MATH-103501

This question paper consists of 3 printed pages, each of which is identified by the reference MATH-1035

Only approved basic scientific calculators may be used.

Examination for the Module MATH-1035

(June 2007)

Analysis

Time allowed: 3 hours

Attempt two questions from Section A, and two questions from Section B. All questions carry equal marks.

In this paper, the symbols $\mathbb{N}$,$\mathbb{Z}$,$\mathbb{Z}_n$,$\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the sets of natural numbers, integers, integers mod n, real numbers and complex numbers. "ln" denotes the natural logarithm.

If a question says "determine", "find", "prove" or "show", you will only get full marks if you give full reasons for your answers, including proofs if appropriate. If a question says "write down", a correct answer with no explanation is enough for full marks.

Section A.

1. (a) Perform the complex number operations:

   i. $\frac{1+i}{1+2i}$.

   ii. $(\frac{1+i}{1+2i})^3$.

   (b) Prove by Mathematical Induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in \mathbb{N}$.

   (c) Write down the polar forms for the five complex numbers z satisfying $z^5 = 32$. Draw their positions on the Argand diagram.

   (d) Show that the sequence $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing, and bounded below by zero. What is its limit?

2. (a) Use Mathematical Induction to prove that for all $n \in \mathbb{N}$, the sum $\sum_{i=1}^{n} i(i+1)$ is equal to $\frac{1}{3}n(n+1)(n+2)$.

   (b) On $\mathbb{R}$, define an operation $*$ by

   $$a * b = 2a + 2b.$$

   Show that the operation $*$ is commutative but not associative.

   (c) Give an example of an operation $*$, on a set S of your choice, which is associative but not commutative. Explain briefly why your example has these properties.

(d) Write down an English sentence, as simple and understandable as possible, which is equivalent to the mathematical statement "$(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[y < x]$". Is this statement true or false? Why is that?

(e) Write down a mathematical sentence (no English words allowed) expressing the (true) statement that every complex number has a complex square root.

3. (a) Use Euclid's algorithm, showing your working, to find the greatest common divisor of 157 and 54.

(b) Use your working in part (a) to express g.c.d.(157,54) in the form $157a + 54b$ $(a, b \in \mathbb{Z})$.

(c) Use Euclid's algorithm, showing your working, to find the greatest common divisor of the real polynomials $X^3 - 5X + 2$ and $X^3 - X^2 - X - 2$.

(d) Showing your working, solve the following linear equations in the field $\mathbb{Z}_7$ of integers mod 7:

$$\begin{aligned} 2x \quad +y \quad +z &= 0 \pmod{7} \\ x \quad + \quad 3z &= 3 \pmod{7} \\ x \quad -y \quad +5z &= 0 \pmod{7}. \end{aligned}$$

Give answers $x, y, z \in \{0, 1, 2, 3, 4, 5, 6\}$ - and check that they are correct! [*Hint: it may be helpful to write down a complete multiplication table for $\mathbb{Z}_7$, which should not take you very long.*]

Section B.

4. (a) (i) State the *Completeness axiom* as applied to the real numbers.

(ii) Write down, if they exist, the *least upper bound, the greatest lower bound, the maximum element and the minimum element* of the set

$$\{1 - \frac{2}{n} - \frac{3}{n^2} : n \in \mathbb{Z}^+\}.$$

(b) (i) Write down the value of $\lim_{n\to\infty}(\frac{n^2 + 2n + 3}{4n^2 + 5n + 6})^3$.

(ii) Determine the value of $\lim_{n\to\infty}(\frac{n+1}{n})^{3n+2}$. (Show your working.)

(c) (i) Let $\{a_n\}$ be an infinite sequence. Is the following a correct definition of $\lim_{n\to\infty} a_n = l$ correct? If not, write down a correct version.

$\lim_{n\to\infty} a_n = l$ if and only if for each positive real number N there exists a real number ϵ such that $|a_N - l| < \epsilon$.

Continued ...

(ii) Using the correct (N, ϵ) definition of limit, prove *from first principles* that

$\lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3}$

(d) Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

Prove that $\lim_{n\to\infty} a_n b_n = ab$.

5. (a) Use the partial fraction method to prove that the series $\sum_{n=2}^{\infty} \frac{2}{n(n+2)}$ is convergent. To what value does it converge?

(b) Prove that *if* the series $\sum_{n=1}^{\infty} a_n$ is convergent *then* $\lim_{n\to\infty} a_n = 0$.

Write down an example which shows that the converse of this statement is false.

(c) Determine the convergence or divergence of each of the following series, stating the tests you use.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2n}{3+4n}$; (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2n}{3+4n+5n^2}$; (iii) $\displaystyle\sum_{n=10}^{\infty} \frac{(-1)^n}{n^3-1}$; (iv) $\displaystyle\sum_{n=3}^{\infty} \frac{(n!)^2}{(2n)!}$.

(d) Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{5^n}$.

6. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a functions and let $a \in \mathbb{R}$. Give the definition of:

f is differentiable at a.

Prove that, if f is differentiable at a, then f is continuous at a.

Give an example of a function which is continuous at all $a \in \mathbb{R}$ but is not differentiable at 1.

(b) State (A) Rolle's Theorem and (B) the Mean Value Theorem. Using Rolle's Theorem (without proof) give a proof of the Mean Value Theorem.

(c) Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - 4x + \sin x - 1$. Show that the equation $f(x) = 0$ has a solution in the closed interval [0,3]. State any general theorems that your argument uses.

(d) Find the limit, if it exists, of:

$\lim_{x\to 0} \dfrac{\ln(1+2x)}{\sin 3x}$. Give reasons for your answer.

End.