Physics of the Earth and Planetary Interiors, 75 (1993) 317–327
Elsevier Science Publishers B.V., Amsterdam

A geodynamo model incorporating a finitely conducting inner core

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(Received 16 April 1992; revision accepted 3 July 1992)

ABSTRACT

Hollerbach, R. and Jones, C.A., 1993. A geodynamo model incorporating a finitely conducting inner core. Phys. Earth Planet. Inter., 75: 317–327

A direct spectral solution is presented for the mean-field geodynamo equations. The momentum equation, with inertia neglected but viscosity retained, is solved in a spherical shell. The induction equation is solved in a similar fashion, and includes Ohmic dissipation in the finitely conducting inner core. The existence of a second Taylor's constraint is pointed out, requiring the integrated Lorentz torque on the inner core to vanish in the limit of vanishing viscosity. In the slightly supercritical regime this constraint is shown to be violated, resulting in a differential rotation of the inner core actively driving a portion of the outer core flow. In the more strongly supercritical regime this constraint is shown to be satisfied, the process of adjustment involving a minimization of the Ohmic dissipation in the inner core.

1. Introduction

The Earth's magnetic field is created in its metallic core by so-called 'dynamo action', in which fluid motions generate a magnetic field. The electromotive force (EMF) induced by this motion of conducting material across magnetic field lines generates currents which sustain that field. In this way, the dynamo converts the mechanical energy of the motion into magnetic energy and dissipates it via Ohmic heating. The ultimate source of the energy is presently believed to be compositional convection (Braginsky, 1963; Loper and Roberts, 1983).

This metallic core, in turn, consists of a solid inner core of radius $r_i = 1220$ km, surrounded by a liquid outer core of radius $r_o = 3480$ km. (As an aside, the compositional convection mentioned above involves a crystallization process whereby the inner core very gradually grows at the expense of the outer core.) Now, considering that the inner core is solid, the fluid motions that constitute one half of the dynamo process are necessarily confined to the outer core. Consequently, many dynamo models neglect the inner core altogether. Indeed, models in geometries as diverse as spheres (Hollerbach and Ierley, 1991) and infinite plane layers (Soward and Jones, 1983) have yielded broadly similar results, making the neglect of an inner core not unreasonable.

Nevertheless, one would not anticipate the inner core to be entirely negligible. Considering that the composition of the inner and outer cores is (almost) the same, one would also expect their conductivities to be similar. Thus, although the inner core is not directly involved in the fluid half of the dynamo process, it is involved in the electromagnetic half. In fact, the electromagnetic coupling between the inner and outer cores indirectly turns out to involve the inner core in the fluid half as well. It is this interaction that is the subject of this paper. Finally, as an aside, note

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that if one allows for a weakly conducting mantle, there is also the possibility of electromagnetic coupling between the core and the mantle; this issue is not pursued here, but see, for example, Busse (1991).

2. Statement of the electromagnetic problem

The scaled mean-field induction equations are

$$\frac{\partial \boldsymbol{B}_i}{\partial t} = \sigma^{-1} \nabla^2 \boldsymbol{B}_i \tag{2.1}$$

in the inner core, and

$$\frac{\partial \boldsymbol{B}_o}{\partial t} = \nabla^2 \boldsymbol{B}_o + \nabla \times (\alpha \boldsymbol{B}_o) + \nabla \times (\boldsymbol{U} \times \boldsymbol{B}_o) \tag{2.2}$$

in the outer core, where $\boldsymbol{B}$ is the large-scale axisymmetric magnetic field, and $\boldsymbol{U}$ is the large-scale axisymmetric fluid flow considered in the next section. σ is the ratio of inner to outer core conductivities, and is of order unity. α is a parameterization of the small-scale non-axisymmetric flow, introduced in the mean-field theory of Steenbeck et al. (1966). According to Cowling's (1934) theorem, some form of non-axisymmetric flow must be present, either explicitly or implicitly as is the case here, to achieve any dynamo action at all.

Decomposing as

$$\boldsymbol{B} = \nabla \times \left[A(r, \theta) \hat{\boldsymbol{e}}_\phi \right] + B(r, \theta) \hat{\boldsymbol{e}}_\phi \tag{2.3a}$$

$$\boldsymbol{U} = \nabla \times \left[\psi(r, \theta) \hat{\boldsymbol{e}}_\phi \right] + v(r, \theta) \hat{\boldsymbol{e}}_\phi \tag{2.3b}$$

where (r, θ, ϕ) are spherical coordinates, the induction equations become (Proctor, 1977)

$$\frac{\partial A_i}{\partial t} = \sigma^{-1} D^2 A_i \tag{2.4a}$$

$$\frac{\partial B_i}{\partial t} = \sigma^{-1} D^2 B_i \tag{2.4b}$$

and

$$\frac{\partial A_o}{\partial t} = D^2 A_o + \alpha B_o + N(\psi, A_o) \tag{2.5a}$$

$$\frac{\partial B_o}{\partial t} = D^2 B_o + \hat{\boldsymbol{e}}_\phi \cdot \nabla \times \left[\alpha \nabla \times (A_o \hat{\boldsymbol{e}}_\phi) \right] + M(v, A_o) - M(B_o, \psi) \tag{2.5b}$$

where

$$D^2 = \nabla^2 - (r \sin \theta)^{-2} \tag{2.6}$$

$$N(X, Y) = \hat{\boldsymbol{e}}_\phi \cdot \left[\nabla \times (X \hat{\boldsymbol{e}}_\phi) \times \nabla \times (Y \hat{\boldsymbol{e}}_\phi) \right] \tag{2.7}$$

$$M(X, Y) = \hat{\boldsymbol{e}}_\phi \cdot \nabla \times \left[X \hat{\boldsymbol{e}}_\phi \times \nabla \times (Y \hat{\boldsymbol{e}}_\phi) \right] \tag{2.8}$$

The free-decay ($\alpha = 0$, $\boldsymbol{U} = 0$) modes of (2.4) and (2.5) have as inner core solution

$$F = \exp(-\lambda^2 t) P_l^{(1)}(\cos \theta) j_l(\sqrt{\sigma} \lambda r) \tag{2.9}$$

and as outer core solution

$$F = \exp(-\lambda^2 t) P_l^{(1)}(\cos \theta) f(r) \tag{2.10a}$$

where

$$f(r) = C_1 j_l(\lambda r) + C_2 y_l(\lambda r) \tag{2.10b}$$

Here F stands for either A or B. $P_l^{(1)}(\cos \theta)$ are associated Legendre functions, and j_l and y_l are the spherical Bessel functions (Abramowitz and Stegun, 1968). The eigenvalues λ are determined by the various matching conditions. For A the matching conditions at the inner core boundary turn out to be continuity of the r and θ components of the magnetic field, yielding

$$j_l(\sqrt{\sigma} \lambda r_i) f_A'(r_i) = \sqrt{\sigma} \lambda j_l'(\sqrt{\sigma} \lambda r_i) f_A(r_i) \tag{2.11a}$$

and at the outer core boundary matching to an external potential field yields

$$f_A'(r_o) + \frac{l+1}{r_o} f_A(r_o) = 0 \tag{2.11b}$$

For B the matching conditions at the inner core boundary turn out to be continuity of the ϕ component of the magnetic field and the θ component of the associated electric field, yielding

$$\left[\sqrt{\sigma} \lambda r_i j_l'(\sqrt{\sigma} \lambda r_i) + j_l(\sqrt{\sigma} \lambda r_i) \right] f_B(r_i) = \sigma j_l(\sqrt{\sigma} \lambda r_i) \left[r_i f_B'(r_i) + f_B(r_i) \right] \tag{2.12a}$$

and at the outer core boundary matching to zero yields

$$f_B(r_o) = 0 \tag{2.12b}$$

In each case requiring that solutions exist for non-trivial C_1 and C_2 then yields a rather tedious algebraic equation whose (numerically determined) roots are the eigenvalues λ. The details are of no interest here and are not presented.

It is of some interest, however, to consider briefly the inner core matching conditions in the two limits $\sigma \to 0$ and $\sigma \to \infty$ (although we will later take $\sigma = 1$). For $\sigma \to 0$, (2.11a) and (2.12a) reduce, respectively, to

$$f'_A(r_i) - \frac{l}{r_i} f_A(r_i) = 0 \tag{2.13a}$$

$$f_B(r_i) = 0 \tag{2.13b}$$

For $\sigma \to \infty$, (2.11a) and (2.12a) reduce, respectively, to

$$f_A(r_i) = 0 \tag{2.14a}$$

$$f'_B(r_i) + \frac{1}{r_i} f_B(r_i) = 0 \tag{2.14b}$$

(Implicit in all these reductions is the assumption that $\lambda = O(1)$, that is, that the time-scale is the outer core Ohmic decay time.)

The significance of (2.13) and (2.14) is that they only involve the outer core radial structure $f(r)$. Unlike (2.11a) and (2.12a), they can thus be used as boundary conditions, rather than matching conditions. Indeed, previous dynamo models that incorporate an inner core at all usually consider either of the limits $\sigma = 0$ or $\sigma = \infty$, and merely impose either (2.13) or (2.14) as the appropriate boundary conditions. Although we have derived (2.13) and (2.14) by considering free decay modes, these would in fact be the appropriate boundary conditions in general.

If $\sigma = 0$, the magnetic field in the (now non-conducting) inner core adjusts instantaneously to changes on the boundary. If $\sigma = \infty$, the magnetic field never penetrates the (now perfectly conducting) inner core at all. In neither case must one include the inner core in one's calculations; merely imposing appropriate boundary conditions on the outer core is sufficient. However, when $\sigma \approx 1$, that is clearly not the case. The evolution of the field in the inner core is determined not only by changes on the boundary, but also by its own past history. It must thus be explicitly included in the calculation.

3. Statement of the fluid dynamical problem

Neglecting inertia, the scaled momentum equation in the outer core is

$$2\hat{\boldsymbol{k}} \times \boldsymbol{U} = -\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \epsilon \nabla^2 \boldsymbol{U} \tag{3.1}$$

where again $\boldsymbol{B}$ is the large-scale magnetic field, $\boldsymbol{U}$ is the large-scale fluid flow, p is the pressure, and $\hat{\boldsymbol{k}}$ is the unit vector parallel to the axis of rotation. The Ekman number $\epsilon \ll 1$, reflecting the essentially inviscid nature of the flow (at least throughout the bulk of the fluid). Using the decomposition (2.3), the momentum equation becomes

$$2\frac{\partial \psi}{\partial z} + \epsilon D^2 v = -N(B, A) \tag{3.2a}$$

$$2\frac{\partial v}{\partial z} - \epsilon (D^2)^2 \psi = M(B, B) + M(D^2 A, A) \tag{3.2b}$$

where

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{3.3}$$

The no-slip boundary conditions associated with (3.2) are for ψ

$$\psi = \frac{\partial \psi}{\partial r} = 0 \qquad \text{at } r = r_i, r_o \tag{3.4}$$

For v one must be a little more careful, since the inner core will not necessarily co-rotate with the mantle, to which our coordinate system refers. So, we must take

$$v = \Omega r_i \sin\theta \qquad \text{at } r = r_i \tag{3.5a}$$

$$v = 0 \qquad \text{at } r = r_o \tag{3.5b}$$

where Ω is the solid-body rotation rate of the inner core, and is to be determined as part of the solution.

Neglecting inertia as in (3.1), the condition that determines Ω is clearly that the total torque on the inner core vanishes (Gubbins, 1981). The integrated viscous torque is just

$$\epsilon 2\pi r_i^3 \int_0^\pi r \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\bigg|_{r=r_i} \sin^2\theta \, d\theta \tag{3.6}$$

and the electromagnetic torque is (Landau and Lifshitz, 1960)

$$2\pi r_i^3 \int_0^{\pi} B_\phi B_r \big|_{r=r_i} \sin^2\theta \, d\theta \qquad (3.7)$$

Neglecting any possible topographic torques, (3.5a) must thus be supplemented by the torque balance

$$\epsilon \int_0^{\pi} r \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\bigg|_{r=r_i} \sin^2\theta \, d\theta = -\int_0^{\pi} B_\phi B_r \big|_{r=r_i} \sin^2\theta \, d\theta \qquad (3.8)$$

to determine Ω.

It is through this electromagnetic coupling between the inner and outer cores that the inner core actively becomes involved in the outer core fluid flow. Note also that this coupling is entirely absent in either of the limits $\sigma = 0$ or $\sigma = \infty$: if $\sigma = 0$, $B_\phi|_{r=r_i} = 0$ from (2.13b); if $\sigma = \infty$, $B_r|_{r=r_i} = 0$ from (2.14a). The potential presence of this Lorentz torque is probably the single most important feature of a finitely conducting inner core. Furthermore it is the need to have a viscous torque to balance this torque that precludes the use of the linear friction introduced in Hollerbach and Ierley (1991).

According to (3.8), in the limit of vanishing viscosity $\epsilon \to 0$, one must have

$$\int_0^{\pi} B_\phi B_r \big|_{r=r_i} \sin^2\theta \, d\theta = 0 \qquad (3.9)$$

which is seen to be a second Taylor's constraint. As first noted by Taylor (1963), integrating (3.2a) over the vertical extent of the fluid shell, and applying the no-normal-flow boundary condition, results in Taylor's constraint

$$\int_{z_1}^{z_2} N(B, A) \, dz = 0 \qquad (3.10)$$

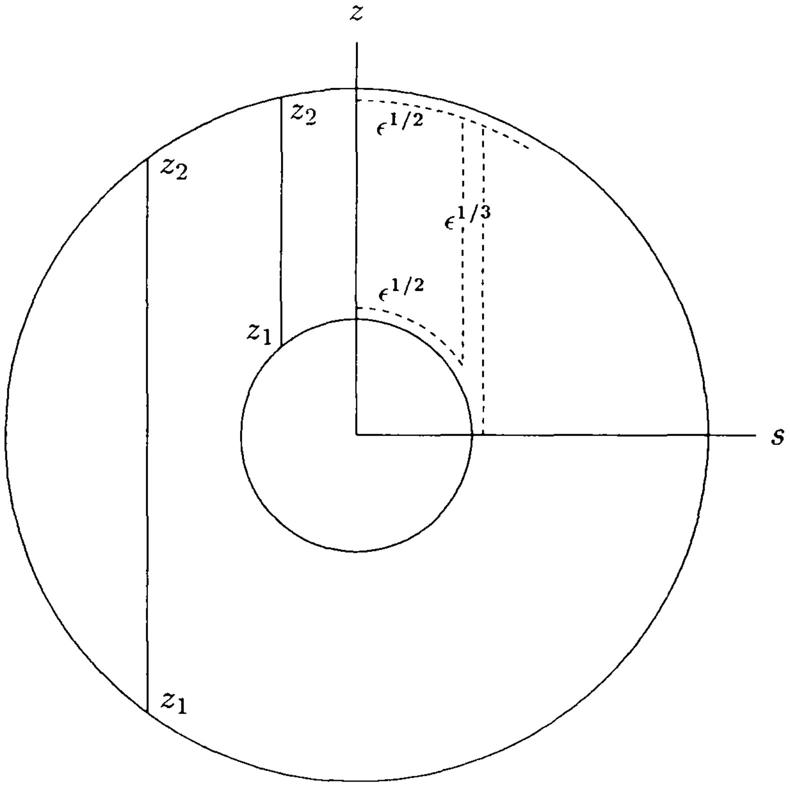

Fig. 1. On the left, the geostrophic contours in (3.10); on the right, a sketch of the boundary layer structure deduced by Proudman and by Stewartson.

(3.10) asserts that the integrated Lorentz torque on geostrophic contours, concentric cylindrical shells parallel to the axis of rotation, must vanish on each such cylinder simply because in the limit of vanishing viscosity there is nothing to balance it. (3.9) is an exactly analogous statement applied not to the geostrophic contours of the fluid outer core, but to the whole of the solid inner core, and one would expect that its importance in determining the magnetic field and fluid flow could be as profound as the more familiar constraint (3.10).

If (3.10) is not satisfied, the non-vanishing Lorentz torque will induce a geostrophic flow, consisting of these concentric cylindrical shells in solid-body (z-independent) rotation. Because this flow is opposed only by viscosity, only weak forcing is required to excite it to $O(1)$, and the resulting scaling of the field turns out to be $O(\epsilon^{1/4})$. Similarly, if (3.9) is not satisfied, the non-vanishing torque will induce a differential rotation of the inner core, which is really just a special case of a geostrophic flow, and the field again ought to scale as $O(\epsilon^{1/4})$. The field must thus adjust itself in such a way as to satisfy (3.9) and (3.10); if it cannot it remains viscously controlled.

4. Discussion of previous work

The earliest relevant work is that of Proudman (1956) and Stewartson (1966), who considered the fluid flow between two concentric spheres rotating about the same axis with almost the same angular velocity. The boundary layer structure they deduced consists of an Ekman layer of thickness $\epsilon^{1/2}$ on the inner and outer boundaries, and, on the particular geostrophic cylinder circumscribing the inner sphere, a rather intricate Stewartson layer of innermost thickness $\epsilon^{1/3}$. A sketch is shown in Fig. 1.

The addition of a magnetic field might be expected to complicate matters a great deal, considering that the presence of a magnetic field in a conducting fluid imparts a certain 'stiffness' to it. Gilman and Benton (1968) considered modified Ekman layers, and Ingham (1969) considered modified Stewartson layers, and found the structure to be largely unchanged for order one magnetic fields. In particular, the Ekman layers are still active in the sense that they control which, of an infinite number of possible geostrophic flows satisfying the interior (inviscid) equations, is actually selected; the Stewartson layers are still passive in the sense that they provide a return flow between the inner and outer Ekman layers, and smooth out various discontinuities, but do not control any aspects of the interior flow.

In a sphere (of unit radius), the asymptotic expansion for the particular geostrophic flow selected by the Ekman layer turns out to be

$$v_g = \epsilon^{-1/2} \frac{z_T^{1/2}}{2} \int_{-z_T}^{+z_T} N(B, A)\, \mathrm{d}z \qquad (4.1)$$

where $\pm z_T = \pm(1 - s^2)^{1/2}$ are the vertical boundaries of the geostrophic cylinder at cylindrical radius s (Tough and Roberts, 1968). It is the presence of this flow that limits the field to $O(\epsilon^{1/4})$ unless (3.10) is satisfied, since its contribution to the energy balance can be shown to be

$$-\epsilon^{-1/2} \int_0^1 \left[\frac{\mathrm{d}}{\mathrm{d}s} \left(s^2 \int_{-z_T}^{+z_T} B \frac{\partial A}{\partial z} \mathrm{d}z \right) \right]^2 z_T^{1/2} / s^3 \mathrm{d}s \le 0 \qquad (4.2)$$

and so it cannot act as a source, as first noted by Childress (1969). In the spherical shell under consideration here, the equivalent asymptotic expansion for the geostrophic flow has not been derived, and is anticipated to be considerably more complicated. One might, however, expect that its contribution to the energy balance would be similarly negative-semidefinite, vanishing only when (3.9) and (3.10) are both satisfied.

There is an energetic result one can prove for any ϵ, without recourse to the small ϵ asymptotic expansions. Taking the dot product of (2.2) with $\boldsymbol{B}$, and (3.1) with $\boldsymbol{U}$ and adding, one obtains

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} |\boldsymbol{B}|^2 &= \boldsymbol{B} \cdot [\nabla \times (\alpha \boldsymbol{B})] - |\nabla \times \boldsymbol{B}|^2 - \epsilon |\nabla \times \boldsymbol{U}|^2 \\ &\quad + \nabla \cdot [\boldsymbol{B} \times (\nabla \times \boldsymbol{B})] \\ &\quad + \nabla \cdot [\epsilon \boldsymbol{U} \times (\nabla \times \boldsymbol{U}) + (\boldsymbol{U} \times \boldsymbol{B}) \times \boldsymbol{B} - \boldsymbol{U} p] \end{aligned} \qquad (4.3)$$

Integrating (4.3) over the volume of the outer core, the divergence terms become surface integrals. The surface terms involving $\boldsymbol{U}$ clearly contribute nothing at the outer core boundary, where $\boldsymbol{U}=0$. At the inner core boundary they turn out to contribute $-\epsilon 16/3\pi r_i^3\Omega^2$, upon using (3.8). Taking into account also the magnetic energy in the inner core and external insulator, the surface terms involving only $\boldsymbol{B}$ turn out to yield a surface integral involving α at the inner core boundary. The final result is

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_V |\boldsymbol{B}|^2 \mathrm{d}V &= \int_{V_o} \boldsymbol{B}\cdot[\nabla\times(\alpha\boldsymbol{B})]\,\mathrm{d}V - \int \alpha B_\phi B_\theta \mathrm{d}S\,|_{r=r_i} \\ &\quad - \int_{V_o} |\nabla\times\boldsymbol{B}|^2 \mathrm{d}V - \int_{V_i} \sigma^{-1} |\nabla\times\boldsymbol{B}|^2 \mathrm{d}V \\ &\quad - \epsilon\int_{V_o} |\nabla\times\boldsymbol{U}|^2 \mathrm{d}V - \epsilon\tfrac{16}{3}\pi r_i^3\Omega^2 \qquad (4.4)\end{aligned}$$

where the volume V is all of space, V_i is the inner core, and V_o is the outer core. The only source is thus the kinematically prescribed α-effect. The Ohmic dissipation in the inner and outer cores, and the viscous dissipation are negative-definite. In particular, the zonal flow $v\hat{\boldsymbol{e}}_\phi$, of which the geostrophic flow is a part, once again cannot act as a source. If we had included some kinematically or dynamically prescribed buoyancy forcing in (3.1), the meridional circulation $\nabla\times(\psi\hat{\boldsymbol{e}}_\phi)$ could act as a source, but the zonal flow still could not.

5. Numerical solution

As noted above, the asymptotic expansion of the momentum equation in a spherical shell has not been derived. Instead, the direct numerical solution of Hollerbach (unpublished, 1992), explicitly resolving all boundary layers, is implemented. The discussion of the previous section is nevertheless important, not only in interpreting the final output, but also in guiding the numerical implementation. After all, it is the need to resolve the increasingly thin boundary layer structure that limits how small one can set ϵ, and it is the potential presence of the $O(\epsilon^{1/2})$ advective time-scale associated with the geostrophic flow that determines how small one must choose one's time-step. Thus, at least a rough idea of the asymptotic results is essential in developing a successful numerical solution.

Therefore, restricting attention to pure dipole solutions [a], for which A is symmetric and B is antisymmetric about the equator, in the inner core we expand the magnetic field as

$$A_i = \sum_{n=1}^{N_i}\sum_{m=1}^{M_i/2+1} a_{nm}^{(i)}(t) T_{2m-1}(x_i) P_{2n-1}^{(1)}(\cos\theta) \qquad (5.1a)$$

$$B_i = \sum_{n=1}^{N_i}\sum_{m=1}^{M_i/2+1} b_{nm}^{(i)}(t) x_i T_{2m-1}(x_i) P_{2n}^{(1)}(\cos\theta) \qquad (5.1b)$$

where the $P_l^{(1)}(\cos\theta)$ are associated Legendre functions, T are Chebyshev polynomials, and $x_i = r/r_i$ is the radial coordinate normalized to (0, 1). Note that the expansion in x_i automatically satisfies A_i, $B_i \to 0$ as $x_i \to 0$, and also the requirement that A_i be antisymmetric and B_i be symmetric in r.

In the outer core we expand the magnetic field as

$$A_o = \sum_{n=1}^{N_i}\sum_{m=1}^{M_i+2} a_{nm}^{(o)}(t) T_{m-1}(x_o) P_{2n-1}^{(1)}(\cos\theta) \qquad (5.2a)$$

$$B_o = \sum_{n=1}^{N_i}\sum_{m=1}^{M_i+2} b_{nm}^{(o)}(t) T_{m-1}(x_o) P_{2n}^{(1)}(\cos\theta) \qquad (5.2b)$$

[a] A referee has asked us to comment on the preference of dipole over quadrupole solutions. Apart from the observational fact that the Earth's magnetic field is predominantly dipolar, there is essentially no justification. Roberts (1972) has shown the linear kinematic eigenvalues for the onset of dynamo action to be similar for both parities, and Hollerbach (1991) has shown the subsequent non-linear evolution to be similar as well. We restrict attention to pure dipole solutions merely to reduce the computational complexity.

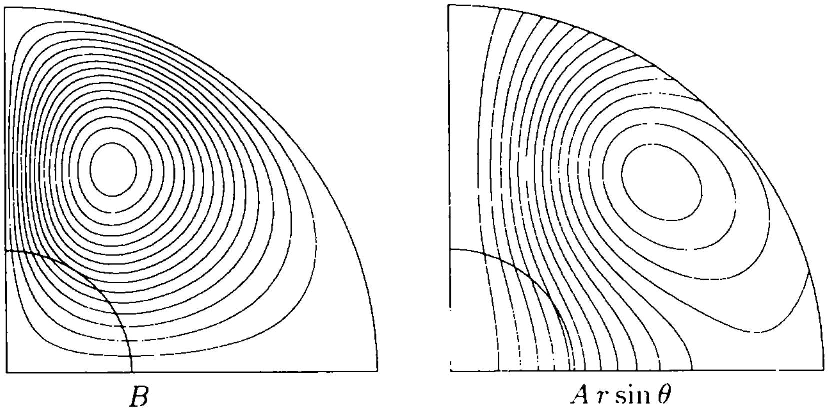

Fig. 2. The kinematic eigensolution at $\alpha_c = 5.15$.

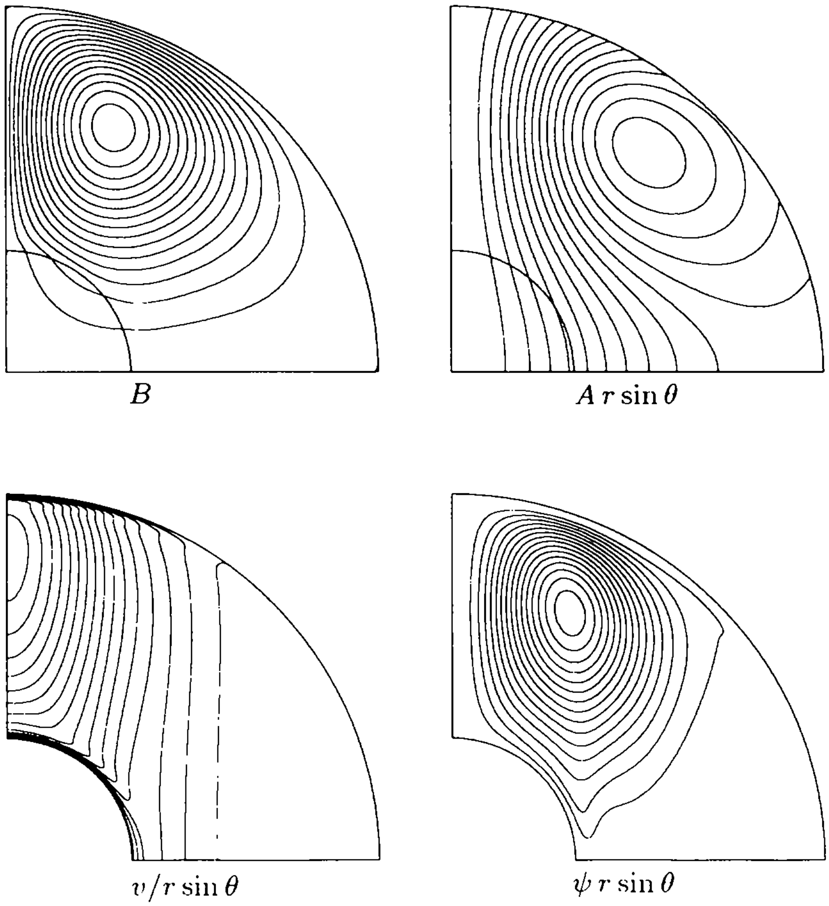

Fig. 3. Contour plots of the magnetic field and the associated angular velocity $v/r \sin\theta$ and meridional circulation $\psi r \sin\theta$ when $\alpha_0 = 5.5$. $B_{max} = 1.15$, $(Ar \sin\theta)_{max} = 0.21$, $(v/r \sin\theta)_{max} = -21.3$, $\Omega = 0.6$, $(\psi r \sin\theta)_{max} = -0.12$.

where

$$r = \frac{r_o + r_i}{2} + \frac{r_o - r_i}{2} x_o \qquad (5.3)$$

determines x_o, the radial coordinate normalized to $(-1, 1)$ across the gap. (Incidentally, we take $r_i = 1/2$, $r_o = 3/2$, for a radius ratio of 1/3.)

The induction eqns. (2.4) and (2.5) are then time-stepped by a second-order Runge–Kutta method, modified to treat the diffusive terms implicitly. Calculations in the angular coordinate are done pseudospectrally, in the radial coordinate by collocation. (2.4) is collocated at the $M_1/2$ zeros of $T_{M1+1}(x_i)$ on (0, 1), and (2.5) at the M_1 zeros of $T_{M_1}(x_o)$ on $(-1, 1)$. The 'extra' coefficients ($m = M_1/2 + 1$ in (5.1), and $m = M_1 + 1$, $M_1 + 2$ in (5.2)) are determined by the two matching conditions at $r = r_i$ and the boundary condition at $r = r_o$.

The outer core fluid flow is expanded as

$$\psi = \sum_{n=1}^{N_2} \sum_{m=1}^{M_2+4} \psi_{nm} T_{m-1}(x_o) P_{2n}^{(1)}(\cos\theta) \qquad (5.4a)$$

$$v = \sum_{n=1}^{N_2} \sum_{m=1}^{M_2+2} v_{nm} T_{m-1}(x_o) P_{2n-1}^{(1)}(\cos\theta) \qquad (5.4b)$$

and at each time-step of the induction equations the momentum equation (3.2) is solved as in Hollerbach (1992), explicitly resolving any Ekman or Stewartson layers that may develop.

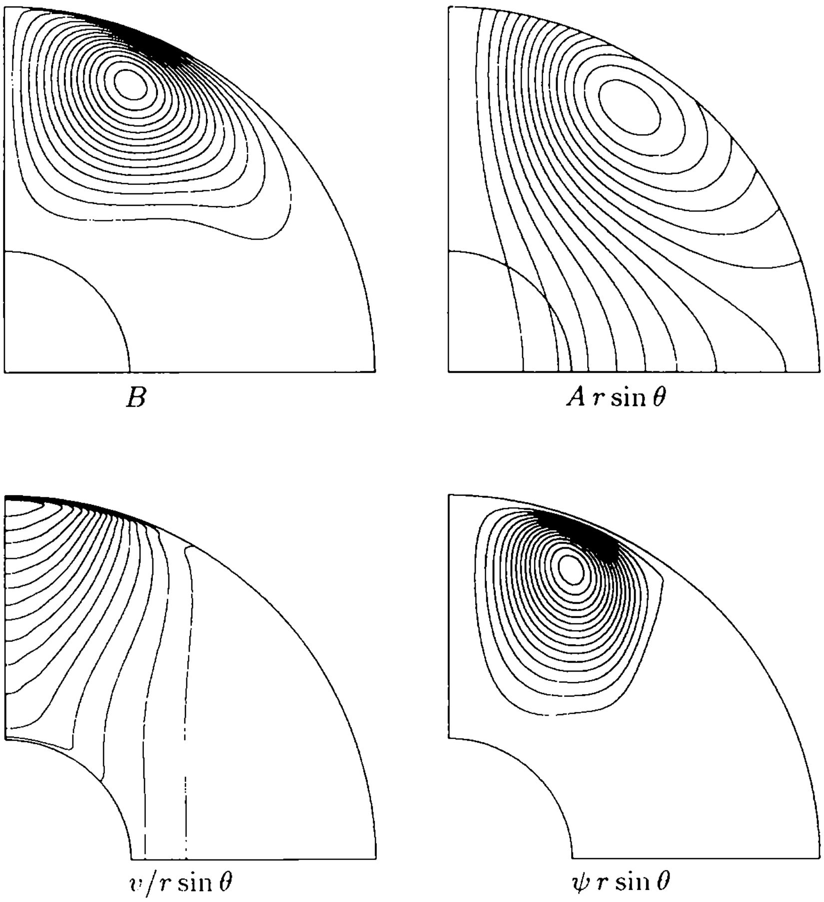

Fig. 4. Contour plots of the magnetic field and the associated angular velocity $v/r \sin\theta$ and meridional circulation $\psi r \sin\theta$ when $\alpha_0 = 8.0$. $B_{max} = 1.82$, $(Ar \sin\theta)_{max} = 0.35$, $(v/r \sin\theta)_{max} = -76.9$, $\Omega = -9.8$, $(\psi r \sin\theta)_{max} = -0.34$.

Finally, the truncation used in this work is $N_1 = 20$, $M_1 = 24$, $N_2 = 40$, $M_2 = 50$. The fluid flow $\boldsymbol{U}$ is thus seen to have twice the resolution of the magnetic field. The reason of course is that most of the boundary layer structure is anticipated to be in the flow rather than in the magnetic field, which after all is subject to order one Ohmic diffusion. This resolution turns out to be sufficient to reduce the viscosity to $\epsilon = 2.5 \times 10^{-4}$. Even this is not as small as one would like, certainly nowhere near geophysically realistic values, but it is sufficient to see the emergence of the boundary layer structure.

6. Results

The choice of α we will consider is $\alpha = \alpha_0 \cos\theta$, uniformly distributed radially across the gap. The critical value for the onset of dynamo action is $\alpha_c = 5.15$. Figure 2 shows the infinitesimal kinematic eigensolution, which does not satisfy either of the constraints (3.9) or (3.10). In the slightly supercritical regime one would thus expect there to be a significant geostrophic flow, including a differential rotation of the inner core. Figure 3 shows the equilibrated field and the associated flow at $\alpha_0 = 5.5$. The zonal flow is indeed dominated by the z-independent geostrophic flow, and the angular velocity of the inner core is a local maximum, indicating that the electromagnetic torque is actively driving it and thereby also a portion of the outer core flow. If the inner core were merely passively responding to the outer core flow, its angular velocity could hardly be greater than that of all of the immediately surrounding fluid.

That the kinematic eigensolution fails to satisfy either of the two Taylor's constraints is hardly surprising, since it is independent of any dynamical considerations of the momentum equation. However, in the increasingly supercritical regime, as the non-linear coupling between the flow and the field becomes more and more important, it is conceivable, as postulated by Malkus and Proctor (1975), that the field will evolve into a configuration where the constraints are satisfied. Figure 4 shows the equilibrated field and the associated flow at $\alpha_0 = 8.0$. The zonal flow is no longer dominated by the geostrophic flow, and the angular velocity of the inner core is not a local maximum. The field is clearly evolving into a configuration where (3.9) is satisfied; the electromagnetic torque $\tau = -0.005$ at $\alpha_0 = 8.0$ is in fact less than the torque $\tau = 0.017$ at $\alpha_0 = 5.5$, even though the field is greater.

TABLE 1
A detailed balance of the energy eqn. (4.4) for the two cases $\alpha_0 = 5.5$ and 8.0

	$\alpha_0 = 5.5$	$\alpha_0 = 8.0$
$\int_{V_o} \boldsymbol{B}\cdot[\nabla\times(\alpha\boldsymbol{B})]\,\mathrm{d}V$	91.06389	311.74996
$-\int \alpha B_\phi B_\theta\,\mathrm{d}S\big\|_{r=r_i}$	0.64477	−0.22479
$-\int_{V_o} \|\nabla\times\boldsymbol{B}\|^2 \mathrm{d}V$	−90.71815	−300.52757
$-\int_{V_i} \sigma^{-1} \|\nabla\times\boldsymbol{B}\|^2 \mathrm{d}V$	−0.26102	−0.02390
$-\epsilon\int_{V_o} \|\nabla\times\boldsymbol{U}\|^2 \mathrm{d}V$	−0.72926	−10.92355
$-\epsilon\frac{16}{3}\pi r_i^3\Omega^2$	−0.00019	−0.05041
Total	+0.00004	−0.00026

Note that the totals do indeed add up to zero, as they must for converged steady-state solutions.

It is apparent that the field is satisfying (3.9) by expelling the toroidal component $B\hat{\boldsymbol{e}}_\phi$ from the inner core entirely. It is in fact adjusting to minimize the Ohmic dissipation in the inner core, which requires B to be zero and A to match to an internal potential field. This would suggest, incidentally, that if one wishes to neglect the finite conductivity of the inner core, the limit $\sigma = 0$ would be more appropriate than the limit $\sigma = \infty$. Table 1 shows the contributions of the six terms that make up the right-hand side of the energy eqn. (4.4). Aside from being a useful check on the numerical implementation, it clearly demonstrates how the field is evolving to minimize the Ohmic dissipation in the inner core. Although the Ohmic dissipation in the outer core increases by a factor of three, the dissipation in the inner core decreases by a factor of ten. The dominant balance in both cases is between the α-effect forcing and the Ohmic dissipation in the outer core. Note also the considerable enhancement of the viscous dissipation as the degree of supercriticality is increased.

On reflection, it is perhaps not surprising that the field should adjust to the constraint (3.9) by minimizing the Ohmic dissipation in the inner core — the field in the inner core can be maintained against dissipation only by diffusion across the core boundary, and the outer core field is apparently incapable of doing so without simultaneously generating a non-vanishing torque and thereby a strong differential rotation. The drain on the magnetic energy produced by that differential rotation is unsustainable, and so the field in the inner core simply dies away for lack of support, leaving only a dissipationless internal potential field behind.

7. Conclusion

In this work we have addressed a number of consequences of the finite conductivity of the Earth's inner core, and how it affects the dynamo processes in the outer core. Most importantly, the electromagnetic coupling between finitely conducting inner and outer cores leads to a Lorentz torque on the inner core. Because this torque is opposed only by viscosity, even a weak torque can easily lead to an $O(1)$ differential rotation of the inner core, just as even a weak torque on the geostrophic contours of the outer core can easily lead to an $O(1)$ geostrophic flow. To prevent an infinite response, in the limit of vanishing viscosity these integrated torques must then vanish, leading to the two Taylor's constraints (3.9) and (3.10). In accordance with the Malkus–Proctor hypothesis, in the supercritical regime the field is seen to adjust in such a way that they do tend to vanish. The adjustment occurs in such a way that the Ohmic dissipation in the inner core is minimized.

In the slightly supercritical regime, where this adjustment has not yet fully taken place, the field is expected to scale as $O(\epsilon^{1/4})$, whereas in the more strongly supercritical regime, where it has taken place, the field is expected to scale as $O(1)$. On the basis of previous work (Hollerbach and Ierley, 1991) exploring the limit of asymptotically small viscosity, the transition typically occurs at perhaps 10% supercritical, and so one would expect our solutions at $\alpha_0 = 5.5$ and $\alpha_0 = 8.0$ to fall into these two regimes, respectively. However, this previous work also demonstrated that to achieve the asymptotic limit conclusively requires viscosities considerably smaller than the $\epsilon = 2.5 \times 10^{-4}$ used here. It is thus not possible to verify the scalings of the field, and indeed the field amplitudes shown here are quite similar.

Nevertheless, the qualitative features of the adjustment, and the consequences for the flow, can clearly be seen. At $\alpha_0 = 5.5$ the zonal flow is dominated by the geostrophic flow; at $\alpha_0 = 8.0$ it is not. As part of the adjustment process, the inner Ekman layer also seems to become much less conspicuous. Note also that in both cases the structure in the zonal flow induces a certain structure in the toroidal field, despite the order one Ohmic diffusion: for $\alpha_0 = 5.5$ at the inner Ekman layer; for $\alpha_0 = 8.0$ at the outer Ekman layer. Gubbins (1981) provided a simple example demonstrating analytically how a discontinuity in the zonal flow will induce just such 'kinks' in the toroidal field. Finally, note the lack of any appreciable Stewartson layer in the meridional circulation; evidently the 'stiffening' effect of the field quite effectively suppresses such boundary layers.

One final point worth commenting on concerns the particular choice of α used here; after all, one could no doubt expend considerable effort exploring other choices. Not suprisingly, the magnetic field tends to be concentrated where α is largest. Therefore, by concentrating α at the inner core boundary one could presumably enhance the effect of the inner core, whereas by concentrating α at the outer core boundary one could diminish it. However, in view of the fact that α is just a parameterization of the effects of the small-scale convective motions, that seems rather pointless. It is much more sensible to proceed to compute these convective motions and thereby produce a self-consistent dynamo, which is the ultimate objective of this project. The potential significance of the inner core for the real geodynamo should then emerge from that calculation.

Acknowledgements

We thank Andrew Soward and Paul Roberts for valuable discussions on the dynamics of the inner core. This work was funded by the Science and Engineering Research Council under grant number GR/E93251.

References

Abramowitz, M. and Stegun, I.A. (Editors), 1968. Handbook of Mathematical Functions. Dover, New York.

Braginsky, S.I., 1963. Structure of the F layer and reasons for convection in the Earth's core, Sov. Phys. Dokl., 149: 8–10.

Busse, F.H., 1991. Problems of planetary dynamo theory. In: E.R. Priest and A.W. Hood (Editors), Advances in Solar System Magnetohydrodynamics. Cambridge University Press, Cambridge, pp. 51–58.

Childress, S., 1969. A class of solutions of the magnetohydrodynamic dynamo problem. In: S.K. Runcorn (Editor), The Application of Modern Physics to the Earth and Planetary Interiors. Wiley, London, pp. 629–648.

Cowling, T.G., 1934. The magnetic field of sunspots. Mon. Not. R. Astron. Soc., 94: 39–48.

Gilman, P.A. and Benton, E.R., 1968. Influence of an axial magnetic field on the steady linear Ekman boundary layer. Phys. Fluids, 11: 2397–2401.

Gubbins, D., 1981. Rotation of the inner core. J. Geophys. Res., B86: 11695–11699.

Hollerbach, R., 1991. Parity coupling in α^2-dynamos. Geophys. Astrophys. Fluid Dyn., 60: 245–260.

Hollerbach, R. and Ierley, G.R., 1991. A modal α^2-dynamo in the limit of asymptotically small viscosity. Geophys. Astrophys. Fluid Dyn., 56: 133–158.

Ingham, D.B., 1969. Magnetohydrodynamic flow in a container. Phys. Fluids, 12: 389–396.

Landau, L.D. and Lifshitz, E.M., 1960. Electrodynamics of Continuous Media. Pergamon, Oxford.

Loper, D.E. and Roberts, P.H., 1983. Compositional convection and the gravitationally powered dynamo. In: A.M. Soward (Editor), Stellar and Planetary Magnetism. Gordon and Breach, New York, pp. 297–327.

Malkus, W.V.R. and Proctor, M.R.E., 1975. The macrodynamics of α-effect dynamos in rotating fluids. J. Fluid Mech., 67: 417–443.

Proctor, M.R.E., 1977. Numerical solutions of the nonlinear α-effect dynamo equations. J. Fluid Mech., 80: 769–784.

Proudman, I., 1956. The almost-rigid rotation of viscous fluid between concentric spheres. J. Fluid Mech., 1: 505–516.

Roberts, P.H., 1972. Kinematic dynamo models. Philos. Trans. R. Soc. London, Ser. A, 272: 663–698.

Soward, A.M. and Jones, C.A., 1983. α^2-dynamos and Taylor's constraint. Geophys. Astrophys. Fluid Dyn., 27: 87–122.

Steenbeck, M., Krause, F. and Rädler, K.-H., 1966. A calculation of the mean emf in an electrically conducting fluid in turbulent motion, under the influence of Coriolis forces. Z. Naturforsch., A21: 369–376.

Stewartson, K., 1966. On almost rigid rotations. J. Fluid Mech., 26: 131–144.

Taylor, J.B., 1963. The magnetohydrodynamics of a rotating fluid and the Earth's dynamo problem. Proc. R. Soc. London, Ser. A, 274: 274–283.

Tough, J.G. and Roberts, P.H., 1968. Nearly symmetric hydromagnetic dynamos. Phys. Earth Planet Inter., 1: 288–296.