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What can the observed rotation of the Earth's inner core reveal about the state of the outer core?

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SUMMARY

The discovery that the Earth's inner core is rotating relative to the mantle has prompted a number of authors to reinvestigate the dynamics of inner core rotation. These models include a highly idealized analytical one by Aurnou, Brito & Olson (1996), as well as a fully 3-D numerical one by Glatzmaier & Roberts (1996). In this work I present a model intermediate between these two extremes. In particular, I retain the simplicity of the model of Aurnou *et al.* by kinematically prescribing a thermal wind and poloidal magnetic field. By doing so it is possible to vary the strengths of these quantities at will, and thereby explore the dependence of the inner core's rotation rate on them more thoroughly than in the model of Glatzmaier & Roberts, where these quantities emerge as part of the solution, and one therefore has far less control over their strengths. However, as in the model of Glatzmaier & Roberts, the full back-reaction of the magnetic field on the fluid flow in the outer core is included. It is found that if one includes this effect, the relationship between the inner core's rotation rate and the strength of the thermal wind is more complicated than that found by Aurnou *et al.*, who did not include it. As a result, while the observed rotation of the inner core certainly gives a rigorous lower bound on the maximum difference in angular velocity throughout the outer core, that maximum difference could be as much as an order of magnitude greater. Finally, it is also pointed out that, because of the particular nature of the torque balance that determines the inner core's rotation rate, it is difficult, if not impossible, to use that observed rate to obtain precise bounds on the magnetic field strength deep within the core.

Key words: Earth's core, Earth's magnetic field.

1 INTRODUCTION

The seismically inferred prograde rotation of the Earth's solid inner core, at the surprisingly rapid rate of several degrees per year (Song & Richards 1996; Su, Dziewonsky & Jeanloz 1996; Creager 1997; see also Song 1997 for a review), is a tremendously exciting development in core dynamics, because it is the first time we have been able to look deep down into the core. In contrast, direct observations of the Earth's magnetic field (Bloxham, Gubbins & Jackson 1989) can only be projected down to the core–mantle boundary at the very top of the core, and the resulting inferences about the fluid flow (Bloxham & Jackson 1991) are then also limited to the top of the core. However, it is conceivable that the field and flow deep within the core could be quite different; indeed, a number of geodynamo models (Glatzmaier & Roberts 1995a,b; Jones, Longbottom & Hollerbach 1995) seem to suggest that they are. It is therefore of some interest to consider the question what the observed rotation of the inner core can reveal about the field and flow deep within the core. By considering the torque balance that determines the inner core's rotation rate, and analysing the adjustment to it in a particularly simple model, I will attempt to address this question.

The equation governing the inner core's rotation rate is

$$C\,\frac{d\Omega_{\rm IC}}{dt}=\Gamma\,, \tag{1}$$

where C is its polar moment of inertia and Γ is the total axial torque acting upon it (Glatzmaier & Roberts 1996). The importance of inertia in this balance is difficult to assess. On the relatively long timescales on which the geodynamo evolves, it is certainly negligible (Gubbins 1981), but on the decadal timescales covered by the seismic data it is not. Since we do not yet know the extent to which the observed rotation is time-dependent, we do not know whether inertia is important or not. In this work I will consider the simplest possible model

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in which no time dependence at all is included, but see also Aurnou, Brito & Olson (1998) for a time-dependent model. With inertia thus neglected, (1) becomes simply

$$\Gamma=0\,, \tag{2}$$

stating that the total torque on the inner core must vanish. This total torque, in turn, consists of two terms, a viscous torque and a magnetic torque. [We are assuming here that the inner core is perfectly axisymmetric. If it is slightly asymmetric, topographic and gravitational torques can also play a role. See e.g. Buffett (1996, 1997) for a discussion of some of these effects.] Now, the viscous torque cannot be very large, because the viscosity of the fluid outer core is so small; even if one assumes a turbulent rather than molecular viscosity one still obtains an extremely small Ekman number. By considering the usual dynamics of the Ekman layer at the inner core boundary (ICB), one finds that the viscous torque cannot exceed $O(E^{1/2})$. That then implies that the magnetic torque, which nominally could be $O(1)$, must in fact also be $O(E^{1/2})$.

That is, the inner core's rotation rate is determined rather indirectly by the requirement that the magnetic torque acting on it must almost completely cancel itself. In the geodynamo simulation of Glatzmaier & Roberts (1996), hereafter referred to as GR96, this is accomplished by having the inner core dragged along at some suitably averaged value of the angular velocity of the fluid just above the ICB, what they termed the 'synchronous motor' mechanism. If this mechanism is generally applicable, it immediately suggests that the observed rotation of the inner core actually tells us quite a lot about the fluid flow in the outer core, namely that the average angular velocity of the fluid just above the ICB must also be several degrees per year. However, it also suggests that the inner core's rotation tells us almost nothing about the magnetic field in the outer core, because even if we could somehow invert the observed rotation rate to obtain the net magnetic torque, that torque would only be the tiny residual left over after most of the torque has cancelled itself, and therefore could not be used to infer the average field strength at the ICB.

Indeed, in a recent model of inner core rotation by Aurnou *et al.* (1996), hereafter referred to as ABO, the equilibrium rotation rate was always approximately 86 per cent of the imposed thermal wind, completely independent of the imposed field strength, again implying that in this particular model at least, one can infer everything about the flow but nothing about the field. (Actually, ABO did go on to infer something about the field, a point I will return to later.) The purpose of the model presented here, which in many ways complements and extends the model of ABO, is to explore this synchronous motor mechanism in more detail, and thereby further understand the potentially quite complicated relationship between the inner core's rotation rate and the field and flow just above the ICB.

2 EQUATIONS

The appropriately scaled equations governing the fluid flow $\mathbf{U}$ and the magnetic field $\mathbf{B}$ in the outer core are

$$2\hat{\mathbf{k}}\times\mathbf{U}=-\nabla p+E\nabla^2\mathbf{U}+(\nabla\times\mathbf{B})\times\mathbf{B}+\Theta\mathbf{r}\,, \tag{3}$$

$$\frac{\partial}{\partial t}\mathbf{B}=\nabla^2\mathbf{B}+\nabla\times(\mathbf{U}\times\mathbf{B})\,, \tag{4}$$

where Θ is the buoyancy. As in previous work exploring the influence of the inner core on the geodynamo (Hollerbach & Jones 1993b, 1995), this buoyancy will be kinematically prescribed, thereby driving a given thermal wind. We should also point out that we are neglecting the inertia of the fluid outer core in (3), just as we previously neglected the inertia of the solid inner core in (2). The restriction to relatively long timescales is the same in both cases.

Restricting attention to purely axisymmetric solutions, and using the non-divergence conditions $\nabla\cdot\mathbf{U}=\nabla\cdot\mathbf{B}=0$ to decompose as

$$\mathbf{U}=\nabla\times(\psi\hat{\mathbf{e}}_\phi)+v\hat{\mathbf{e}}_\phi\,, \tag{5a}$$

$$\mathbf{B}=\nabla\times(A\hat{\mathbf{e}}_\phi)+B\hat{\mathbf{e}}_\phi\,, \tag{5b}$$

the momentum equation (3) becomes

$$2\frac{\partial}{\partial z}\psi+ED^2v=-N(B,A)\,, \tag{6a}$$

$$2\frac{\partial}{\partial z}v-ED^4\psi=M(B,B)+M(D^2A,A)+\frac{\partial}{\partial\theta}\Theta\,, \tag{6b}$$

and the induction equation (4) becomes

$$\frac{\partial}{\partial t}A=D^2A+N(\psi,A), \tag{7a}$$

$$\frac{\partial}{\partial t}B=D^2B+M(v,A)-M(B,\psi)\,, \tag{7b}$$

where

$$D^2=\nabla^2-(r\sin\theta)^{-2}\,, \tag{8a}$$

$$M(X,\,Y)=\hat{\mathbf{e}}_\phi\cdot\nabla\times[X\hat{\mathbf{e}}_\phi\times\nabla\times(Y\hat{\mathbf{e}}_\phi)]\,, \tag{8b}$$

$$N(X,Y)=\hat{\mathbf{e}}_\phi\cdot[\nabla\times(X\hat{\mathbf{e}}_\phi)\times\nabla\times(Y\hat{\mathbf{e}}_\phi)]\,. \tag{8c}$$

Further expanding ψ and v in terms of associated Legendre functions,

$$\psi=\sum_n\psi_n(r)P^{(1)}_{2n}(\cos\theta)\,, \tag{9a}$$

$$v=\sum_n v_n(r)P^{(1)}_{2n-1}(\cos\theta)\,, \tag{9b}$$

the boundary conditions associated with (6) become

$$\psi_n=\frac{\partial}{\partial r}\psi_n=0\,,\qquad\text{for all }n\,, \tag{10a}$$

$$v_n=0\,,\qquad\text{for all }n\,, \tag{10b}$$

at the outer boundary, and

$$\psi_n=\frac{\partial}{\partial r}\psi_n=0\,,\qquad\text{for all }n\,, \tag{11a}$$

$$v_n=0\,,\qquad\text{for }n\neq1\,, \tag{11b}$$

$$v_n=-\Omega_{\mathrm{IC}}r_i\,,\qquad\text{for }n=1\,, \tag{11c}$$

at the inner boundary.

These boundary conditions represent matching to $\mathbf{U}=0$ at the CMB and to $\mathbf{U}=\Omega_{\mathrm{IC}}\,r_i\sin\theta\,\hat{\mathbf{e}}_\phi$ at the ICB, where Ω_{IC} is the rotation rate of the inner core, and is to be determined as part of the solution. We therefore need one additional equation to determine Ω_{IC}, which once again is precisely the constraint (2) that the sum of the viscous and magnetic torques on the inner

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core must vanish, yielding

$$\frac{4}{3}\,Er\,\frac{\partial}{\partial r}\left(\frac{v_1}{r}\right)\bigg|_{r=r_i}=\int_0^{\pi} B_\phi B_r\bigg|_{r=r_i}\sin^2\theta\,d\theta\,, \tag{12}$$

as in Hollerbach & Jones (1993a). It is at this stage that we obtain the result mentioned above, that the magnetic torque on the inner core must vanish with vanishing E. The reason it must only vanish as $O(E^{1/2})$, and not as $O(E)$, as one might naively expect from looking at (12), is that the thickness of the Ekman layer at the ICB scales as $E^{1/2}$, so the radial derivative in (12) contributes an additional $E^{-1/2}$, so the viscous torque balancing the magnetic torque can be as large as $O(E^{1/2})$. Finally, it is also worth mentioning that in practice (12) is used as the boundary condition on v_1, and (11c) then determines $\Omega_{\rm IC}$, even though originally (11c) came from the boundary condition on v_1 and (12) came from the equation for $\Omega_{\rm IC}$. That these two equations have interchanged roles in this way is merely a reflection of the fact that this torque balance (2) determines $\Omega_{\rm IC}$ in a very roundabout manner.

Similarly expanding A and B as [note incidentally that both here and in (9) we are imposing a particular equatorial symmetry, corresponding to pure dipole solutions]

$$A=\sum_n A_n(r)P^{(1)}_{2n-1}(\cos\theta)\,, \tag{13a}$$

$$B=\sum_n B_n(r)P^{(1)}_{2n}(\cos\theta)\,, \tag{13b}$$

the boundary conditions associated with (7) become

$$\frac{\partial}{\partial r}A_n+\frac{2n}{r}A_n=B_0 f(n)\,, \tag{14a}$$

$$B_n+\epsilon\frac{\partial}{\partial r}B_n=0 \tag{14b}$$

at the outer boundary, and

$$\frac{\partial}{\partial r}A_n-\frac{2n-1}{r}A_n=0\,, \tag{15a}$$

$$\frac{\partial}{\partial r}B_n-\frac{2n}{r}B_n=0 \tag{15b}$$

at the inner boundary, for all n in both cases.

The boundary condition (14a) matches the poloidal field to an external potential field. The inhomogeneous term $B_0 f(n)$, where we will take

$$f(1)=-1.5\,,\qquad f(2)=0.525\,, \tag{16}$$

and $f(n)=0$ for all other n, represents an externally imposed potential field, with the parameter B_0 measuring the strength of that field (that is, B_0^2 is the Elsasser number measuring the poloidal field strength). We are thus not maintaining the poloidal field by the usual α-effect [see e.g. Hollerbach (1996a) for a discussion of this effect], but by this externally imposed field. This will turn out to have two advantages: first, it allows us much greater control over the strength of the field; and second, it allows us to make a direct comparison with ABO, who also imposed an external field.

The boundary condition (14b) matches the toroidal field to a weakly conducting mantle, with the parameter ϵ representing the total conductance of a thin conducting layer at the base of the mantle (Love & Bloxham 1994). In fact, varying ϵ in the range from 0 (perfectly insulating) to 0.01 (already too strongly conducting) had practically no effect on the results described below, indicating that in this model at least, it makes no difference whether the torque balance at the CMB is dominated by viscous or magnetic coupling.

Finally, the boundary conditions (15) match the field to a *steady-state* field in the finitely conducting inner core. If one really wanted to follow the detailed evolution of the field, one would have to solve for the field in the inner core as well, as in Hollerbach & Jones (1993a,b, 1995). However, if one is only interested in the final steady-state equilibration, as we will be in this work, one can simply impose the boundary conditions (15) instead.

One last, perhaps somewhat technical, but nonetheless important point to make about these equations is the level of truncation used in the numerical solution, and in particular how it varies with Ekman number. On the one hand one would naturally like to reduce E as much as possible to see this adjustment to the constraint of vanishing magnetic torque,

$$\int_0^{\pi} B_\phi B_r\big|_{r=r_i}\sin^2\theta\,d\theta=O(E^{1/2})\,, \tag{17}$$

as clearly as possible. On the other hand, reducing E inevitably requires increasing the radial truncation in order to resolve the increasingly thin Ekman layers. In general, reducing E also requires increasing the angular truncation in order to resolve structures such as Stewartson layers [which are eventually suppressed for sufficiently strong magnetic fields (Hollerbach 1994, 1996b; Kleeorin *et al.* 1997), but which nevertheless do seem to cause problems if one does not increase the angular truncation as well]. The result is that one is hard pressed to reduce E much beyond 10^{-5}, which unfortunately is often not enough to see the emergence of a clear asymptotic limit.

Glatzmaier & Roberts (1995a,b, 1996) have attempted to get around this limitation by introducing a so-called hyperviscosity, in which the viscosity depends on the spherical harmonic degree, increasing with increasing degree. Because the higher modes are thus more strongly damped, one does not need to increase the angular truncation as one reduces E (although one still needs to increase the radial truncation), so one can reduce E considerably further. In this work we also employ a hyperviscosity, of the form

$$E(l)=E_0\,1.25^{l-1}\,, \tag{18a}$$

where l is the spherical harmonic degree [that is, according to (9), $l=2n$ for ψ_n and $l=2n-1$ for v_n]. This form is in fact (considerably) less than the Glatzmaier & Roberts form,

$$E(l)=E_0(1+0.075l^3)\,, \tag{18b}$$

for $l\leq 36$, but thereafter rises quite sharply. The result is that one can reduce E_0 (when we refer to E in subsequent sections, we really mean E_0) down to 10^{-7}, and at a truncation of only 24 angular modes (for each of ψ and v; that is, up to $l=48$) times 240 radial modes [as in Hollerbach & Jones (1993a), the radial structure is expanded in Chebyshev polynomials].

There is, of course, the concern that by using hyperviscosities of this form, one will still not see the emergence of a clear asymptotic limit, because the higher, relatively strongly damped modes are contributing significantly to various balances. For example, Sarson, Jones & Longbottom (1998) have shown that the adjustment to Taylor's (1963) constraint does not seem to improve when one introduces a hyperviscosity, presumably because these higher modes, for which viscosity is still important, are contributing sufficiently

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to disrupt the adjustment to a true Taylor state, where viscosity should be completely unimportant. Indeed, we will also find that we never obtain a state that is completely independent of E_0, presumably for the same reason. Nevertheless, we will be able to show some of the effects that the adjustment to Taylor's constraint has on the outer core flow, and we show that the inclusion of these dynamics is the single biggest difference between this model and that of ABO.

Finally, it is also worth pointing out that, as troubling as the use of a hyperviscosity potentially is (see also Zhang & Jones 1997), one can be quite certain that it will not disrupt this torque balance on the inner core. The reason is that, according to (12), the *only* contribution to the integrated viscous torque comes from the lowest mode v_1, which really does see just the Ekman number E_0. That is, the E in (17) really is just E_0, and not some (substantially greater) effective average E. Indeed, we will see that by reducing E_0 down to 10^{-7} we obtain the asymptotic limit (17) quite clearly.

3 RESULTS

We begin by prescribing the buoyancy gradient

$$\frac{\partial}{\partial\theta}\,\Theta = -2\Theta_0 r \sin\theta\cos\theta\exp(-1.6s^6)\,, \tag{19}$$

where $s \equiv r\sin\theta/r_i$ is the scaled cylindrical radius. According to (6b), in the limit of small E this will drive a thermal wind

$$\frac{\partial}{\partial z}\,v_{\rm T} \approx \frac{1}{2}\frac{\partial}{\partial\theta}\,\Theta\,. \tag{20}$$

Fig. 1 shows the induced thermal wind at the finite value $E = 10^{-6}$, and it is indeed very similar to the asymptotic form (20). It is not identical, of course, because (20) only holds exactly for infinitesimal E, but $E = 10^{-6}$ is already sufficiently small that the solution for $E = 10^{-7}$ looks essentially the same (as shown in Fig. 6 below). We thus see that by prescribing a buoyancy we have prescribed a thermal wind, and this thermal wind is essentially independent of E, provided $E \lesssim 10^{-6}$.

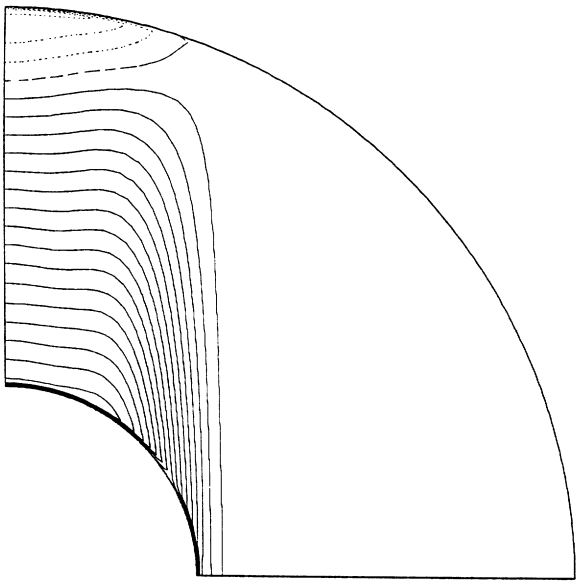

Figure 1. The prescribed thermal wind $\Omega_{\rm T}$. Solid contours represent eastward flow relative to the mantle.

If we now compare this thermal wind with that of ABO, we note first that both are confined within the inner core tangent cylinder. ABO's thermal wind is in fact discontinuous across the tangent cylinder, jumping from $\partial v_{\rm T}/\partial z = c$ inside to $\partial v_{\rm T}/\partial z = 0$ outside. In their analytical model such a discontinuity causes no great difficulty, but in our numerical model it would, so we smooth this transition out somewhat by having the factor $\exp(-1.6s^6)$ in (19), rather than a step function.

A more significant difference is probably that their thermal wind is everywhere eastwards, whereas ours is eastwards only deep within the core, but westwards at the top of the core. From a practical, computational point of view, the reason this comes about is that we do not get to specify $v_{\rm T}$, only $\partial v_{\rm T}/\partial z$, with the constant of integration in (20) being determined by the details of the numerical solution. So, having specified a form for $\partial\Theta/\partial\theta$, as in (19), we simply have to accept whatever constant of integration comes out of the numerical solution of (6). However, from a geophysical point of view, it is probably just as well that $\Omega_{\rm T}$ should come out this way, eastwards at the ICB but westwards at the CMB; the dynamically determined thermal wind of GR96 has this same property. [Fig. 2(b) of GR96 shows the total zonal flow, but according to Glatzmaier (personal communication, 1997) just the thermally driven part of it would look similar.]

Turning to the effect that the thermal wind has on the inner core's rotation rate, we note that in the ABO model the inner core is swept along at precisely 100 per cent of the maximum difference in angular velocity throughout the outer core; that is, the angular velocity is everywhere eastwards, and the inner core's angular velocity is maximally eastwards. In contrast, in this model the inner core is still swept along in an eastward direction, but because of this feature that the thermal wind is westwards at the top of the core, at only approximately 50 per cent of the maximum difference in angular velocity $\Delta\Omega_{\rm T} \equiv \Theta_0$ throughout the outer core. In both cases we are then interested in the effect that a magnetic field will have on this inner core rotation rate.

Fig. 2 shows the imposed poloidal field, which once again is imposed via the inhomogeneous boundary conditions (14a), with $f(n)$ given by (16). Taking only $f(1) = -1.5$ and $f(n) = 0$ for all other n would yield a purely axial field of strength B_0, which is precisely what ABO imposed. However, it is known (Fearn & Proctor 1992) that a purely axial field is a highly degenerate special case when one considers the adjustment to Taylor's constraint in the outer core. Since we noted above that the inclusion of these Taylor's constraint dynamics is the single biggest difference between this model and that of ABO, we do not want to restrict ourselves to such a special case. It is by taking $f(2) = 0.525$ in addition to $f(1) = -1.5$ that we add some curvature to our imposed field, and thereby avoid this degeneracy. In all other respects, though, our imposed field is very similar to ABO's, and so we should expect similar adjustments to occur. (It is important to emphasize, though, that neither ABO's nor this field are intended to be accurate representations of the true geomagnetic field, and since all subsequent results depend to a certain extent on the form of this imposed field, they should be interpreted as illustrations of what could happen rather than as predictions of what will happen.)

That some type of adjustment must occur for sufficiently large B_0 is evident when one considers Fig. 3, which shows

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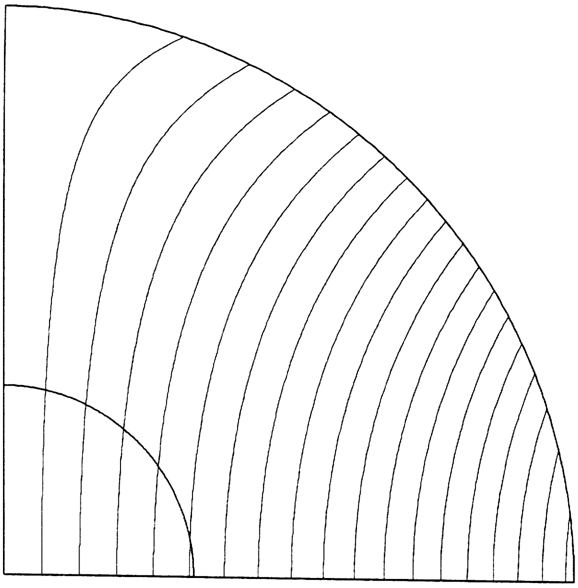

Figure 2. The prescribed poloidal field A. The lines of force are clockwise-directed.

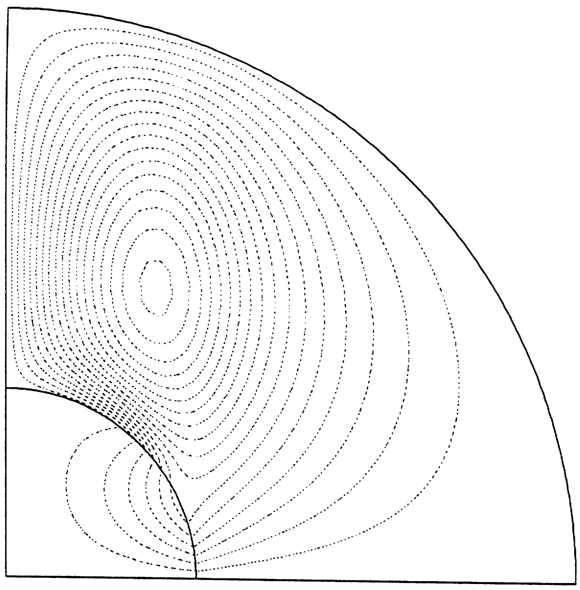

Figure 3. The toroidal field induced by the action of the prescribed thermal wind on the prescribed poloidal field, with the dashed contours representing the westward-directed field.

the induced toroidal field. That is, now that we have a given poloidal field (Fig. 2) and a given thermal wind (Fig. 1), the thermal wind will draw out the poloidal field to produce a toroidal field via the ω-effect, the term $M(v, A)$ in (7b). It is seen in Fig. 3 that B_ϕ is everywhere negative (westwards) and in Fig. 2 that B_r is everywhere positive (outwards). Therefore, far from cancelling itself, the magnetic torque on the inner core is everywhere of the same negative sign. The magnetic torque is thus of order B_0^2, and so it is clear that as soon as B_0^2 exceeds $O(E^{1/2})$ the magnetic torque will overwhelm the viscous torque, and will force the inner core to rotate slower than the thermal wind alone would make it do. This suppression of the rotation will continue until it brings about a state where the magnetic torque is beginning to cancel itself, so that it can once again be balanced by the viscous torque.

Fig. 4 shows this adjustment in the inner core's rotation rate as one gradually increases B_0^2, for E ranging from 10^{-5} to 10^{-7}, and for the three values $\Delta\Omega_T = 10$, 100, 1000. The units of $\Delta\Omega_T$—radians per magnetic diffusion time—work out so that 1000 corresponds to approximately one degree per year. The largest value of $\Delta\Omega_T$ is thus in the geophysically realistic range. One notices first that for sufficiently small B_0^2, Ω_{IC} is indeed approximately 50 per cent of $\Delta\Omega_T$. (The reason there is still some slight variation with E in this range is that Ω_T still varies slightly with E, as noted above.) However, as one then increases B_0^2 beyond $O(E^{1/2})$, Ω_{IC} begins to decrease. Note how this adjustment begins considerably earlier for $E = 10^{-7}$ than for $E = 10^{-5}$. That is, as soon as B_0^2 exceeds $O(E^{1/2})$, the (initially uniformly negative) magnetic torque is indeed suppressing the inner core's rotation from that which would have been caused by the thermal wind alone. However, in sharp contrast with the model of ABO, in which this suppression of the rotation amounted to a fixed 14 per cent, independent of any further increase in B_0^2, here we find that it continues, until for sufficiently large B_0^2 it exceeds 100 per cent; that is, the inner core is rotating in the *opposite* direction to that which would result from the thermal wind. Although this counter-rotation only seems to occur for very large B_0^2 ($B_0^2 \approx 10^{-1}$ is geophysically reasonable), we clearly would like some sort of explanation for this very counterintuitive phenomenon. (See also Gubbins 1981 for a similar phenomenon.)

First, however, we will consider the cancellation that occurs in the magnetic torque as a consequence of this adjustment in the inner core's rotation rate. Fig. 5 shows this adjustment in $\Gamma_{\mathbf{B}}$, for the same parameter values as in Fig. 4. It is seen that for sufficiently small B_0^2, $\Gamma_{\mathbf{B}}$ is proportional to $\Delta\Omega_T\, B_0^2$. However, as one increases B_0^2 beyond $O(E^{1/2})$, $\Gamma_{\mathbf{B}}$ does indeed

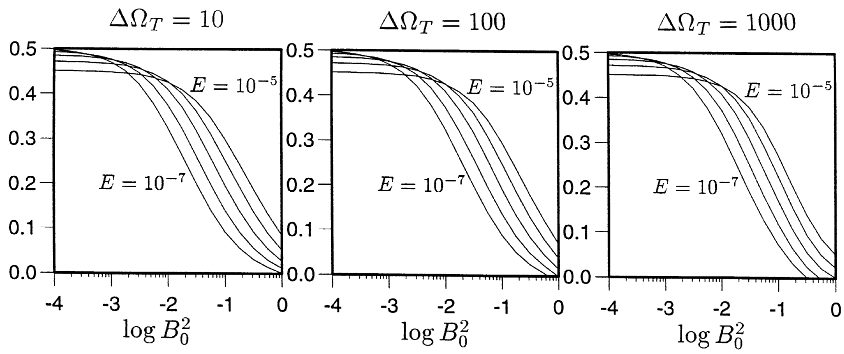

Figure 4. $\Omega_{IC}/\Delta\Omega_T$ as a function of $\log B_0^2$, for E from 10^{-5} to 10^{-7}, and for $\Delta\Omega_T = 10$, 100, 1000.

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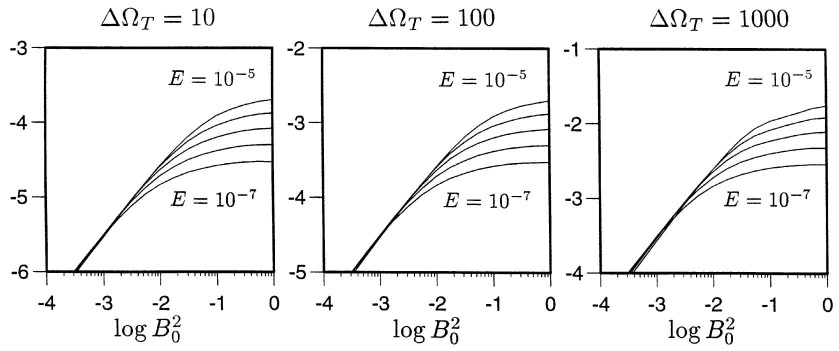

Figure 5. $\text{Log}|\Gamma_{\mathbf{B}}|$ as a function of $\log B_0^2$, for E from 10^{-5} to 10^{-7}, and for $\Delta\Omega_{\rm T} = 10$, 100, 1000.

ultimately level off at a value proportional to $\Delta\Omega_{\rm T}\, E^{1/2}$, as required by (17). (That both the magnetic and the viscous torques should also be proportional to $\Delta\Omega_{\rm T}$ is hardly surprising: a stronger thermal wind can certainly exert a proportionately greater viscous torque, and a stronger thermal wind will also induce a proportionately greater toroidal field, and hence magnetic torque.)

One might note, incidentally, that whereas the torque in Fig. 5 is indeed approaching a sensible asymptotic limit as E tends to zero, the inner core rotation in Fig. 4 is not becoming independent of E as it ought to. As pointed out earlier, the reason for this lack of convergence is presumably due to the use of the hyperviscosity disrupting the adjustment to a true Taylor state. For sufficiently small E one should eventually obtain an inner core rotation independent of E, even with a hyperviscosity; the fact that $E = 10^{-7}$ is still not sufficiently small is thus a reminder that one must be careful in interpreting not just the results of this model, but also those of the more complicated models (Glatzmaier & Roberts 1995a,b, 1996; Kuang & Bloxham 1997) that operate at even greater Ekman numbers.

With this cautionary note in mind, we wish to consider the detailed structure of the flow and the field, to see why the inner core's rotation is suppressed so strongly, and also how this levelling off of the magnetic torque comes about. Fig. 6 shows contour plots of the total angular velocity Ω, just the magnetically driven part $\Omega - \Omega_{\rm T}$, and the toroidal field B for $E = 10^{-7}$, $\Delta\Omega_{\rm T} = 1000$ and $B_0^2 = 10^{-3}$, 10^{-2}, 10^{-1}, 10^0. The meridional circulation ψ and the poloidal field A are not shown, ψ because it never appears to play much of a role in these results, and A because it never changes very much from the imposed field shown in Fig. 2. Note that according to (7a) A is only ever affected by ψ, so the relative insignificance of ψ and the relative constancy of A do indeed go hand in hand. Of course, in the real geodynamo the poloidal field is also free to adjust to a much greater extent than allowed here, with the probable result that some of the other adjustments we have obtained would not have to be nearly so substantial to still achieve the necessary torque balance.

Turning first to the toroidal field, we immediately see how the levelling off of $\Gamma_{\mathbf{B}}$ comes about: as B_0^2 increases we get a region of oppositely directed flux near the equator, so we do indeed have cancellation between strongly negative and positive contributions. In contrast, in the model of ABO the toroidal field is simply expelled from the inner core entirely, so the magnetic torque vanishes not only when integrated, but also point-wise. There is thus never any cancellation between oppositely signed contributions. It seems unlikely that this expulsion of the toroidal field would be the generic way of satisfying the requirement that $\Gamma_{\mathbf{B}}$ tends to zero, since it is so much more restrictive than it needs to be, but it is worth noting that the earlier numerical models of Hollerbach & Jones (1993a,b, 1995) also showed such an expulsion of B from the inner core. It is thus not just an artefact of the highly idealized analytic model of ABO. Of course, it is quite possible that in the real geodynamo both effects are at work: $\Gamma_{\mathbf{B}}$ tending to zero may well come about partly by this expulsion of B, which already reduces the torque point-wise, and partly by this cancellation of positive and negative contributions, which reduces it even further when integrated. Finally, it is also worth pointing out that in fully 3-D models the integration is not only over θ, as in (12), but over ϕ as well, with further scope for cancellation as a result. For example, in the model of Kuang & Bloxham (1997) the quadratic interactions of the non-axisymmetric fields play a vital role in this cancellation (Bloxham, personal communication, 1998).

Turning next to the angular velocity, if we consider first the total Ω we can see the synchronous motor mechanism of GR96 in operation; it is particularly clear at $B_0^2 = 1$ how the inner core is rotating at some averaged value of the fluid just above it. To understand why, even with this synchronous motor mechanism in operation, the rotation rates of both the inner core and correspondingly the fluid just above it are nevertheless increasingly suppressed, it helps if we consider instead only the magnetically driven part $\Omega - \Omega_{\rm T}$. From the very strong vertical alignment of the contour lines, we see immediately that the magnetically induced flow is predominantly a z-independent or geostrophic flow. Such a flow is exactly what we would expect to obtain while the field is in the process of adjusting to Taylor's constraint in the outer core [see e.g. Hollerbach (1996a) for a discussion of the relationship between Taylor's constraint and the geostrophic flow]. This strong geostrophic flow then simply carries the inner core along with it without being particularly influenced by the inner core torque balance. That is, the inner core torque balance determines the rotation of the inner core relative to the fluid immediately above the ICB, but that fluid itself may be rotating relative to the CMB, and if that rotation depends in some very complicated way on the various parameters in the problem, then so will the rotation of the inner core relative to the CMB.

The presence of this geostrophic flow thus constitutes the single biggest difference between this model and that of ABO. In this model the fluid flow in the outer core also adjusts quite

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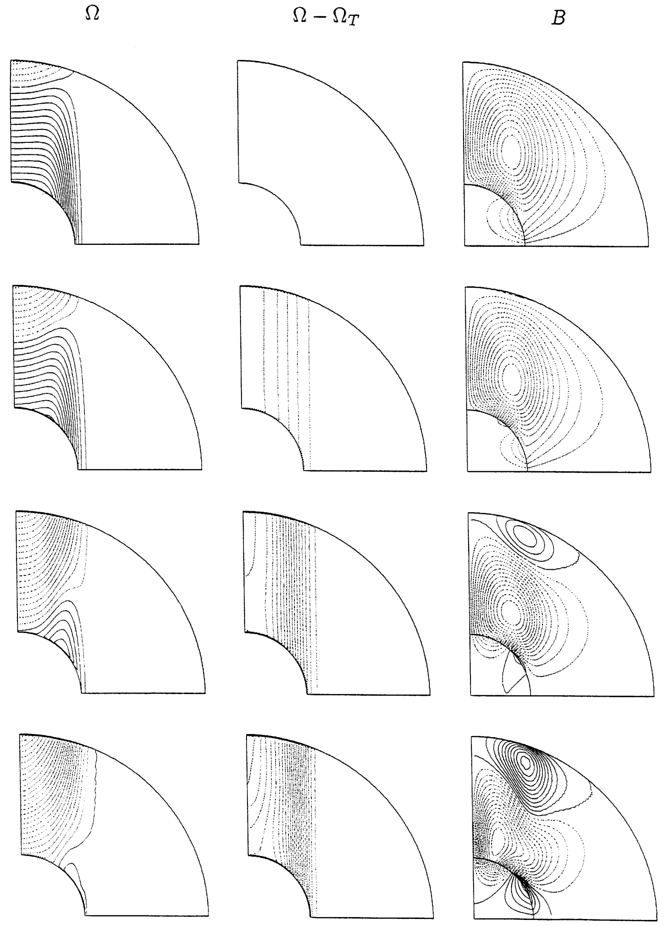

Figure 6. Contour plots of the total angular velocity Ω, the magnetically driven part $\Omega - \Omega_T$, and the toroidal field B, for $E = 10^{-7}$, $\Delta\Omega_T = 1000$, and from top to bottom $B_0^2 = 10^{-3}, 10^{-2}, 10^{-1}, 10^0$. Note how similar the (essentially purely kinematic) Ω at $B_0^2 = 10^{-3}$ is to the thermal wind for $E = 10^{-6}$ shown in Fig. 1, thus verifying that Ω_T is essentially independent of E for $E \lesssim 10^{-6}$.

strongly; in theirs it does not. Indeed, if we examine Taylor's constraint, which may be written as

$$\frac{1}{s^2}\frac{d}{ds}\left[s^2 \int_{z_B}^{z_T} B_\phi B_s \, dz\right] + \frac{1}{\cos\theta} B_\phi B_r \Big|_{z_B}^{z_T} = 0 \qquad (21)$$

(see e.g. Fearn & Proctor 1992), we see that in their model it is satisfied identically; the integral vanishes because of the purely axial imposed field, so $B_s = 0$ everywhere, and the boundary terms vanish because $B_\phi = 0$ everywhere on the boundaries, at the CMB because that is the boundary condition at the insulating mantle, and at the ICB because of the point-wise vanishing of the magnetic torque on the inner core. We thus see that the dynamics of Taylor's constraint are highly non-generic in the model of ABO, with no adjustment at all required in the outer core fluid flow. In general, some sort of adjustment will almost certainly be required, which, as indicated above, will then also affect the rotation of the inner core relative to the mantle.

Indeed, even after this adjustment to Taylor's constraint has taken place, once $B_0^2 \approx O(1)$, the ageostrophic magnetic wind will also come into play. The adjustment to Taylor's constraint should in fact occur as soon as B_0^2 exceeds $O(E^{1/2})$, just like the adjustment to the constraint (17). In principle it should thus be possible to separate the effects of the geostrophic flow induced by the adjustment to Taylor's constraint, and the ageostrophic flow induced by the field even after the adjustment to Taylor's constraint has taken place. However, because Taylor's constraint involves the whole of the fluid outer core, which sees the whole range of effective Ekman numbers (18a), we inevitably have the adjustment to Taylor's constraint also spread out over the entire range $B_0^2 \gtrsim O(E^{1/2})$ right up to $B_0^2 \approx O(1)$, and are thus not able to separate clearly when the two effects occur. Nevertheless, in Fig. 6 for $B_0^2 \geq 10^{-1}$ there is indeed an increasingly strong magnetic wind, as evidenced by the increasingly strong departure from purely vertically aligned contours in $\Omega - \Omega_T$. This magnetic wind was also not included in the model of ABO, but in general it too will affect the rotation of the inner core relative to the mantle.

4 DISCUSSION

In this work we have considered the general question what determines the inner core's rotation rate, both with respect to the fluid just above the ICB, and with respect to the mantle. We have noted once again that it is the need to satisfy the constraint of vanishing magnetic torque on the inner core that determines its rotation rate relative to the fluid just above the ICB, and that this constraint is satisfied by having the inner core rotate at some suitably averaged value of the fluid just

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above the ICB, in perfect agreement with the synchronous motor mechanism of GR96. So, returning to the question we originally set out to address, namely what the observed rotation of the inner core actually tells us, we can conclude that it does indeed tell us what the average angular velocity of the fluid just above the ICB is. However, we should be careful not to conclude from this that the maximum difference in angular velocity throughout the outer core will then also be a few degrees per year; it will of course be at least this much, but it could be considerably greater. In the model of ABO, the relationship $\Omega_{\rm IC} \approx 0.86\Delta\Omega_{\rm T}$ was such that the maximum difference in angular velocity was indeed always comparable to $\Omega_{\rm IC}$, but in this model we found that at the geophysically reasonable value $B_0^2 \approx 10^{-1}$ the maximum difference in angular velocity can be as much as 10 times greater than $\Omega_{\rm IC}$. In Fig. 2(b) of GR96 this maximum difference is also about four times $\Omega_{\rm IC}$. The observed rotation of the inner core thus gives us a rigorous lower bound on the difference in angular velocity throughout the outer core, but it could be an order of magnitude greater.

Turning next to the question what the rotation of the inner core can tell us about the strength of the field within the core, ABO argued that because the strength of the poloidal field is in fact known, and because in their model the strength of the differential rotation is also known, one can then deduce the strength of the toroidal field simply by considering the effect of the given differential rotation on the given poloidal field. While it is true that this will give a reasonable estimate for the toroidal field strength, it is important to emphasize that this estimate is not nearly as rigorous as the above bound on the angular velocity. The difficulty is not so much that the angular velocity is only bounded below (that, after all, would merely mean that the toroidal field is also only bounded below); rather, it is that the poloidal field strength is not in fact so well known. That is, the poloidal field that is known is the field at the CMB, but the field that is relevant for these estimates is the field at the ICB, and the two could be quite different. In Fig. 3(b) of GR96, for example, it is considerably stronger at the ICB than at the CMB, due to the presence of field lines closed entirely within the core. If one could be certain that the field will always be at least as strong at the ICB as at the CMB, one could still obtain a rigorous lower bound on the toroidal field strength, which could then be combined with upper bounds derived from a consideration of magnetic instabilities (Zhang & Fearn 1993) to produce a very tight estimate. Unfortunately, however, one cannot be certain that the poloidal field will always be at least as strong at the ICB as at the CMB; Fig. 4 of Glatzmaier & Roberts (1995a), for example, shows a solution where the field at the CMB is very similar to that of GR96, but the field at the ICB is completely different, being far weaker, and even in the opposite direction (relative to that at the CMB). The relationship between the poloidal field strengths at the CMB and the ICB is thus not known for certain, so these estimates of the toroidal field strength are just estimates rather than rigorous bounds.

Finally, there is one effect that, if it should turn out to be seismically detectable, would allow one to obtain a better estimate of the poloidal and hence also the toroidal field strengths at the ICB. It is highly likely that the inner core does not rotate at a uniform rate, but rather undergoes (potentially quite substantial) torsional oscillations superimposed on some average rotation rate. [See also Zatman & Bloxham (1997) for a discussion of torsional oscillations in the outer core.] If one could detect these oscillations, and in particular their typical timescales, it would reveal a great deal about the field strengths at the ICB. For example, GR96 report a numerical experiment in which they artificially induce such an oscillation (by instantaneously setting $\Omega_{\rm IC}$ to zero and observing the response) and find its natural timescale to be about 2 yr. One can be quite certain that if their fields at the ICB had been stronger, this timescale would have been shorter, even though the average rotation rate might well have been the same (see e.g. Fig. 12c of Aurnou *et al.* 1998). The reason is that whereas the average rate only sees the tiny residual left over after most of the magnetic torque has cancelled itself, and is thus largely insensitive to the precise field strength, the instantaneous rate sees the temporary departures from this state of almost perfect cancellation, and is thus considerably more sensitive to the field strength. We can conclude, therefore, that if we had reliable information about the timescales of these fluctuations about the average rotation rate, we could infer more about the field strengths at the ICB than we can knowing only the average rotation rate. As the seismic detection of the inner core's rotation rate improves and (one hopes) eventually begins to provide us with this further information, we should then be able to develop more sophisticated models of inner core rotation that make use of this information. We can thus look forward to exciting new insights into the state of the field and flow deep within the core.

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