurcation Coefficients

The equation system (6) can be solved order by order for the p-linear U^p_{qr} (Chossat and Guyard, 1996). The computation, at the lower orders, gives the below expressions.

linear coefficients

$$\begin{aligned} \lambda^{(i)} &= \left\langle L_1\zeta_0^{(i)}, \zeta_0^{*(i)} \right\rangle \\ \eta^{(i)} &= \left\langle L_2\zeta_0^{(i)}, \zeta_0^{*(i)} \right\rangle \end{aligned} \quad \text{and} \quad \lambda_i = \lambda^{(i)}\tilde{\lambda} + \eta^{(i)}\tilde{\eta}.$$

quadratic coefficients

$$\begin{aligned} a &= -\left\langle M\left(\zeta_0^{(2)}, \zeta_0^{(2)}\right), \zeta_0^{*(2)} \right\rangle \\ b &= \frac{1}{2}\left\langle M\left(\zeta_0^{(3)}, \zeta_0^{(3)}\right), \zeta_0^{*(2)} \right\rangle \\ c &= -\sqrt{5}\left\langle M\left(\zeta_0^{(2)}, \zeta_0^{(3)}\right), \zeta_0^{*(3)} \right\rangle \end{aligned}$$

cubic coefficients

$$
\begin{aligned}
e11 &= -2\left\langle M\left(\zeta_0^{(2)}, SM\left(\zeta_0^{(2)},\zeta_0^{(2)}\right)\right),\zeta_0^{*(2)}\right\rangle \\
e12 &= -2\left\langle M\left(\zeta_0^{(2)}, SM\left(\zeta_0^{(3)},\zeta_0^{(3)}\right)\right),\zeta_0^{*(2)}\right\rangle - 4\left\langle M\left(\zeta_0^{(3)}, SM\left(\zeta_0^{(2)},\zeta_0^{(3)}\right)\right),\zeta_0^{*(2)}\right\rangle \\
e21 &= -2\left\langle M\left(\zeta_0^{(3)}, SM\left(\zeta_0^{(2)},\zeta_0^{(2)}\right)\right),\zeta_0^{*(3)}\right\rangle - 4\left\langle M\left(\zeta_0^{(2)}, SM\left(\zeta_0^{(2)},\zeta_0^{(3)}\right)\right),\zeta_0^{*(3)}\right\rangle \\
e22 &= -2\left\langle M\left(\zeta_3^{(3)}, SM\left(\zeta_0^{(3)},\zeta_0^{(3)}\right)\right),\zeta_3^{*(3)}\right\rangle - 4\left\langle M\left(\zeta_0^{(3)}, SM\left(\zeta_0^{(2)},\zeta_3^{(3)}\right)\right),\zeta_3^{*(3)}\right\rangle \\
d &= \frac{e_{22}}{18} + \frac{1}{9}\left\langle M\left(\zeta_0^{(3)}, SM\left(\zeta_0^{(3)},\zeta_0^{(3)}\right)\right),\zeta_0^{*(3)}\right\rangle .
\end{aligned}
$$

where S is the *pseudo-inverse* of L_0. More precisely, S is the operator such that $SY = X$, where $Y = L_0X$ and $X, Y \in Im(L_0)$.

RESULTS

In order to obtain the "pure" solutions $x = \pm\sqrt{-\frac{e_{11}}{\lambda_2}}$ and $y = \pm\sqrt{-\frac{e_{22}}{\lambda_3}}$, it is necessary that:

- a close to zero,
- $e_{11} < 0$ and $e_{22} < 0$.

Furthermore, in order to obtain the existence for heterocline cycles, it is required that

$$bc > 0.$$

All these conditions are satisfied if $g^* = h^*$. It is not our case thus we have to compute numerically these coefficients.

The coefficients are shown in Figures 2 and 3 as a function of the Prandtl number $\mathcal{P}_r$ ($-3 < log P_r < 4$). The cubic coefficients e_{11} and e_{22} are negative for all values of $\mathcal{P}_r$. The quadratic coefficient a has the same order that cubic coefficients for the range $\mathcal{P}_r < 10^{-1}$. Otherwise, a is negligible. And a vanishes for $\mathcal{P}_r \cong 0.237$ and changes its sign. The coefficient b holds the same sign for all Prandtl values. On the other hand, c changes its sign for Prandtl close to the unity. So, for $\mathcal{P}_r > 1$, the product bc is negative, we cannot find cycles.

Hence, the value $Pr = 0.237$ is very interesting because a is close to zero and the requirements of heterocline cycles existence are satisfied. We give in Table 1 the coefficients values for this case.

A second interesting point is the degeneracy case for $Pr =\simeq 0.95$ for which $c = 0$. The dynamic issues from this bifurcation is poorly known. A recent study for the interaction (1,2) has been treated by Porter and Knobloch (2001).

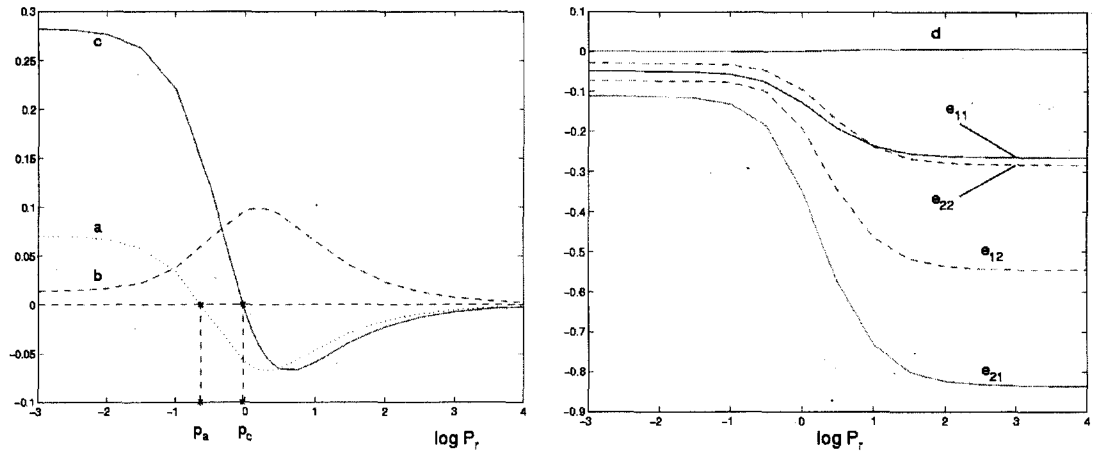

Fig. 2. Quadratic coefficients a, b et c vs $\log \mathcal{P}_r$ and cubic coefficients e_{11}, e_{12}, e_{21}, e_{22} et d vs $\log \mathcal{P}_r$

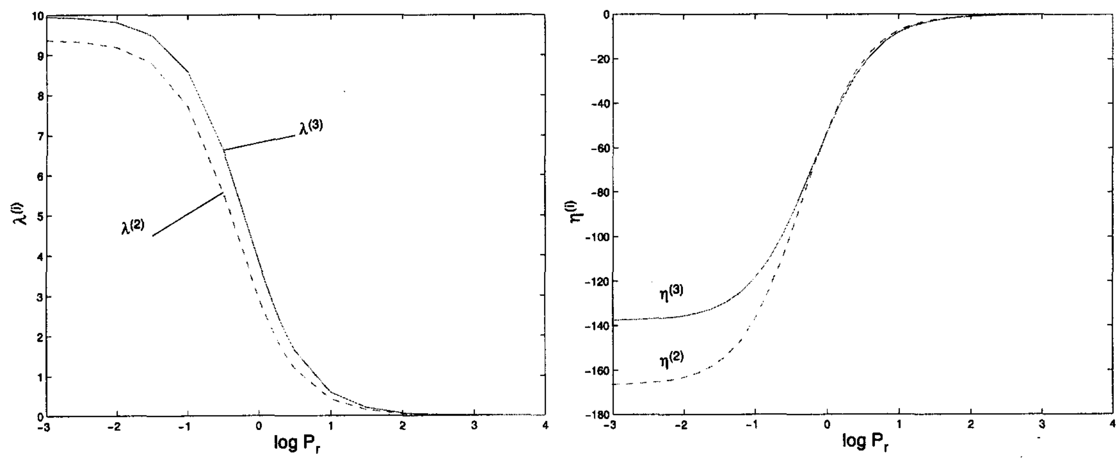

Fig. 3. $\lambda^{(2)}$ and $\lambda^{(3)}$ coefficients vs $\log \mathcal{P}_r$ and $\eta^{(2)}$ and $\eta^{(3)}$ coefficients vs $\log \mathcal{P}_r$

Table 1. Coefficients values for $\mathcal{P}_r = 0.237$

a	b	c
$-4.25\ 10^{-7}$	$5.93\ 10^{-2}$	$1.52\ 10^{-1}$

$\lambda^{(2)}$	$\lambda^{(3)}$	$\eta^{(2)}$	$\eta^{(3)}$
6.23	7.26	−43.1	−37.5

e_{11}	e_{12}	e_{21}	e_{22}	d
$-6.87\ 10^{-2}$	$-8.97\ 10^{-2}$	$-1.66\ 10^{-1}$	$-4.12\ 10^{-2}$	$3.89\ 10^{-4}$

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