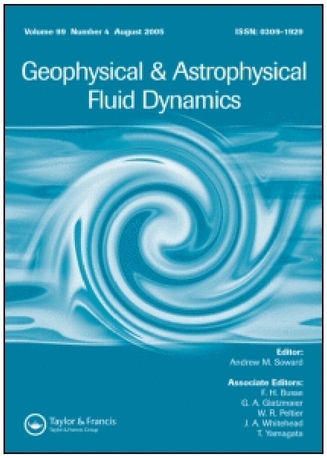

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The dynamical balance in semi-Taylor states

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THE DYNAMICAL BALANCE IN SEMI-TAYLOR STATES

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Anufriev *et al.* (1995) have recently presented a mean field geodynamo model in which the magnetic field appears to scale as $\mathbf{B} = O(E^{1/8})$, where the Ekman number E is a measure of viscous to Coriolis forces. They called these new solutions semi-Taylor states, as they seem to be intermediate between Ekman states having $\mathbf{B} = O(E^{1/4})$ and true Taylor states having $\mathbf{B} = O(1)$, both familiar in the literature. I present a balance which will yield precisely such an $O(E^{1/8})$ scaling, and in the process demonstrate that this is indeed the only intermediate state one may have. I suggest how one might reconsider this particular solution of Anufriev *et al.*, in light of this balance. Finally, I discuss the possible geophysical implications of these new solutions.

Keywords: Earth's core; Taylor's constraint

1. INTRODUCTION

The Earth's magnetic field is created by the 'dynamo action' of fluid motions in its liquid iron outer core. It is generally recognized that rotation plays a crucial role in this process. It is the influence of the Earth's rotation that leads to dynamically distinct Ekman and Taylor states. In this note I derive a balance which yields an intermediate state, having features in common with both Ekman and Taylor states. Such a semi-Taylor state, as it has been called, has recently been obtained numerically by Anufriev *et al.* (1995).

The appropriately scaled mean field geodynamo equations for the magnetic field $\mathbf{B}$ and the fluid flow $\mathbf{U}$ are,

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla^2\mathbf{B} + \nabla\times(\alpha\mathbf{B}) + \nabla\times(\mathbf{U}\times\mathbf{B}), \tag{1}$$

$$2\hat{\mathbf{k}}\times\mathbf{U} = -\nabla p + E\nabla^2\mathbf{U} + (\nabla\times\mathbf{B})\times\mathbf{B} + \Theta\,\mathbf{r}, \tag{2}$$

where α is a parameterization of the small-scale turbulent flow, and Θ is the buoyancy, which then drives a differential rotation ω. In these mean field models α and Θ (or equivalently α and ω) are kinematically prescribed, and one simply follows the evolution of $\mathbf{B}$ and $\mathbf{U}$. See, for example, one of the reviews by Soward (1991), Fearn (1995), or Hollerbach (1996) for a fuller discussion of mean field theory.

The parameter $E(=\nu/\Omega L^2)$ in (2) is known as the Ekman number, and is a measure of viscous to Coriolis forces. In the Earth's core the Ekman number is extremely small, perhaps $O(10^{-12})$, indicating the powerful influence of rotation. Although one might be tempted, in view of this small value, to set $E\equiv 0$, one may not be able to do so, for reasons that will become clearer in a moment. Instead, an asymptotic analysis of (2) in the limit $E\to 0$ yields the solution

$$\mathbf{U} = v_g(s)\hat{\mathbf{e}}_\phi + \mathbf{U}_\mathrm{M} + \mathbf{U}_\mathrm{T}, \tag{3}$$

where the purely zonal geostrophic flow

$$v_g(s) = E^{-1/2}\frac{(1-s^2)^{1/4}}{4\pi s}\int_{C(s)}\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right]_\phi dS, \tag{4}$$

and these contours of integration $C(s)$ consist of concentric cylindrical shells parallel to the axis of rotation. The ageostrophic components of the flow are

$$\mathbf{U}_\mathrm{M} = -\frac{1}{2}\int^z \nabla\times\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right]dz', \tag{5}$$

$$\mathbf{U}_\mathrm{T} = -\frac{1}{2}\int^z \nabla\times(\Theta\mathbf{r})dz'. \tag{6}$$

Again, the details of this asymptotic analysis are discussed more fully in any review of geodynamo theory.

We see then that according to (4), in the limit $E\to 0$ the geostrophic flow $v_g(s)$ will diverge, unless Taylor's (1963) constraint is satisfied, at least to within $O(E^{1/2})$, on each such cylindrical shell $C(s)$,

$$\int_{C(s)} [(\nabla\times\mathbf{B})\times\mathbf{B}]_\phi dS = O(E^{1/2}). \tag{7}$$

It is at this stage that we obtain the classical distinction between Ekman states and Taylor states (see, for example, Jones 1991). In an Ekman state (7) is trivially satisfied by having $\mathbf{B} = O(E^{1/4})$, whereas in a Taylor state (7) is less trivially satisfied by having $\mathbf{B} = O(1)$, but having sufficient internal cancellation in the integration over $C(s)$. Thus, in the Ekman regime the nonlinear equilibration of (1) is entirely through the geostrophic flow (4), whereas in the Taylor regime it is primarily through the ageostrophic flow (5), with the geostrophic flow merely enforcing Taylor's constraint.

This distinction between Ekman and Taylor states has traditionally been considered to be quite precise, and the transition from one to the other has been considered in some detail (Malkus and Proctor, 1975; Hollerbach and Ierley, 1991). However, as indicated above, Anufriev *et al.* (1995) have obtained a numerical solution for which "the magnetic field decreases very weakly as $O(\nu^{1/8})$ when $\nu \to 0$," where the viscosity ν is of course just the Ekman number E in the notation here. Now this is really a very peculiar scaling. On the one hand, it is sufficiently small that one would think the ageostrophic nonlinearity should be negligible, as it is in the Ekman regime. On the other hand, if one does neglect it, and includes only the geostrophic nonlinearity, one obtains only the Ekman regime scaling. The purpose of this note therefore is to demonstrate how one may obtain a $\mathbf{B} = O(E^{1/8})$ scaling from these equations.

2. ASYMPTOTIC SCALINGS

We begin by noting that, having expanded the momentum equation (2) as in (3)–(6), the induction equation (1) takes the schematic form

$$\frac{\partial}{\partial t} Y_i = L_{ij} Y_j + \varepsilon^{-1} C^{(g)}_{ijkl} Y_j Y_k Y_l + C^{(a)}_{ijkl} Y_j Y_k Y_l, \tag{8}$$

where $\varepsilon \equiv E^{1/2}$. The linear term $L_{ij} Y_j$ comes from the terms $\nabla^2\mathbf{B}$, $\nabla\times(\alpha\mathbf{B})$ and $\nabla\times(\mathbf{U}_\mathrm{T}\times\mathbf{B})$, where once again α and $\mathbf{U}_\mathrm{T}$ are kinematically prescribed. The cubically nonlinear terms $\varepsilon^{-1}C^{(g)}_{ijkl}Y_jY_kY_l$ and $C^{(a)}_{ijkl}Y_jY_kY_l$ come from the geostrophic flow (4) and the ageostrophic flow (5), when these are substituted back into the term $\nabla\times(\mathbf{U}\times\mathbf{B})$. That the governing equations take this form (8) is most easily seen in a Galerkin model such as that of Hollerbach and Ierley (1991), where the Y_i are just the poloidal and toroidal expansion coefficients, but it is equally true in a finite-difference model such as that of Anufriev *et al.* (1995), where the Y_i are the poloidal and toroidal components of the field at the various grid points. So, in this work we want to explore the different ways in which the solutions Y of these governing equations (8) may scale with this parameter $\varepsilon \equiv E^{1/2}$.

When one solves these equations numerically, as was done by Hollerbach and Ierley (1991) and Anufriev *et al.* (1995), one timesteps them to some sort of equilibrium for different values of ε, and simply sees how the solutions appear to scale with ε. This is how Hollerbach and Ierley obtained solutions scaling as $\varepsilon^{1/2}$ (the Ekman state) and ε^0 (the Taylor state), and how Anufriev *et al.* obtained this new solution scaling as $\varepsilon^{1/4}$ (the semi-Taylor state). However, if one is trying to understand the dynamical balances underlying these various scalings, it is best to imagine a formal expansion in powers of ε,

$$Y = \varepsilon^{\gamma}\sum_{n=0}^{\infty}\varepsilon^{n\delta}Y^{(n)}, \tag{9}$$

where ε^{γ} is thus the leading order scaling, and $\varepsilon^{\gamma+n\delta}$ the successive higher order corrections. What we are trying to discover is what values of γ (and δ) are allowed as solutions of (8).

Having expanded the Y's as in (9), the linear terms $\partial Y_i/\partial t$ and $L_{ij}Y_j$ will give rise to terms scaling as

$$\varepsilon^{\gamma},\quad \varepsilon^{\gamma+\delta},\quad \varepsilon^{\gamma+2\delta},\quad \ldots, \tag{10a}$$

the nonlinear term $\varepsilon^{-1}C^{(g)}_{ijkl}Y_jY_kY_l$ will give rise to terms scaling as

$$\varepsilon^{3\gamma-1},\quad \varepsilon^{3\gamma-1+\delta},\quad \varepsilon^{3\gamma-1+2\delta},\quad \ldots, \tag{10b}$$

and the nonlinear term $C^{(a)}_{ijkl}Y_jY_kY_l$ will give rise to terms scaling as

$$\varepsilon^{3\gamma},\quad \varepsilon^{3\gamma+\delta},\quad \varepsilon^{3\gamma+2\delta},\quad \ldots. \tag{10c}$$

There are then a number of different ways we can balance these various terms against one another, resulting in a number of different values for γ.

Most simply, we could balance them as

$$\begin{array}{llll} \varepsilon^{\gamma}, & \varepsilon^{\gamma+\delta}, & \varepsilon^{\gamma+2\delta}, & \ldots, \\ \varepsilon^{3\gamma-1}, & \varepsilon^{3\gamma-1+\delta}, & \varepsilon^{3\gamma-1+2\delta}, & \ldots, \\ & \varepsilon^{3\gamma}, & \varepsilon^{3\gamma+\delta}, & \ldots. \end{array} \tag{11}$$

Equating $\gamma = 3\gamma - 1$ and $\gamma + \delta = 3\gamma$ yields $\gamma = 1/2, \delta = 1$. That is, we have recovered the familiar Ekman state, scaling as $\varepsilon^{1/2}$.

However, suppose now that at leading order the solution satisfies Taylor's constraint, that is,

$$\sum_{ijkl} C^{(g)}_{ijkl} Y^{(0)}_i Y^{(0)}_j Y^{(0)}_k Y^{(0)}_l = 0. \tag{12}$$

Note, incidentally, that we have actually used the dot product of $Y^{(0)}_i$ with (8) to obtain a single quartic constraint, corresponding not to Taylor's constraint directly, but to the energy dissipated through the geostrophic flow. One can show (Hollerbach and Ierley, 1991), that this energy contribution is non-positive, vanishing only when Taylor's constraint is satisfied. This single quartic constraint (12) is thus indeed equivalent to Taylor's constraint, even if it is expressed in a slightly different form from (7). To recover the more familiar form, as in Hollerbach and Ierley one could in principle use the theory of quadratic forms to convert this one quartic constraint into many quadratic constraints. It is in this latter form that (12) would be most easily recognizable as Taylor's constraint; these quadratic constraints one would obtain are in fact precisely the projection of (7) onto one's Galerkin or finite-difference expansion.

So, if (12) is satisfied, that eliminates the leading order geostrophic terms scaling as $\varepsilon^{3\gamma-1}$ in (10b); if these terms exactly balance themselves [for each i in (8)], there is no need to balance them against anything else. Once these leading order geostrophic terms are thus eliminated, however, we are able to balance the remaining terms in several different ways.

We could balance them as

$$\begin{array}{llll} \varepsilon^{\gamma}, & \varepsilon^{\gamma+\delta}, & \varepsilon^{\gamma+2\delta}, & \ldots, \\ \varepsilon^{3\gamma-1+\delta}, & \varepsilon^{3\gamma-1+2\delta}, & \varepsilon^{3\gamma-1+3\delta}, & \ldots, \\ \varepsilon^{3\gamma}, & \varepsilon^{3\gamma+\delta}, & \varepsilon^{3\gamma+2\delta}, & \ldots. \end{array} \tag{13}$$

Equating $\gamma = 3\gamma - 1 + \delta$ and $\gamma = 3\gamma$ yields $\gamma = 0, \delta = 1$. That is, we have recovered the familiar Taylor state, scaling as ε^0.

However, we could equally well balance them as

$$\begin{array}{llll} \varepsilon^{\gamma}, & \varepsilon^{\gamma+\delta}, & \varepsilon^{\gamma+2\delta}, & \ldots, \\ \varepsilon^{3\gamma-1+\delta}, & \varepsilon^{3\gamma-1+2\delta}, & \varepsilon^{3\gamma-1+3\delta}, & \ldots, \\ & \varepsilon^{3\gamma}, & \varepsilon^{3\gamma+\delta}, & \ldots. \end{array} \tag{14}$$

Equating $\gamma = 3\gamma - 1 + \delta$ and $\gamma + \delta = 3\gamma$ yields $\gamma = 1/4, \delta = 1/2$. That is, we have obtained the not-so-familiar semi-Taylor state, scaling as $\varepsilon^{1/4}$.

Furthermore, we see that these three balances, corresponding to the Ekman state, the Taylor state, and the semi-Taylor state, are the only possible balances (at least in the absence of additional boundary layers, as in Braginsky's (1975) model-Z for example, with its current layers). That is, this semi-Taylor state, scaling as $\varepsilon^{1/4}$, is indeed the only state intermediate between the Ekman state scaling as $\varepsilon^{1/2}$ and the Taylor state scaling as ε^0, a point apparently not appreciated by Anufriev *et al.* Their numerical solution didn't just happen to follow an $\varepsilon^{1/4}$ scaling; once it yielded an intermediate state at all, according to the theory developed here that state had to follow precisely an $\varepsilon^{1/4}$ scaling, and nothing else.

3. DYNAMICAL BALANCES

Having obtained these two states, both satisfying the constraint (12), we would like to compare and contrast them in some detail. (We will not consider the Ekman state in such detail, as its dynamical balance is so straightforward.) It is important to emphasize first of all that satisfying Taylor's constraint at leading order, as in (12), does not determine the amplitude of one's solution, since it is a homogeneous constraint. Instead, satisfying (12) merely eliminates the leading order geostrophic flow, and thereby allows one's solutions to grow beyond the $O(\varepsilon^{1/2})$ amplitude at which this leading order geostrophic flow would otherwise equilibrate them, as in the Ekman regime. However, satisfying (12) says nothing about how far beyond $O(\varepsilon^{1/2})$ one's solutions will grow. In a Taylor state the solution grows to an O(1) amplitude, in a semi-Taylor state only to an $O(\varepsilon^{1/4})$ amplitude. The question then is, why this difference? Indeed, this is really the most puzzling feature of this semi-Taylor state. Just like a true Taylor state, it satisfies Taylor's constraint at leading order, and yet it does not make the transition to a true Taylor state. Why?

We can at least begin to answer this question by considering not just the general scalings associated with these two states, but their detailed dynamical balances as well. So, according to the scalings derived above, in a Taylor state

$$\mathbf{B} = \mathbf{B}_0 + E^{1/2}\mathbf{B}_1 + E\,\mathbf{B}_2 + \dots, \tag{15}$$

and in a semi-Taylor state

$$\mathbf{B} = E^{1/8}\mathbf{B}_0 + E^{3/8}\mathbf{B}_1 + E^{5/8}\mathbf{B}_2 + \dots. \tag{16}$$

Since both of these solutions must satisfy Taylor's constraint at leading order, we must have

$$\int_{C(s)} \left[(\nabla\times\mathbf{B}_0)\times\mathbf{B}_0\right]_\phi dS = 0 \tag{17}$$

for both.

The geostrophic flow is then given by

$$v_g = (v_{01} + v_{10}) + E^{\delta/2}(v_{02} + v_{11} + v_{20}) + \dots, \tag{18}$$

where

$$v_{ij} \equiv \frac{(1-s^2)^{1/4}}{4\pi s}\int_{C(s)} \left[(\nabla\times\mathbf{B}_i)\times\mathbf{B}_j\right]_\phi dS, \tag{19}$$

and once again δ equals 1 for a Taylor state and 1/2 for a semi-Taylor state. Incidentally, an immediate consequence is thus that the first nonzero contribution to the geostrophic flow is O(1) in both regimes (indeed, in the Ekman regime as well), in perfect agreement with Anufriev *et al.* (1995), who note that "the geostrophic [flow] does not seem to depend on viscosity." Similarly, the ageostrophic flow is given by

$$\mathbf{U}_\mathrm{M} = \mathbf{U}_{00} + E^{1/2}(\mathbf{U}_{01} + \mathbf{U}_{10}) + \ldots \tag{20}$$

in a Taylor state, and by

$$\mathbf{U}_\mathrm{M} = E^{1/4}\mathbf{U}_{00} + E^{1/2}(\mathbf{U}_{01} + \mathbf{U}_{10} + \ldots \tag{21}$$

in a semi-Taylor state, where

$$\mathbf{U}_{ij} \equiv -\frac{1}{2}\int^{z} \nabla\times\left[(\nabla\times\mathbf{B}_i)\times\mathbf{B}_j\right]dz'. \tag{22}$$

Separated out order by order, the dynamical balances in the induction equation thus become

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B}_0 =& \nabla^2\mathbf{B}_0 + \nabla\times(\alpha\mathbf{B}_0) + \nabla\times(\mathbf{U}_\mathrm{T}\times\mathbf{B}_0)\\ &+ \nabla\times[(v_{01} + v_{10})\hat{\mathbf{e}}_\phi\times\mathbf{B}_0] + \nabla\times(\mathbf{U}_{00}\times\mathbf{B}_0),\end{aligned} \tag{23 a}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B}_1 =& \nabla^2\mathbf{B}_1 + \nabla\times(\alpha\mathbf{B}_1) + \nabla\times(\mathbf{U}_\mathrm{T}\times\mathbf{B}_1)\\ &+ \nabla\times[(v_{01} + v_{10})\hat{\mathbf{e}}_\phi\times\mathbf{B}_1] + \nabla\times(\mathbf{U}_{00}\times\mathbf{B}_1)\\ &+ \nabla\times[(v_{02} + v_{11} + v_{20})\hat{\mathbf{e}}_\phi\times\mathbf{B}_0]\\ &+ \nabla\times[(\mathbf{U}_{01} + \mathbf{U}_{10})\times\mathbf{B}_0],\end{aligned} \tag{23 b}$$

in a Taylor state, and

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B}_0 =& \nabla^2\mathbf{B}_0 + \nabla\times(\alpha\mathbf{B}_0) + \nabla\times(\mathbf{U}_\mathrm{T}\times\mathbf{B}_0)\\ &+ \nabla\times[(v_{01} + v_{10})\hat{\mathbf{e}}_\phi\times\mathbf{B}_0],\end{aligned} \tag{24 a}$$

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B}_1 =& \nabla^2\mathbf{B}_1 + \nabla\times(\alpha\mathbf{B}_1) + \nabla\times(\mathbf{U}_\mathrm{T}\times\mathbf{B}_1)\\ &+ \nabla\times[(v_{01} + v_{10})\hat{\mathbf{e}}_\phi\times\mathbf{B}_1]\\ &+ \nabla\times[(v_{02} + v_{11} + v_{20})\hat{\mathbf{e}}_\phi\times\mathbf{B}_0]\\ &+ \nabla\times(\mathbf{U}_{00}\times\mathbf{B}_0),\end{aligned} \tag{24 b}$$

in a semi-Taylor state (and similarly at higher order).

In attempting to solve either (23) or (24), one might proceed as follows: since $\mathbf{B}_1$ is not yet determined in (23a) and (24a), the combination $(v_{01}+v_{10})$ is unknown, so we may relabel it as some new unknown $\hat{v}_1$, say, to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B}_0 =& \nabla^2\mathbf{B}_0 + \nabla\times(\alpha\mathbf{B}_0) + \nabla\times(\mathbf{U_T}\times\mathbf{B}_0)\\ &+ \nabla\times(\hat{v}_1\hat{\mathbf{e}}_\phi\times\mathbf{B}_0) + \nabla\times(\mathbf{U}_{00}\times\mathbf{B}_0)\end{aligned} \tag{25}$$

in a Taylor state, or

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B}_0 =& \nabla^2\mathbf{B}_0 + \nabla\times(\alpha\mathbf{B}_0) + \nabla\times(\mathbf{U_T}\times\mathbf{B}_0)\\ &+ \nabla\times(\hat{v}_1\hat{\mathbf{e}}_\phi\times\mathbf{B}_0)\end{aligned} \tag{26}$$

in a semi-Taylor state. In solving these equations for $\mathbf{B}_0$, we choose $\hat{v}_1(s)$ such that $\mathbf{B}_0$ also satisfies Taylor's constraint (17) throughout its evolution (this may not always be possible, but for the moment we will assume it is).

Then, having solved for $\mathbf{B}_0$, we note that since $\mathbf{B}_2$ is not yet determined in (23b) and (24b), the combination $(v_{02}+v_{11}+v_{20})$ is unknown, so we may relabel it as some new unknown $\hat{v}_2$, say, to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B}_1 =& \nabla^2\mathbf{B}_1 + \nabla\times(\alpha\mathbf{B}_1) + \nabla\times(\mathbf{U_T}\times\mathbf{B}_1)\\ &+ \nabla\times(\hat{v}_1\hat{\mathbf{e}}_\phi\times\mathbf{B}_1) + \nabla\times(\mathbf{U}_{00}\times\mathbf{B}_1)\\ &+ \nabla\times(\hat{v}_2\hat{\mathbf{e}}_\phi\times\mathbf{B}_0) + \nabla\times[(\mathbf{U}_{01}+\mathbf{U}_{10})\times\mathbf{B}_0]\end{aligned} \tag{27}$$

in a Taylor state, or

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{B}_1 =& \nabla^2\mathbf{B}_1 + \nabla\times(\alpha\mathbf{B}_1) + \nabla\times(\mathbf{U_T}\times\mathbf{B}_1)\\ &+ \nabla\times(\hat{v}_1\hat{\mathbf{e}}_\phi\times\mathbf{B}_1)\\ &+ \nabla\times(\hat{v}_2\hat{\mathbf{e}}_\phi\times\mathbf{B}_0) + \nabla\times(\mathbf{U}_{00}\times\mathbf{B}_0)\end{aligned} \tag{28}$$

in a semi-Taylor state. In solving these equations for $\mathbf{B}_1$, we choose $\hat{v}_2(s)$ such that $\mathbf{B}_0$ and $\mathbf{B}_1$ also satisfy

$$v_{01}+v_{10}=\hat{v}_1, \tag{29}$$

where we recall that $\hat{v}_1$ has indeed already been determined at the previous order.

Proceeding in this fashion, one can imagine sequentially solving (23) or (24) up to any order. At each order n we note that there is a part of the geostrophic flow that involves $\mathbf{B}_{n+1}$, and is therefore still undetermined. Labelling this undetermined geostrophic flow $\hat{v}_{n+1}(s)$, we attempt to solve for $\mathbf{B}_n$. In the process we choose $\hat{v}_{n+1}(s)$ such that $\mathbf{B}_0$ through $\mathbf{B}_n$ do indeed yield the previously determined geostrophic flow $\hat{v}_n(s)$, as in (29). In fact, the first step in applying this procedure to (23), namely choosing $\hat{v}_1(s)$ in (25) in such a way that $\mathbf{B}_0$ also satisfies (17), is exactly the procedure suggested by Taylor himself. We have thus not only extended this procedure to all orders, but applied it to both Taylor and semi-Taylor states.

In fact, considering how little success people have had in obtaining even just Taylor states in this way [see, for example, Fearn and Proctor (1987) for one such attempt], it is perhaps not surprising that one is likely to have even less success in obtaining semi-Taylor states. For one notices that unlike the nonlinear equation (25), the 'linear' equation (26) does not yet determine the amplitude of $\mathbf{B}_0$. [Of course, in reality both (25) and (26) are nonlinear, because $\hat{v}_1 \equiv v_{01} + v_{10}$ is itself a quadratic function of $\mathbf{B}_0$ and $\mathbf{B}_1$.] This suggests that whereas one may succeed in obtaining Taylor states in this sequential fashion, one will almost certainly fail in obtaining semi-Taylor state. In a Taylor state one can solve for $\mathbf{B}_0$ first and $\mathbf{B}_1$ later, but in a semi-Taylor state one cannot; one must solve for both simultaneously. Note, for example, how γ is obtained by balancing just the first order terms in the Taylor scaling (13), but in the semi-Taylor scaling (14) it is only obtained by balancing first and second order terms. That the amplitude of $\mathbf{B}_0$ is not determined by (26) alone is merely a reflection of this fact.

So we see that the nonlinear equilibration in the semi-Taylor regime is wonderfully subtle, relying on a delicate balance between the first and second order terms. In fact, this balance is so delicate that in general it is probably not accessible at all. Imagine a solution evolving according to either (25) or (26). In (25) the nonlinear equilibration through the ageostrophic flow $\mathbf{U}_{00}$ is so straightforward that the solution cannot diverge to infinity. In contrast, in (26) the nonlinear equilibration through the geostrophic flow $\hat{v}_1$ is so subtle that the solution probably can diverge to infinity. But if that occurs one's scaling has broken down; that is, one has just made the transition from the semi-Taylor regime to the Taylor regime.

For the semi-Taylor regime to endure, we see that the solution must adjust itself in such a way that the amplitude of $\mathbf{B}_0$, which again can only be determined by considering first and second order terms simultaneously, does not diverge. In general, for time-dependent solutions it is very difficult to discover what that might imply about the structure of the solution. However, for time-independent solutions it is very easy to discover what that implies; for (26) to have non-trivial steady-state solutions, a certain solvability condition must be met [for the discretised version of (26) it would be that the determinant of the matrix representing the operators on the right-hand side must vanish].

We have then the following scenario: if $\hat{v}_1$, which we remember is given in terms of $\mathbf{B}_0$ and $\mathbf{B}_1$, is able to adjust itself such that this solvability condition is satisfied, the semi-Taylor state will be accessible; if $\hat{v}_1$ is not able to adjust itself in this way, the semi-Taylor state will not be accessible. We thus see, first of all, that generically the semi-Taylor state is probably not accessible (which would explain why it has not been observed earlier). And second, we see that whether the semi-Taylor state is accessible or not depends not only on $\hat{v}_1$, that is, on the solution itself, but also on all the other operators on the right-hand side of (26). In particular, it also depends on the kinematically prescribed forcing α and $\mathbf{U}_\mathbf{T}$. Upon reflection, that is probably not surprising, since the only features (apart from the details of the numerical implementation) that distinguish models such as Hollerbach and Ierley (1991) or Anufriev *et al.* (1995) are the details of the kinematically prescribed terms. There must therefore be some connection between the prescribed input and the nature of the output one obtains.

4. CONCLUSION

In this work we have considered the different scalings with E one may obtain as solutions of the geodynamo equations when these are written in the asymptotic form (8). We found that if Taylor's constraint is not satisfied at leading order, one will inevitably obtain an Ekman state, with the magnetic field scaling as $O(E^{1/4})$. However, if Taylor's constraint is satisfied at leading order, so that the leading order geostrophic flow vanishes, one may obtain larger fields. If the ageostrophic flow enters at first order, one obtains a true Taylor state, with the field scaling as O(1); if the ageostrophic flow enters at second order, one obtains a semi-Taylor state, with the field scaling as $O(E^{1/8})$. This may seem counterintuitive, that a higher order ageostrophic contribution should equilibrate the field at a smaller value, but it is not. If the ageostrophic flow is so efficient (in some vaguely defined sense) that it already has an effect on the field at second rather than first order, one might expect it to equilibrate the field sooner rather than later in its growth beyond $O(E^{1/4})$.

By considering the detailed dynamical balances at the various orders, we also began to address the question of what very special adjustment in the solution is required to obtain such a semi-Taylor state. We discovered that although this procedure of sequentially solving order by order may just work in the Taylor regime, it almost certainly will not work in the semi-Taylor regime. Nevertheless, the equations we obtained in this way were quite useful in giving us a clue as to the nature of this required adjustment. Also, these equations may be extremely useful in analysing some particular semi-Taylor state *a posteriori*. The results presented here would suggest revisiting the solution of Anufriev *et al.* and at least attempting to separate out $\mathbf{B}_0$ and $\mathbf{B}_1$. Considering that the expansion (16) tells us not only that the leading order solution scales as $O(E^{1/8})$, but also that the next order solution scales as $O(E^{3/8})$, it should be possible to (approximately anyway) separate out the first two terms. One could then insert these terms into (24), and thereby perhaps gain further insight into the nature of this required adjustment, at least for this particular solution. It seems clear that in order to understand these semi-Taylor solutions, one must begin by understanding this subtle balance between the first and second order terms, and this very delicate adjustment that makes this balance possible.

Finally, we should comment briefly on the possible geophysical implications of this semi-Taylor state. Although Anufriev *et al.* have suggested that the semi-Taylor state might itself be a plausible candidate for the geodynamo, in light of its non-generic nature that seems unlikely. It seems unlikely that the basic field would be in a non-generic state throughout its entire evolution. Instead, this semi-Taylor state could describe temporary departures from that basic field. For example, Hollerbach and Jones (1995) presented a solution exhibiting a sudden collapse in the field strength. They interpreted that collapse as a temporary transition from the Taylor regime back to the Ekman regime, caused by a temporary violation of Taylor's constraint. However, we now see that it's not even necessary to violate Taylor's constraint to cause a sudden collapse in the field strength. The balances presented here demonstrate that it should be possible to satisfy Taylor's constraint at leading order throughout the entire evolution of one's solution, and still have transitions between different scalings, and thus different field strengths. One can imagine a solution evolving primarily in the (generic) Taylor regime, with transitions to the (non-generic) semi-Taylor regime occurring only on those rare occasions when that regime becomes temporarily accessible. The semi-Taylor state may thus be relevant to geophysical phenomena such as excursions, despite the fact that it is non-generic. Or perhaps one should even say that it may be relevant to phenomena such as excursions precisely because it is non-generic. Considering that excursions are relatively rare events, it seems more likely that they would be caused by non-generic rather than by generic features.

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