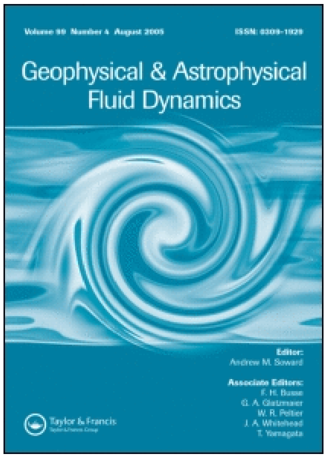

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On the energetics of magnetic instabilities

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ON THE ENERGETICS OF MAGNETIC INSTABILITIES

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The Earth's magnetic field is created by fluid motions in its core, but it may also be destroyed by fluid motions. Indeed, it has been suggested by Zhang and Fearn (1994, 1995) that the onset of magnetic instability should provide an upper bound on the geomagnetic field. It is thus of considerable interest to investigate the possible instabilities of the large-scale field. See, for example, Fearn (1993) and numerous references therein. In this note I derive the energy balance associated with purely magnetic instabilities, and suggest how it might be useful in understanding them.

The non-dimensional equations for the magnetic field **B** and the fluid flow **U** are (Fearn, 1994)

$$R_0\frac{D\mathbf{U}}{Dt} + 2\hat{\mathbf{k}} \times \mathbf{U} = -\nabla p + E\nabla^2\mathbf{U} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla^2\mathbf{B} + \nabla \times (\mathbf{U} \times \mathbf{B}). \tag{2}$$

If we were interested in the generation of the field, we would need to include the buoyancy Θ as well. However, if we are interested in purely magnetic instabilities of the existing field, these two equations will suffice. Linearizing them about some existing large-scale field **B**, we obtain the magnetic instability equations for the small-scale perturbations **b** and **u**:

$$R_0\frac{\partial \mathbf{u}}{\partial t} + 2\hat{\mathbf{k}} \times \mathbf{u} = -\nabla p + E\nabla^2\mathbf{u} + (\nabla \times \mathbf{b}) \times \mathbf{B} + (\nabla \times \mathbf{B}) \times \mathbf{b}, \tag{3}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla^2\mathbf{b} + \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{4}$$

Now, following a similar derivation in Hollerbach and Jones (1993), one can derive an associated energy equation. Taking the sum of the dot products of (3) with **u** and

(4) with $\mathbf{b}$, one obtains after a few vector identities,

$$\frac{1}{2}\frac{\partial}{\partial t}[|\mathbf{b}|^2 + R_0|\mathbf{u}|^2] = \mathbf{u}\cdot(\mathbf{J}\times\mathbf{b}) + \nabla\cdot[(\mathbf{u}\times\mathbf{B})\times\mathbf{b}]$$

$$-|\nabla\times\mathbf{b}|^2 + \nabla\cdot[\mathbf{b}\times(\nabla\times\mathbf{b})]$$

$$-E|\nabla\times\mathbf{u}|^2 + \nabla\cdot[E\mathbf{u}\times(\nabla\times\mathbf{u}) - \mathbf{u}p], \tag{5}$$

where $\mathbf{J} = \nabla\times\mathbf{B}$ is the existing large-scale current density.

Integrating (5) over the interior of the core, the divergence terms become surface integrals. Assuming no-slip boundary conditions, the surface terms involving $\mathbf{u}$ contribute nothing. The boundary conditions actually imposed in various studies (Fearn, 1993) are admittedly usually not no-slip, but the true boundary conditions are no-slip. Taking into account the magnetic energy in the exterior of the core, the surface term involving only $\mathbf{b}$ also contributes nothing further. The final result is then

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V [|\mathbf{b}|^2 + R_0|\mathbf{u}|^2]dV = \int_V \mathbf{u}\cdot(\mathbf{J}\times\mathbf{b})dV$$

$$-\int_V [|\nabla\times\mathbf{b}|^2 + E|\nabla\times\mathbf{u}|^2]dV, \tag{6}$$

where the volume V is now all of space, but of course all terms except $|\mathbf{b}|^2$ only contribute within the core.

The necessary condition for the instability to grow is then that

$$\int_V \mathbf{u}\cdot(\mathbf{J}\times\mathbf{b})dV \geqslant \int_V [|\nabla\times\mathbf{b}|^2 + E|\nabla\times\mathbf{u}|^2]dV, \tag{7}$$

stating that the energy which the instability is able to extract from the large-scale structure must be sufficient to overcome its own ohmic and viscous losses. Note, incidentally, that according to (7) magnetic instabilities are instabilities not so much of the field $\mathbf{B}$ as of the current $\mathbf{J} = \nabla\times\mathbf{B}$.

I believe this result (7) may prove useful in understanding the pattern of magnetic instabilities. It is already evident from (7) that the larger the current $\mathbf{J}$, the more likely it is that instabilities will occur. Probably far more useful, however, is the directional information contained in (7); since $\mathbf{u}\cdot(\mathbf{J}\times\mathbf{b})$ will be maximised if these three vectors are mutually orthogonal, there should be a preferred orientation to the instability. Considering that the toroidal and poloidal fields studied by Zhang and Fearn (1994, 1995) have very different directions of imposed current $\mathbf{J}$, this analysis would suggest that very different directions of $\mathbf{u}$ and $\mathbf{b}$ should result. This, together with the geometrical constraints already imposed on $\mathbf{u}$ by the Taylor–Proudman theorem, may help in interpreting the structure of these magnetic instabilities.

References

Fearn, D. R., "Magnetic instabilities in rapidly rotating systems," in: *Solar and Planetary Dynamos* (Eds. M. R. E. Proctor, P. C. Matthews and A. M. Rucklidge), Cambridge University Press (1993).

Fearn, D. R., "Nonlinear planetary dynamos," in: *Stellar and Planetary Dynamos* (Eds. M. R. E. Proctor and A. D. Gilbert), Cambridge University Press (1994).

Hollerbach, R. and Jones, C. A., "A geodynamo model incorporating a finitely conducting inner core," *Phys. Earth and Planet. Int.* **75**, 317–327 (1993).

Zhang, K. and Fearn, D. R., "Hydromagnetic waves in rapidly rotating spherical shells generated by toroidal decay modes," *Geophys. Astrophys. Fluid Dynam.* **77**, 133–157 (1994).

Zhang, K. and Fearn, D. R., "Hydromagnetic waves in rapidly rotating spherical shells generated by poloidal decay modes," *Geophys. Astrophys. Fluid Dynam.* **81**, 193–209 (1995).