Amenable and weakly amenable Banach algebras with compact multiplication

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Abstract. We investigate amenable and weakly amenable Banach algebras with compact multiplication. Any amenable Banach algebra with compact multiplication is biprojective. As a consequence, every semisimple such algebra which has the approximation property is a topological direct sum of full matrix algebras. In the radical case no such structure theorem is at hand. We also investigate Banach algebras which have a bounded approximate identity consisting of normalized powers of an element x. Any such Banach algebra is either unital or radical; if the algebra is also generated by x, it is weakly amenable. We construct a radical example with compact multiplication which moreover is an integral domain. This furnishes a new example of a commutative, weakly amenable, non-amenable, radical Banach algebra.

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0. Introduction

In many aspects, amenability can be thought of as a finiteness condition. For instance, a locally compact group G is finite if and only if every closed cofinite ideal of $L^1(G)$ has an identity, but is amenable if and only if each such ideal has a bounded approximate identity. Also the finite-dimensional, semisimple Banach algebras $\mathfrak{A}$ are "almost" characterized by the fact that $H^1(\mathfrak{A}, E) = \{0\}$ for each Banach $\mathfrak{A}$-bimodule E, where "almost" means that no counterexample has been found for the last quarter of a century and that if one existed it would have to be very pathological in regard to its Banach space geometry, see [31], [35]. The amenable Banach algebras $\mathfrak{A}$ are defined to satisfy the weaker demand that $H^1(\mathfrak{A}, E^*) = \{0\}$ for each Banach $\mathfrak{A}$-bimodule E. This latter definition has turned out to be an extremely fruitful concept in the theory of Banach algebras: it is strong enough to allow the development of a rich theory, yet weak enough to encompass a wide range of important examples. For instance, the group algebra of a locally compact group is amenable if and only if the group is amenable [21], and a C^*-algebra is amenable if and only if it is nuclear [10, 18]. To characterize the amenable Banach algebras within a given class of Banach algebras is an active area of research: it is still undetermined for which Banach spaces E, the algebra $\mathcal{K}(E)$ of compact operators on E is amenable [17], and for several years it was an open problem as to whether a radical, amenable Banach algebra existed. An example of such an algebra, non-commutative, was finally presented in [29]. At the time this paper was written, no commutative example was known, but see the final remark in §5.

Another way of imposing a finiteness condition on a Banach algebra $\mathfrak{A}$ is to demand that it has compact multiplication, that is, for each $a \in \mathfrak{A}$ the operators L_a and R_a of multiplication by a from the left and from the right, respectively, are compact. (This condition is also known as the algebra being completely continuous, see §8.7 of [26].) It is therefore reasonable to hope that if amenability and compact multiplication team up, then the resulting class of Banach algebras should have a rich and strong structure theory. As we shall see in this paper, this is indeed true in the semisimple case (under an additional, but mild hypothesis).

The interplay of compactness of multiplication and amenability in the commutative semisimple case has been discussed in [37] and [24]. Here we are interested in the radical case – whether an amenable, radical, commutative Banach algebra with compact multiplication can exist. Should this be so, such an algebra would have to display novel and interesting structural phenomena. Although we are so far unable to construct such an algebra, we construct (§4) a *weakly* amenable, radical, commutative Banach algebra with compact multiplication. The only previously known examples of weakly amenable, radical, commutative Banach algebras were given in [11], and were obtained as quotients $I(E)/\overline{J(E)}$ for certain sets E not of synthesis in $L^1(\mathbb{R})$. A direct attempt to obtain an amenable example in this manner will fail – if I is a closed amenable ideal in $L^1(G)$ for some locally compact abelian group G, then $\operatorname{hull}(I)$ is necessarily of synthesis, [23]. We show that under suitable conditions our construction yields an integral domain, in which case our example is of necessity not amenable, and is different from the the earlier example. However, it is conceivable that an appropriate modification of the construction may yield an amenable algebra.

The example in §4 was not constructed with amenability in mind. Rather, it was prompted by the idea that, for quasinilpotent x, analytic properties of the sequence $(\|x^n\|)$ might be related to algebraic properties of x.

Such a relationship can be seen in the radical Banach algebras of power series $\ell^1(\mathbb{N},\omega)$. It is shown in [4] that for $x \in \ell^1(\mathbb{N},\omega)$, the map $y \mapsto xy$ $\bigl((y \in \ell^1(\mathbb{N},\omega)\bigr)$ being compact implies that $\lim_{n\to\infty} \|x^{n+1}\|/\|x^n\| = 0$ and that this occurs if and only if x belongs to a certain standard ideal of $\ell^1(\mathbb{N},\omega)$. Subsequently, it was shown in [39] that for x in the Volterra algebra $L^1(0,1)$, we have either (1) $\lim_{n\to\infty} \|x^{n+1}\|/\|x^n\| = 0$ or (2) $\limsup_{n\to\infty} \|x^{n+k}\|/\|x^n\| > 0$ for all k. This answered a question of Dales and, interestingly, the proof relies on knowing the ideals in the Volterra algebra. Later, it was shown in [34] that in fact (1) holds for every x in the Volterra algebra and the proof relies on the existence of a non-zero derivation on $L^1(0,1)$. This use of algebraic techniques to prove what are essentially analytic facts is an intriguing refinement of the spectral radius formula.

Now multiplication in the Volterra algebra is compact and so a natural question is whether there can be an algebra $\mathfrak{A}$ with compact multiplication such that, for some $x \in \mathfrak{A}$, $x^{n+1}/\|x^n\| \not\to 0$. Compactness of multiplication implies that the subset $\{x^n/\|x^n\|\}$ of the multiplier algebra of $\mathfrak{A}$ is pre-compact with respect to the strong operator topology. Hence, if $x^n/\|x^n\| \not\to 0$ this set has non-zero accumulation points. The extreme case would be for the identity to be a strong-operator accumulation point of $\{x^n/\|x^n\|\}$ and it is just such an example which is constructed in §4. That algebras having this property are weakly amenable follows by recasting the argument in [34]. These algebras may therefore play an important role in the further exploration of the connection between analytic properties of

($\|x^n\|$) and algebraic properties of x.

1. Preliminaries

Let $\mathfrak{A}$ be a Banach algebra, E a Banach $\mathfrak{A}$-bimodule. Then E^* is a Banach $\mathfrak{A}$-bimodule under the actions :

$$\langle a\cdot m,\xi\rangle=\langle m,\xi\cdot a\rangle,\langle m\cdot a,\xi\rangle=\langle m,a\cdot\xi\rangle \qquad (a\in\mathfrak{A},\xi\in E,m\in E^*)\,.$$

A *derivation* $D:\mathfrak{A}\to E$ is a (bounded) linear map such that

$$D(ab)=D(a)\cdot b+a\cdot D(b) \qquad (a,b\in\mathfrak{A})\,.$$

The derivation D is *inner* if it is of the form $a\mapsto a\cdot\xi-\xi\cdot a$ for some $\xi\in E$. The cohomology space $H^1(\mathfrak{A},E)$ is the quotient of the space of derivations by the inner derivations, and in many situations triviality of this space is of considerable importance. In particular, $\mathfrak{A}$ is *contractible* if, for every Banach $\mathfrak{A}$-bimodule E, $H^1(\mathfrak{A},E)=\{0\}$, *amenable* if, for every Banach $\mathfrak{A}$-bimodule E, $H^1(\mathfrak{A},E^*)=\{0\}$, and *weakly amenable* if $H^1(\mathfrak{A},\mathfrak{A}^*)=\{0\}$. In the case that $\mathfrak{A}$ is commutative, then weak amenability is equivalent to every derivation into a commutative bimodule being zero [3, Theorem 1.5]. In the finite dimensional case amenability and contractibility clearly coincide, and are the same as semisimplicity. From the point of view of the structure theorem in §3 below, such an algebra is a finite direct sum $\bigoplus_k M_{n_k}$ of full matrix algebras.

Note that a Banach algebra $\mathfrak{A}$ is amenable if and only if it has an *approximate diagonal*, that is, a bounded net $(m_i)\subset\mathfrak{A}\hat{\otimes}\mathfrak{A}$ such that for each $x\in\mathfrak{A}$, $m_ix-xm_i\to 0,\pi(m_i)x\to x$, where $\pi:\mathfrak{A}\hat{\otimes}\mathfrak{A}\to\mathfrak{A}$ is the natural product map [8, Theorem 43.9]. Analogously, $\mathfrak{A}$ is contractible if and only if it possess a diagonal in the obvious sense. The Banach algebra $\mathfrak{A}$ is *biprojective* if the product map π has a bounded right $\mathfrak{A}$-bimodule inverse. Biprojective plus unital is equivalent to contractible.

A *weight sequence* is a sequence $(\omega_n)_{n\geq 1}$ of real numbers such that

$$\begin{aligned}&\omega_n>0 \text{ for all } n\geq 1;\\ &\omega_{n+m}\leq\omega_n\,\omega_m \text{ for all } n,m\geq 1;\\ &\omega_n^{1/n}\to 0 \text{ as } n\to\infty\,.\end{aligned}$$

Given such, the sequence space

$$\ell^1(\omega_n)=\big\{(x_n):\sum_{n=1}^{\infty}|x_n|\omega_n<\infty\big\} \tag{1.1}$$

is a radical Banach algebra under convolution. Conversely, if x is a non-nilpotent, quasinilpotent element of a Banach algebra, then $\omega_n = \| x^n \|$ gives a weight sequence. Properties of such weight sequences in the case of the Volterra algebra are considered in [2], [39].

Much work has been done with algebras of the form (1.1) concerning the interplay between properties of the weight sequence and the ideal structure of the algebra, for example [5, 36], and more recently with homological dimension of the algebra, [33]. One notion that has proved useful is the following. The weight sequence (ω_n) is *regulated at* p, [4], if

$$\omega_{n+p}/\omega_n \to 0 \qquad \text{as } n \to \infty.$$

As noted earlier, in the case of an algebra such as (1.1), it is shown in [5] that multiplication by an element x is compact if and only if $(\| x^n \|)$ is regulated at 1, and for any element x of the Volterra algebra, $(\| x^n \|)$ is regulated at 1, [34].

2. Banach algebras with approximate identities of normalized powers

In §4 below we construct a radical commutative Banach algebra $\mathfrak{A}$ which in particular is generated by an element x for which there is a sequence $(n_k) \subset \mathbb{N}$ such that $(x^{n_k}/\| x^{n_k} \|)$ is a bounded approximate identity for $\mathfrak{A}$. This latter property trivially holds in a unital Banach algebra, we show in this section that in the absence of a unit it necessitates that $\mathfrak{A}$ be radical. But first a result to show the usefulness of the notion.

Theorem 2.1. *Let $\mathfrak{A}$ be a commutative Banach algebra generated by an element x for which there is a sequence $(n_k) \subset \mathbb{N}$ such that $(x^{n_k}/\| x^{n_k} \|)$ is a bounded approximate identity for $\mathfrak{A}$. Then $\mathfrak{A}$ is weakly amenable.*

Proof. The underlying idea is the same as that of the Proposition of [34]. Let $D : \mathfrak{A} \to \mathfrak{A}^*$ be a derivation. By Cohen's factorization theorem, D takes its values in the essential part of $\mathfrak{A}^*$, and consequently $(x^{n_k}/\| x^{n_k} \|)Dy \to Dy$ for each $y \in \mathfrak{A}$. With this in mind, note that

$$\frac{1}{n_k+1} D\left(\frac{x^{n_k+1}}{\| x^{n_k} \|}\right) = \frac{x^{n_k}}{\| x^{n_k} \|} Dx \qquad (k \in \mathbb{N}).$$

Letting $k \to \infty$, it follows that $Dx = 0$. Since $\ker D$ is a closed subalgebra of $\mathfrak{A}$, and since x generates $\mathfrak{A}$, this means that $D = 0$. □

Remark. The same argument works for a derivation of $\mathfrak{A}$ into any commutative module, however, this property is well known to be equivalent to weak amenability, [3]. Although we know that $\mathfrak{A}$ is not always amenable (see §5), there is always a sequence $(d_k)_{k=1}^{\infty}$ in $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ that is almost — except for one property — an approximate diagonal for $\mathfrak{A}$. To see this, for $k \in \mathbb{N}$, let

$$d_k = \frac{1}{n_k - 1} \sum_{j=1}^{n_k - 1} \frac{x^j \otimes x^{n_k - j}}{\| x^{n_k} \|} . \tag{2.1}$$

Then $(d_k)_{k=1}^{\infty} \subset \mathfrak{A}\hat{\otimes}\mathfrak{A}$, and if y is a polynomial in x, $y\pi(d_k) \to y$ and

$$y.d_k - d_k.y = \frac{1}{n_k - 1} \left(\frac{x^{n_k} \otimes y - y \otimes x^{n_k}}{\| x^{n_k} \|} \right) \to 0 .$$

Thus the only property from the definition of an approximate diagonal which, in general, is not clear, is the boundedness of the sequence $(d_k)_{k=1}^{\infty}$ (which is also needed to extend these limit results to all $y \in \mathfrak{A}$). It certainly is bounded if $\mathfrak{A}$ is a uniform algebra (a non-trivial example of such an algebra will soon be given), so that in this case we not only have weak amenability, but amenability. By Sheĭnberg's result on the triviality of amenable, uniform algebras [30], this means that Theorem 2.1 does not help in order to solve the open problem of whether there exists a non-trivial weakly amenable, uniform Banach algebra.

Lemma 2.2. *Let $\mathfrak{A}$ be a Banach algebra with an approximate identity of normalized powers. Suppose that $\theta : \mathfrak{A} \to \mathfrak{B}$ is a contractive homomorphism into another Banach algebra $\mathfrak{B}$, and that θ has dense range. Then $\mathfrak{B}$ also has an approximate identity of normalized powers.*

Proof. Let $(x^{n_k} / \| x^{n_k} \|)$ be a bounded approximate identity for $\mathfrak{A}$. It is clear that $(\theta(x)^{n_k} / \| x^{n_k} \|)$ is a bounded approximate identity for $\theta(\mathfrak{A})$, and hence for $\mathfrak{B}$. In particular, $\theta(x) \neq 0$, and

$$1 \geq \liminf \frac{\| \theta(x)^{n_k} \|}{\| x^{n_k} \|} \geq 1 ,$$

whence $\| x^{n_k} \| / \| \theta(x)^{n_k} \| \to 1$. Thus $(\theta(x)^{n_k} / \| \theta(x)^{n_k} \|)$ is a bounded approximate identity for $\mathfrak{B}$. ∎

Theorem 2.3. *Let $\mathfrak{A}$ be a commutative Banach algebra with an approximate identity of normalized powers. Then $\mathfrak{A}$ is either unital or radical.*

Proof. Suppose that $\mathfrak{A} \neq \operatorname{rad}(\mathfrak{A})$. We first show that $\mathfrak{A}/\operatorname{rad}(\mathfrak{A})$ is unital. Take θ to be the Gelfand map, $\mathfrak{B}$ to be the uniform closure of $\theta(\mathfrak{A})$ in $C_0(X)$, X the locally compact maximal ideal space of $\mathfrak{A}$. By Lemma 2.2 there is $f \in \mathfrak{B}$, $(n_k) \subset \mathbb{N}$ such that $(f^{n_k}/\|f^{n_k}\|)$ is a bounded approximate identity for $\mathfrak{B}$. Assuming that X is not compact, let $x_0 \in X$ be such that $|f(x_0)| = \|f\|$ and choose $x_1 \in X$ such that $|f(x_1)| < \|f\|$. Then we have

$$\frac{f^{n_k}(x_1)}{\|f^{n_k}\|} = \frac{f^{n_k}(x_1)}{|f^{n_k}(x_0)|} \to 0,$$

which is absurd, since $(f^{n_k}/\|f^{n_k}\|)$ is a bounded approximate identity for $\mathfrak{B}$. Thus X is compact, that is, $\mathfrak{B}$, and hence $\mathfrak{A}/\mathrm{rad}(\mathfrak{A})$, is unital.

Thus there is a non-zero idempotent $e \in \mathfrak{A}$ which is an identity mod $\mathrm{rad}(\mathfrak{A})$. Supposing that e is not an identity, so $\mathfrak{A} \neq e\mathfrak{A}$, the map $\psi : a + e\mathfrak{A} \mapsto a(1-e)$ is an topological isomorphism of $\mathfrak{R} = \mathfrak{A}/e\mathfrak{A}$ with the non-zero algebra $\mathfrak{A}(1-e)$. Since $\mathfrak{A}(1-e) \subseteq \mathrm{rad}(\mathfrak{A})$, $\mathfrak{R}$ is a non-zero radical algebra.

Take $\theta, \mathfrak{B}, X$ as before and note that $\theta(e) = 1$. Consider the algebra $\mathfrak{B} \oplus \mathfrak{R}$, with zero product between different summands, and the max norm. The map $\Theta : \mathfrak{A} \to \mathfrak{B} \oplus \mathfrak{R} : a \mapsto \theta(a) \oplus (a + e\mathfrak{A})$ is a contractive homomorphism. Since $\mathfrak{A}(1-e) \cong \mathfrak{A}/e\mathfrak{A}$, it is clear that $0 \oplus \mathfrak{R} \subset \Theta(\mathfrak{A})$, whence $\Theta(\mathfrak{A})$ is dense in $\mathfrak{B} \oplus \mathfrak{R}$.

It follows from Lemma 2.2 that there is a subsequence $(m_k) \subseteq \mathbb{N}$ and an element $z = (f, r) \in \mathfrak{B} \oplus \mathfrak{R}$ with $(z^{m_k}/\|z^{m_k}\|)$ a bounded approximate identity. Clearly $f \neq 0$, so choose $t \in X$ with $\|f\| = |f(t)|$. Since r is quasinilpotent, there is $m \in \mathbb{N}$ such that

$$\|r^n\| < |f(t)|^n = \|f^n\| \quad (n \geq m).$$

Thus

$$\|z^n\| = |f(t)^n| \quad (n \geq m).$$

But since $(z^{m_k}/\|z^{m_k}\|)$ is a bounded approximate identity for $\mathfrak{B} \oplus \mathfrak{R}$, $(r^{m_k}/|f(t)|^{m_k})$ is a bounded approximate identity for $\mathfrak{R}$. However, we have that

$$\frac{r^{m_k}}{|f(t)|^{m_k}} = \left(\frac{r}{|f(t)|}\right)^{m_k} \to 0,$$

whence $\mathfrak{R} = 0$, contrary to assumption. ∎

Corollary 2.4. *Let $\mathfrak{A}$ be a commutative Banach algebra with an approximate identity of normalized powers. If $\mathfrak{A}$ is semisimple with compact multiplication then*

$\mathfrak{A}$ is finite dimensional. Conversely, if $\mathfrak{A}$ is finite dimensional and a sequence of normalized powers of a generator gives an approximate identity, then $\mathfrak{A}$ is semisimple (with compact multiplication).

Proof. Theorem 2.3 and semisimplicity show that $\mathfrak{A}$ has an identity and so is finite dimensional if multiplication is compact.

Conversely, if $\mathfrak{A}$ is finite dimensional (so multiplication is certainly compact), compactness of the unit ball shows $\mathfrak{A}$ is unital. Now $\mathfrak{A}$ is weakly amenable by Theorem 2.1, so that the maximal ideals of $\mathfrak{A}$ must be idempotent. Thus these ideals are unital by [13, Corollary 3.3], and so are of the form $M_i = \mathfrak{A}e_i$ for idempotents $e_1, \dots, e_k$. But then $\operatorname{rad}(\mathfrak{A}) = \mathfrak{A}e_1 \cdots e_k$ is unital and thus zero. $\square$

Corollary 2.5. *Let $\mathfrak{A}$ be a (necessarily infinite dimensional) non-unital commutative Banach algebra with an approximate identity of normalized powers $(x^{n_k}/\parallel x^{n_k} \parallel)$. Then $\parallel x^n \parallel^{1/n} \to 0$ and $x^{n_k}/\left\| x^{n_k - 1} \right\| \to 0$. However, $(\parallel x^n \parallel)$ cannot be regulated at any point.*

Proof. Theorem 2.3 ensures that $\mathfrak{A}$ is radical, hence x is quasinilpotent. Since $\mathfrak{A}$ is non-unital and the ideal $\mathfrak{A}x$ is dense, there is $y \in \mathfrak{A}$ such that any sequence (z_k) in $\mathfrak{A}$ with $xz_k \to y$ is unbounded [9]. But for any subsequence (m_j) of (n_k),

$$x \frac{yx^{m_j - 1}}{\parallel x^{m_j} \parallel} \to y\,,$$

so that $(yx^{m_j-1}/\parallel x^{m_j} \parallel)$ is unbounded, and consequently $(x^{m_j-1}/\parallel x^{m_j} \parallel)$ is also unbounded. Thus $x^{n_k}/\left\| x^{n_k-1} \right\| \to 0$.

Supposing to the contrary that $(\parallel x^n \parallel)$ is regulated at p, we have

$$x^p = \lim_{k\to\infty} x^p \frac{x^{n_k}}{\parallel x^{n_k} \parallel} = \lim_{k\to\infty} \frac{x^{n_k+p}}{\parallel x^{n_k} \parallel} = 0\,,$$

which is absurd. $\square$

Remark. Essentially the same arguments show the results hold under the weakened hypothesis that there is a sequence of normalized powers forming a non-zero multiple of an approximate identity. This latter property has the slight advantage of being stable under continuous homomorphic images.

If a commutative Banach algebra with a bounded approximate identity of normalized powers is semisimple, then it follows from Theorem 2.3 that the algebra is unital, so that we have a trivial such bounded approximate identity. This does not mean that the identity is necessarily the only bounded approximate identity of normalized powers in such an algebra, as the following example shows. Take the null subset of the unit circle

$$X = \left\{ e^{2\pi i\theta} : \theta = \sum_{n=1}^{\infty} x_n 3^{-n^2}, x_n = 0 \text{ or } 1 \right\}.$$

Let x be the identity function on X, and take the commutative, semisimple, unital Banach algebra $\mathfrak{A}$ to be the closure of the polynomials in x under the uniform norm on X. (By Lavrentieff's theorem, [38, Theorem 2.11], $\mathfrak{A} = \mathcal{C}(X)$.) Given $\theta = \sum_{n=1}^{\infty} x_n 3^{-n^2} \in X$, set $\theta_k = \sum_{n=1}^{k} x_n 3^{-n^2}$. Then $3^{k^2}\theta_k$ is an integer, so that

$$e^{2\pi i(3^{k^2}\theta_k)} = 1.$$

Further, $|3^{k^2}(\theta_k - \theta)| \leq 3^{-2k}$. Thus

$$\begin{aligned} \left\| x^{3^{k^2}} - 1 \right\|_X &= \max_{\theta \in X} |x^{3^{k^2}}(\theta) - 1| = \max_{\theta \in X} |e^{2\pi i(3^{k^2}\theta)} - e^{2\pi i(3^{k^2}\theta_k)}| \\ &= \max_{\theta \in X} |e^{2\pi i(3^{k^2}(\theta - \theta_k))} - 1| \leq \max_{|\delta| \leq 3^{-2k}} |e^{2\pi i\delta} - 1| \to 0 \end{aligned}$$

Thus $(x^{3^{k^2}})$ is a bounded approximate identity for $\mathfrak{A}$

More generally, let $\mathfrak{A}$ be a commutative Banach algebra with an approximate identity of normalized powers $(x^{n_k}/\|x^{n_k}\|)$, and an identity e. It is immediate that $x^{n_k}/\|x^{n_k}\| \to e$, so certainly $\|\hat{x}^{n_k}/\|x^{n_k}\| - 1\|_\infty \to 0$. In particular, with $\nu(\cdot)$ denoting spectral radius, $\nu(x)^{n_k}/\|x^{n_k}\| \to 1$. Thus setting $z = x/\nu(x)$, we have (z^{n_k}) is a bounded approximate identity, no normalization needed this time. Since $\hat{z}^{n_k} \to 1$, we have that $\sigma(z) \subset \{\lambda : |\lambda| = 1\}$, and considered as a subset of this circle, $\sigma(z)$ has zero arclength measure. In particular, $\sigma(z)$ is nowhere dense, and so must be totally disconnected.

3. Amenable Banach algebras with compact multiplication

We shall see in this section that an amenable Banach algebra with compact multiplication is automatically biprojective, so that the well developed structure theory of those algebras applies. In fact, we will prove the following somewhat more general statement.

Theorem 3.1. *Let $\mathfrak{A}$ be an amenable Banach algebra with compact multiplication, $I \neq \mathfrak{A}$ a closed ideal of $\mathfrak{A}$. Then the linear map $\Pi : \mathfrak{A}\hat{\otimes}(\mathfrak{A}/I) \to \mathfrak{A}/I$ induced by the module operation of $\mathfrak{A}$ on $\mathfrak{A}/I$ has a bounded bi-module right inverse.*

Proof. Let $(d_\lambda)_{\lambda\in\Lambda}$ be an approximate diagonal for $\mathfrak{A}$. Then

$$d_\lambda = \sum_{k=1}^{\infty} a_k^\lambda \otimes b_k^\lambda$$

for appropriate sequences $(a_k^\lambda), (b_k^\lambda)$ in $\mathfrak{A}$. Let $\mathcal{U}$ be an ultrafilter on Λ which dominates the order filter, and define

$$\Delta : \mathfrak{A}/I \to \mathfrak{A}\hat{\otimes}(\mathfrak{A}/I) : a + I \mapsto \lim_{\mathcal{U}} \sum_{k=1}^{\infty} a_k^\lambda \otimes (b_k^\lambda a + I) \qquad (a \in \mathfrak{A})\,. \tag{3.1}$$

To see that this is well defined, note that for $a \in \mathfrak{A}$ the Cohen factorization theorem shows $a = bc$ for some $b, c \in \mathfrak{A}$. Let L_b and R_{c+I} denote left (resp. right) multiplication on $\mathfrak{A}$ (resp. $\mathfrak{A} + I$) by b (resp. $c + I$). By hypothesis, L_b and R_{c+I} are compact operators. As is shown implicitly in the proof of [14, Theorem 2.6], the tensor product of two compact operators is again compact, so that $L_b \otimes R_{c+I}$ is a compact operator on $\mathfrak{A}\hat{\otimes}(\mathfrak{A}/I)$. Thus

$$\lim_{\mathcal{U}}(L_b \otimes R_{c+I})\Big(\sum_{k=1}^{\infty} a_k^\lambda \otimes (b_k^\lambda + I)\Big) = \lim_{\mathcal{U}} \sum_{k=1}^{\infty} ba_k^\lambda \otimes (b_k^\lambda c + I)$$

exists. Now (d_λ) being an approximate diagonal,

$$\lim_{\lambda}(b.d_\lambda - d_\lambda.b) = 0$$

whence

$$\lim_{\lambda}\Big(\sum_{k=1}^{\infty} ba_k^\lambda \otimes (b_k^\lambda + I) - \sum_{k=1}^{\infty} a_k^\lambda \otimes (b_k^\lambda b + I)\Big) = 0\,. \tag{3.2}$$

Thus we have

$$\begin{aligned}\sum_{k=1}^{\infty} a_k^\lambda \otimes (b_k^\lambda a + I) &= \sum_{k=1}^{\infty} a_k^\lambda \otimes (b_k^\lambda bc + I)\\ &=(Id \otimes R_{c+I})\Big(\sum_{k=1}^{\infty} a_k^\lambda \otimes (b_k^\lambda b + I) - \sum_{k=1}^{\infty} ba_k^\lambda \otimes (b_k^\lambda + I)\Big)\\ &\quad + \sum_{k=1}^{\infty} ba_k^\lambda \otimes (b_k^\lambda c + I)\end{aligned}$$

so that $\lim\limits_{\mathcal{U}} \sum\limits_{k=1}^{\infty} a_k^\lambda \otimes (b_k^\lambda a + I)$ exists (and equals $\lim\limits_{\mathcal{U}} \sum\limits_{k=1}^{\infty} ba_k^\lambda \otimes (b_k^\lambda c + I)$).

It is straightforward to show that Δ is a bounded right module map. Using (3.2) we obtain that Δ is also a left module map. Finally, for $a \in \mathfrak{A}$,

$$\begin{aligned}\Pi(\Delta(a+I)) =& \Pi\big(\lim_{\mathcal{U}} \sum_{k=1}^{\infty} a_k^\lambda \otimes (b_k^\lambda a + I)\big) \\ =& \lim_{\mathcal{U}} \Pi\left(\sum_{k=1}^{\infty} a_k^\lambda \otimes (b_k^\lambda a + I)\right) \\ =& \lim_{\mathcal{U}} \sum_{k=1}^{\infty} (a_k^\lambda b_k^\lambda a + I) = a + I\,.\end{aligned}$$

□

Taking the special case $I = \{0\}$, we obtain:

Corollary 3.2. *An amenable Banach algebra with compact multiplication is biprojective.* □

Remark. Corollary 3.2 is not new, however, we are unable to find a proof in the literature. Yurii Selivanov has informed us that he had previously obtained it (with a different proof), and it is implicitly noted in [20]. Indeed, Barry Johnson has noted another way of viewing the result: if $\mathfrak{A}$ is amenable with virtual diagonal M then compactness of multiplication ensures that $M\mathfrak{A} \subset \mathfrak{A}\hat{\otimes}\mathfrak{A}$, a property investigated in [22]. The map $\Delta : \mathfrak{A} \to \mathfrak{A}\hat{\otimes}\mathfrak{A} : a \mapsto Ma$ then gives biprojectivity.

Corollary 3.3. *For a semisimple, amenable Banach algebra $\mathfrak{A}$ with the approximation property the following are equivalent and imply that $\mathfrak{A}$ is biprojective.*

(i) $\mathfrak{A}$ has compact multiplication;

(ii) $\mathfrak{A}$ is the completion of $\bigoplus_\alpha M_{n_\alpha}$ under some algebra norm, for an appropriate family (n_α) of positive integers, with continuous projections onto the coordinates.

Proof. By Corollary 3.2, (i) implies $\mathfrak{A}$ is biprojective. So by [32], $\bigoplus_\alpha I_\alpha$ is dense in $\mathfrak{A}$, where (I_α) is the family of minimal closed ideals of $\mathfrak{A}$. Further, each I_α is complemented in $\mathfrak{A}$, and I_α is isomorphic to the Banach algebra $\mathcal{N}_{F_\alpha}(E_\alpha)$ of F_α-nuclear operators on E_α, for an appropriate dual pair (E_α, F_α) of Banach spaces. (For details on such algebras see [19, II.2.4].) Being complemented, I_α is itself amenable, whence by [16] the algebra $\mathcal{N}_{F_\alpha}(E_\alpha)$ is finite dimensional, and so is isomorphic to some M_{n_α}. Thus (ii) holds.

Conversely, if (ii) holds, multiplication by any element of the dense subset $\bigoplus_\alpha M_{n_\alpha}$ is finite rank, and so (i) holds. ☐

Remark. The conclusion of Corollary 3.3 becomes false if we only require multiplication to be weakly compact. For let H be an infinite dimensional Hilbert space, $\mathcal{K}(H)$ the algebra of compact operators on H. Then $\mathcal{K}(H)$ is an ideal in its second dual $\mathcal{B}(H)$ and so has weakly compact multiplication, [26], however, $\mathcal{K}(H)$ is not of the structure described in Corollary 3.3. For reflexive Banach algebras multiplication is automatically weakly compact, but it is unknown whether such an amenable algebra is a finite direct sum of full matrix algebras, see [14, 15, 29].

In the not necessarily semisimple case there is no structure result as strong as Corollary 3.3. Some meaningful statements, however, can still be made. Our main tool is the following assertion. Recall that for an algebra $\mathfrak{A}$, and a non-empty subset S of $\mathfrak{A}$, the *right annihilator* of S is defined as

$$\operatorname{rann}(S) = \{a \in \mathfrak{A} : Sa = \{0\}\}.$$

Proposition 3.4. *Let $\mathfrak{A}$ be an amenable Banach algebra with compact multiplication, I a proper closed ideal of $\mathfrak{A}$ such that $\mathfrak{A}$ or $\mathfrak{A}/I$ has the approximation property. Then there is a non-zero bounded left module homomorphism $\theta : \mathfrak{A}/I \to rann(I)$. In particular, $rann(I) \neq \{0\}$.*

Proof. Let $\Delta : \mathfrak{A}/I \to \mathfrak{A}\hat{\otimes}\mathfrak{A}/I$ be a bounded bimodule map which is a right inverse to $\Pi : \mathfrak{A}\hat{\otimes}\mathfrak{A}/I \to \mathfrak{A}/I$. If $x \in \mathfrak{A}/I\backslash\{0\}$, $\Delta(x) \neq 0$. Since either $\mathfrak{A}$ or $\mathfrak{A}/I$ has the approximation property, there is $\phi \in (\mathfrak{A}/I)^*$ such that $\mathrm{id}\otimes\phi : \mathfrak{A}\hat{\otimes}\mathfrak{A}/I \to \mathfrak{A}\otimes\mathbb{C} \cong \mathfrak{A}$ such that $\mathrm{id} \otimes \phi(\Delta(x)) \neq 0$. Evidently $\theta = (\mathrm{id} \otimes \phi) \circ \Delta$ is a non-zero left module homomorphism, and since

$$a\theta(y) = \theta(a.y) = 0 \qquad (a \in I, y \in \mathfrak{A}/I),$$

it is clear that $\theta(\mathfrak{A}/I) \subset \operatorname{rann}(I)$. ☐

Remark. Proposition 3.4 is very similar – in both statement and proof – to Lemma 1.4 of [32]. It seems, however, that Proposition 3.4 is neither implied by nor does it imply that result.

Corollary 3.5. *Let $\mathfrak{A} \not\cong \mathbb{C}$ be a commutative amenable Banach algebra with compact multiplication and which has the approximation property. Then $\mathfrak{A}$ is not an integral domain.*

Proof. By Lomonosov's theorem, [7, Theorem 16.2], $\mathfrak{A}$ has a closed proper ideal I, and by Proposition 3.4, $\operatorname{rann}(I) \neq \{0\}$. Thus every element of I is a divisor of zero. ☐

Remark. Other than $\mathbb{C}$ there are no known commutative amenable Banach algebras which are integral domains, [1, page 362]. Note that Corollary 3.5 is well known in the non-radical case, since $\mathfrak{A}$ will necessarily have idempotents, [37].

4. A radical Banach algebra with an approximate identity of normalized powers

We construct here an example of a commutative Banach algebra $\mathfrak{A}$ with the following properties:

(i) $\mathfrak{A}$ is generated by a single element S;

(ii) a sequence of normalized powers of S is a bounded approximate identity for $\mathfrak{A}$;

(iii) multiplication by S is a quasinilpotent compact operator.

As a consequence:

(iv) $\mathfrak{A}$ is radical (this also follows from Theorem 2.3);

(v) $\mathfrak{A}$ is weakly amenable (by Theorem 2.1);

(vi) $\mathfrak{A}$ has compact multiplication, and multiplication by S is an approximable operator.

Multiplication by S is approximable because it is compact and as a Banach space $\mathfrak{A}$ is isometric to l_1 which has the bounded approximation property. $\mathfrak{A}$ has compact multiplication because multiplication by every power of S is compact, and (i) ensures that these powers are a fundamental set for $\mathfrak{A}$.

Let $S\mathbb{C}[S]$ denote the complex polynomials $p(S)$ in the variable S which have zero constant term. We will pick a basis $(p_i(S))_{i=1}^{\infty}$ for $S\mathbb{C}[S]$ (such that the degree $\deg p_i = i$), and make $S\mathbb{C}[S]$ into a normed space by defining

$$\left\| \sum_{i=1}^{n} \lambda_i p_i(S) \right\| = \sum_{i=1}^{n} |\lambda_i|.$$

The completion $\mathfrak{A}$ of this space is isometric to l_1. The unit ball of $\mathfrak{A}$ is the closed absolutely convex hull of the elements $p_i(S)$; from which it is immediate that $\mathfrak{A}$ is a Banach *algebra* provided

$$\| p_i(S)p_j(S) \| \leq 1$$

for all i and j. In that case, $\mathfrak{A}$ is certainly generated by S. Multiplication by S will be compact provided the following holds: for some sequence $\lambda_i \in [0,1]$ we have

$$\| Sp_i(S) - \lambda_i S \| \to 0.$$

For the condition plainly implies that the absolutely convex hull of the elements $Sp_i(S)$ is totally bounded – a suitable ϵ-net consisting of an $\epsilon/4$-net for the disk $\{\lambda S : |\lambda| \leq 1\}$, summed with an $\epsilon/2$-net for the absolutely convex hull of the (finitely many) elements $Sp_i(S)$ such that $\| Sp_i(S) - \lambda_i S \| \geq \epsilon/4$.

It will be convenient to introduce the truncation operators τ_n, $n \geq 1$ on $S\mathbb{C}[S]$:

$$\tau_n\Big(\sum_{i\geq 1} \lambda_i S^i\Big) = \sum_{i=1}^{n} \lambda_i S^i$$

and the resulting seminorms $\| \cdot \|_n = \| \tau_n(\cdot) \|$. Note that for $n \geq m$,

$$\tau_n\Big(\sum_{i=1}^{m} \lambda_i p_i(S)\Big) = \sum_{i=1}^{m} \lambda_i p_i(S)\,.$$

We will sometimes write p for the polynomial $p(S)$, and $|p|$ will denote the sum of the absolute values of the coefficients of p. We shall also use the interval notation $(n,m) = \{j \in \mathbb{N} : n < j < m\}$, and similarly for $(n,m]$, etcetera.

With these ideas in mind, we begin a recursive construction based on the simple notion of a "valid partial basis".

The sequence $p_1,\ldots,p_n \in S\mathbb{C}[S]$ is a *valid partial basis* for $S\mathbb{C}[S]$ if $\deg p_i = i$, and we have

$$\| p_i(S)p_j(S) \|_n < 1 \tag{4.1}$$

for every $i,j = 1,\ldots,n$. Note the strict inequality here – we must give ourselves a little room to manoeuvre. We claim the following:

Lemma 4.1. *Let $p_1,\ldots,p_n$ be a valid partial basis with $p_1(S) = S$, and let $\eta > 0$. Then there are integers m, N with $N > m > n$ and a valid partial basis $p_1,\ldots,p_N$ extending $p_1,\ldots,p_n$, such that the following hold:*

$$\| S^{n+1} \|_N^{1/n+1} \leq \eta, \text{ and } \| S^{n+1} \|_N \leq \eta \| S^n \|_N\,. \tag{4.2}$$

For all $i \leq n$ and $j \in (n, N]$ there is a $\lambda_{i,j} \in [0,1]$ such that

$$\| p_i(S) p_j(S) - \lambda_{i,j} p_i(S) \|_N \leq \eta\,, \tag{4.3}$$

and

$$p_N(S) \text{ is a monomial } \lambda_N S^N\,. \tag{4.4}$$

Furthermore,

$$\left\| \frac{S^{m+1}}{\| S^m \|_N} - S \right\|_N \leq \eta. \tag{4.5}$$

Using the fundamental Lemma 4.1 we can easily construct the sequence $(p_i)_{i=1}^{\infty}$ – and hence the Banach algebra $\mathfrak{A}$ – that we want. For define $p_1(S) = S$, $n_0 = 1$, and construct $(p_i)_{i=1}^{\infty}$ so that for each $N = n_i$ in an increasing sequence $(n_i)_{i=1}^{\infty}$, the sequence $p_1, \ldots, p_N$ is a valid partial basis extending $p_1, \ldots, p_{n_{i-1}}$ after the manner of Lemma 4.1. The constant $\eta = \eta_i$ used in the transition from n_{i-1} to n_i can be chosen anyhow as long as it tends to zero; we choose $\eta_i = 1/n_{i-1}$. In §5 a more careful choice will be needed for a stronger conclusion.

Theorem 4.2. *Suppose the basis $(p_i)_{i=1}^{\infty}$ is constructed as described in the previous paragraph. Then the norm $\| \cdot \|$ associated with the basis is indeed an algebra norm, and the commutative Banach algebra $\mathfrak{A}$ obtained from it on completion of $S\mathbb{C}[S]$ satisfies the conditions (i), (ii) and (iii) .*

Proof. If $i, j \in \mathbb{N}$, choose k so $i + j \leq n_k$. Because $p_1, \ldots, p_{n_k}$ is a valid partial basis, we have $\| p_i(S) p_j(S) \|_{n_k} < 1$. But in fact, deg $p_i p_j \leq n_k$ so the "truncation" involved in the seminorm $\| \cdot \|_{n_k}$ does not affect the element $p_i p_j$, and so $\| p_i(S) p_j(S) \| = \| p_i(S) p_j(S) \|_{n_k} < 1$. So $\| \cdot \|$ is an algebra norm, and $\mathfrak{A}$ will be the commutative Banach algebra generated by S. For our increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers, (4.5) gives us a value m between $n = n_{k-1}$ and $N = n_k$, such that $\left\| \frac{S^{m+1}}{\| S^m \|_N} - S \right\|_N \leq \eta = 1/n$. Furthermore we know $N > m$, so in fact $\left\| \frac{S^{m+1}}{\| S^m \|_N} - S \right\|_N = \left\| \frac{S^{m+1}}{\| S^m \|} - S \right\|$. Since $\mathfrak{A}$ is generated by S, the sequence of elements $\left(\frac{S^m}{\| S^m \|} \right)$ is a bounded approximate identity for $\mathfrak{A}$.

Now (4.2) tells us that $\| S^{n+1} \|_N^{1/n+1} \leq 1/n$ for each $n = n_{k-1}$ and $N = n_k$. Since $N > n$, we have $\| S^{n+1} \|_N = \| S^{n+1} \|$, and S is quasinilpotent. Finally (4.3) may be used (with $i = 1$) to establish that for $j \in (n_{k-1}, n_k)$ there is a $\lambda_j \in [0,1]$ such that

$$\| S p_j(S) - \lambda_j S \| = \| S p_j(S) - \lambda_j S \|_N \leq 1/n. \tag{4.6}$$

When $j = n_k = N$ it is not true that $\| Sp_j(S) - \lambda_j S \| = \| Sp_j(S) - \lambda_j S \|_N$, so instead we use (4.4) to tell us that $p_N(S) = \mu S^N$ for $\mu = 1/\left\| S^N \right\|$, and then use (4.2) with k replaced by $k+1$ to obtain

$$\| Sp_N(S) \| = \| Sp_N(S) \|_{n_{k+1}} = \left\| S^{N+1} \right\| / \left\| S^N \right\| \leq 1/N.$$

So we in fact have something like (4.6) even for $j = N$, noting that $\lambda_N = 0$. As we have discussed, all this implies that multiplication by S is compact. So (i), (ii) and (iii) are satisfied. □

All we need do, therefore, is prove Lemma 4.1.

Proof of Lemma 4.1.

Take the sequence $p_1, \ldots, p_n$ and define

$$\epsilon = 1 - \max\{\| p_i(S)p_j(S) \|_n : 1 \leq i, j \leq n\} > 0 \tag{4.7}$$

(for $(p_i)_{i=1}^n$ is a valid partial basis), and

$$K = \max\{|p_i|, i = 1, \ldots, n\}\,. \tag{4.8}$$

We then define $m = n+1$, and pick an increasing sequence

$$1 < A < B < C < D. \tag{4.9}$$

A will have to be big, and the sequence A, B, C, D will have to "increase quite rapidly" for the construction to work; for example, we would have no blushes in demanding $C > 10^{10^B}$ if we so required. (For a brief discussion about the philosophy of "rapidly increasing" sequences, see §1 of [27]). Subject to various conditions of "rapid increase" given below, we can choose A, B, C and D how we like and define constants

$$\begin{gathered} \alpha = 1 - \frac{1}{A}, \quad N = (C+1)(n+1) + n\,, \\ \rho_r = 1 - (r-1)/C \qquad (r = 1, \ldots, C+1)\,. \end{gathered} \tag{4.10}$$

Note that the constants α, ϵ, together with $\rho_r (r = 2, \ldots, C)$ and, without loss of generality, η, lie in $(0,1)$; $\rho_1 = 1, \rho_{C+1} = 0$. We shall also need the elementary estimates

$$\prod_{w=1}^{t} \rho_w \leq e^{-\sum_{w=1}^{t}(w-1)/C} \leq \begin{cases} e^{-(t-1)^2/2C} < 1 & t = 1, \ldots, C-1 \\ C^{-1} & t = C \end{cases}. \tag{4.11}$$

We then have the disjoint union

$$(n,N] = \{r(n+1) : r = 1,\ldots,C+1\} \cup \bigcup_{r=1}^{C+1} (r(n+1), r(n+1)+n]\,,$$

and the new polynomials $p_{n+1},\ldots,p_N$ are defined as follows.

For $i = r(n+1)$ with $r = 1,\ldots,C+1$, we define $p_i(S)$ to be the monomial

$$p_i(S) = D^r \alpha^{(r+1)(n+1)} S^i. \tag{4.12}$$

For $i \in (r(n+1), r(n+1)+n]$ with $r = 1,\ldots,C+1$, we write $i_1 = i - r(n+1)$ and define

$$p_i(S) = D^{r-1} B^{2i_1-1} \alpha^{(r+1)(n+1-i_1)} (DS^i - \rho_r S^{i-n-1}). \tag{4.13}$$

For $R = 1,\ldots,C$, we have the following identity which will be useful in verification of property (4.16), *cf.* (4.22) below.

$$\begin{aligned}
&\sum_{r=1}^{R} \rho_R \cdots \rho_{r+1} D^{(r-1)} (DS^{r(n+1)+i} - \rho_r S^{(r-1)(n+1+i})\\
&\quad= \sum_{r=1}^{R} \big(\rho_R \cdots \rho_{r+1} D^r S^{r(n+1)+i} - \rho_R \cdots \rho_{r+1} \rho_r D^{(r-1)} S^{(r-1)(n+1)+i}\big)\\
&\quad= D^R S^{R(n+1)+i} - \rho_R \cdots \rho_2 \rho_1 S^i\,.
\end{aligned}$$

We claim that, provided the sequence A, B, C, D increases fast enough, the sequence $p_1,\ldots,p_N$ is another valid partial basis, satisfying the conditions of Lemma 4.1. To prove this, we must check that the N version of (4.1), and (4.2, 4.3, 4.5) hold, namely:

$$\| p_i(S) p_j(S) \|_N < 1 \text{ for all } i, j \leq N\,; \tag{4.14}$$

$$\| S^{n+1} \|_N^{1/n+1} \leq \eta, \text{ and } \| S^{n+1} \|_N / \| S^n \|_N \leq \eta\,; \tag{4.15}$$

for all $i \leq n$ and $j \in (n,N]$ there is a $\lambda_{i,j} \in [0,1]$ such that

$$\| p_i(S) p_j(S) - \lambda_{i,j} p_i(S) \|_N \leq \eta\,; \tag{4.16}$$

$$\left\| \frac{S^{n+2}}{\| S^{n+1} \|_N} - S \right\|_N \leq \eta\,. \tag{4.17}$$

The fact required for (4.4) that $p_N(S)$ is a monomial follows from (4.13): $\rho_{C+1} = 0$.

Now, (4.15) is straightforward: (4.12) gives $\left\| S^{n+1} \right\|_N = D^{-1}\alpha^{-2(n+1)}$, and $\left\| S^{n+1} \right\|^{1/n+1} = \alpha^{-2}D^{-1/n+1} < \eta$, provided (say) $A > 2$ and $D > (4/\eta)^{n+1}$, our first condition of "rapid increase". Likewise, in view of (4.8) we certainly have $\| S^n \|_N = \| S^n \|_n \geq 1/K$, and so $\left\| S^{n+1} \right\|_N / \| S^n \|_N < \eta$ provided that $D > \alpha^{-2(n+1)}K/\eta$, another condition of "rapid increase".

(4.17) is also easy; by (4.12), $\left\| S^{n+1} \right\|_N = D^{-1}\alpha^{-2(n+1)}$; and $\rho_1 = 1$ so by (4.13),

$$p_{n+2}(S) = B\alpha^{2n}(DS^{n+2} - S).$$

Hence,

$$\frac{S^{n+2}}{\| S^{n+1} \|_N} = D\alpha^{2n+2}S^{n+2} = B^{-1}\alpha^2 p_{n+2}(S) + \alpha^{2n+2}S,$$

and so

$$\left\| \frac{S^{n+2}}{\| S^{n+1} \|_N} - S \right\|_N = B^{-1}\alpha^2 + (1 - \alpha^{2n+2}) .$$

The estimate on the right hand side is at most η provided $B > 2/\eta$ and $1-\alpha^{2n+2} < \eta/2$, for which $A > 4(n+1)/\eta$ is sufficient. Thus we have (4.17), assuming a couple more conditions of "rapid increase".

So we are left with checking (4.14) and (4.16); these will be verified concurrently, with consideration split into six cases depending on the values of i and j. Note we may also suppose that $j < N$.

Proving (4.14, 4.16)**, Case 1:** $i \leq n$**, and** $j = r(n+1)$**.**

Here r is an integer from 1 to $C + 1$ inclusive. In this case we define $\lambda_{i,j} = \alpha^{(r+1)(n+1)} \cdot \prod_{s=1}^{r}\rho_s \in [0, 1]$, and we seek the estimate

$$\| p_i(S)p_j(S) - \lambda_{i,j}p_i(S) \|_N \leq \frac{AK}{B} . \tag{4.18}$$

This is enough to ensure that $\| p_i(S)p_j(S) \|_N < 1$ provided $B > A^2K$, because $\lambda_{i,j} \leq \alpha$ then implies $\| p_i(S)p_j(S) \|_N \leq \alpha + AK/B < 1$. It is also enough to establish (4.16) provided $AK/B < \eta$, or $B > AK/\eta$. So given two more "rapid increase" conditions, (4.18) is enough for our purposes.

Now $S|p_i(S)$, deg $p_i \leq n$, and $|p_i| \leq K$; so it is enough to show that for $i = 1, \ldots, n$, we have

$$\left\| S^i p_j(S) - \lambda_{i,j}S^i \right\|_N \leq A/B, \tag{4.19}$$

which by (4.12) is the same as

$$\left\| D^r \alpha^{(r+1)(n+1)} S^{r(n+1)+i} - \lambda_{i,j} S^i \right\|_N \leq A/B. \tag{4.20}$$

By (4.13), for each $1 \leq s \leq r$ we have

$$p_{s(n+1)+i}(S) = D^{s-1} B^{2i-1} \alpha^{(s+1)(n+1-i)} (D S^{s(n+1)+i} - \rho_s S^{(s-1)(n+1)+i}),$$

hence

$$\left\| D^s S^{s(n+1)+i} - \rho_s D^{s-1} S^{(s-1)(n+1)+i} \right\|_N = \alpha^{(s+1)(i-n-1)} B^{1-2i}. \tag{4.21}$$

Summing (4.21) for $s = 1$ to r with weights $w_s = \prod_{t=s+1}^r \rho_t$, we obtain

$$\begin{aligned} \left\| D^r S^{r(n+1)+i} - \prod_{t=1}^r \rho_t S^i \right\|_N &\leq B^{1-2i} \cdot \sum_{s=1}^r \prod_{t=s+1}^r \rho_t \cdot \alpha^{(s+1)(i-n-1)} \\ &\leq B^{1-2i} \cdot \sum_{s=1}^r \alpha^{(s+1)(i-n-1)} . \end{aligned} \tag{4.22}$$

Now the highest negative power of α in the sum is $(r+1)(i-n-1)$, and the powers differ by at least one for different s, whence

$$\begin{aligned} \left\| D^r S^{r(n+1)+i} - \prod_{t=1}^r \rho_t S^i \right\|_N &\leq B^{1-2i} \alpha^{(r+1)(i-n-1)} \sum_{s=1}^r \alpha^{r-s} \\ &\leq A B^{-1} \alpha^{(r+1)(i-n-1)} \end{aligned}$$

since $\sum_{k=0}^\infty \alpha^k = A$ and $i \geq 1$. Multiplying by $\alpha^{(r+1)(n+1)}$ we get

$$\left\| D^r \alpha^{(r+1)(n+1)} S^{r(n+1)+i} - \lambda_{i,j} S^i \right\|_N \leq A B^{-1} \alpha^{(r+1)i} \leq A B^{-1},$$

so (4.19) is proved and Case 1 is closed.

We remark that in all the other cases below we choose $\lambda_{i,j} = 0$.

Proving (4.14, 4.16), Case 2: $i, j \leq n$.

Since (4.16) involves only i, j with $i \leq n$ and $j > n$, we only need establish (4.14) in this case. We handle this case second instead of (more naturally) first, because the arguments of Case 1 give us useful information: since $\| p_{r(n+1)}(S) p_i(S) \|_N < 1$ for $i = 1, \ldots, n$, we have

$$\| p_{r(n+1)}(S) \cdot q(S) \|_N \leq \| q(S) \|_N$$

for any polynomial $q \in S\mathbb{C}[S]$ with deg $q \leq n$. Because $p_1, \ldots, p_n$ is a valid partial basis we have $\| S^i \|_N \leq \| S \|_N^i = 1$, so for $i = 1, \ldots, n$ we have

$$\| S^i p_{r(n+1)}(S) \|_N \leq 1, \tag{4.23}$$

a statement which is also true when $i = 0$. By (4.12) (with $r = 1$), this gives

$$\left\| D\alpha^{2(n+1)} S^{n+1+i} \right\|_N \leq 1. \tag{4.24}$$

For $i, j \leq n$ we already know that $\| p_i(S) p_j(S) \|_n \leq 1 - \epsilon$, ϵ as in (4.7), so it is sufficient to show $\| (I - \tau_n)\big(p_i(S) p_j(S)\big) \|_N < \epsilon$. Now $|p_i p_j| \leq K^2$ by (4.8), so certainly $|(I - \tau_n)\big(p_i p_j\big)| \leq K^2$, and $(I - \tau_n)\big(p_i p_j\big) \in \operatorname{lin}\{S^{n+k} : 1 \leq k \leq n\}$. Hence (4.24) gives us

$$\| (I - \tau_n)\big(p_i(S) p_j(S)\big) \|_N \leq K^2 D^{-1} \alpha^{-2n-2} < \epsilon$$

as required, provided $D > K^2/\epsilon\alpha^{2n+2}$; a not-too-severe condition of “rapid increase”.

Proving (4.14, 4.16)**, Case 3:** $i \leq n$**, and** $j \in (r(n+1), r(n+1)+n]$**.**

Here again, r is any integer from 1 to $C+1$ inclusive. In this case we seek the simple estimate $\| p_i(S) p_j(S) \|_N < \eta$, which gets us both (4.14) and (4.16) (with $\lambda_{i,j} = 0$).

Writing $p_i(S) = \sum_{k=1}^{i} \mu_k S^k$, we have $p_j(S)$ given by (4.13), and hence with $j_1 = j - r(n+1))$ we have

$$p_i(S) p_j(S) = \sum_{k=1}^{i} \mu_k D^{r-1} B^{2j_1 - 1} \alpha^{(r+1)(n+1-j_1)} \big(D S^{j+k} - \rho_r S^{j+k-n-1}\big).$$

However, $\sum_{k=1}^{n} |\mu_k| \leq K$, so it is enough to show that for each $k = 1, \ldots, n$,

$$\left\| D^{r-1} B^{2j_1 - 1} \alpha^{(r+1)(n+1-j_1)} \big(D S^{j+k} - \rho_r S^{j+k-n-1}\big) \right\|_N < \eta/K. \tag{4.25}$$

SUBCASE 1: $j_1 + k \leq n$.

Then the polynomial on the left is in fact a multiple of $p_{j+k}(S)$; p_{j+k} is given by (4.13), and the multiple in question is $B^{-2k} \cdot \alpha^{(r+1)k} \leq B^{-2}$. Hence (4.25) holds provided $B^2 > K/\eta$.

SUBCASE 2: $j_1 + k > n$.

Then, $j + k \in [(r+1)(n+1), (r+1)(n+1)+n]$. We might possibly have $r = C+1$, a special case with the property that $j+k > N$, so $\| S^{j+k} \|_N = 0$. Since ρ_r is also zero in this case, the left hand side of (4.25) is zero. In the more usual case $r \leq C$, we know by (4.23) that for $i = 0, \ldots, n$, $\| S^i p_{(r+1)(n+1)}(S) \|_N \leq 1$, and by (4.12) this gives

$$\left\| S^{i+(r+1)(n+1)} \right\|_N \leq D^{-r-1} \alpha^{-(r+2)(n+1)} .$$

In the same way,

$$\left\| S^{i+r(n+1)} \right\|_N \leq D^{-r} \alpha^{-(r+1)(n+1)}$$

for all $r = 1, \ldots, C+1$. Choosing i so that $i + (r+1)(n+1) = j+k$, we get

$$\| S^{j+k} \|_N \leq D^{-r-1} \alpha^{-(r+2)(n+1)} \quad \text{and} \quad \| S^{j+k-n-1} \|_N \leq D^{-r} \alpha^{-(r+1)(n+1)}.$$

Accordingly,

$$\begin{aligned} &\left\| D^{r-1} B^{2j_1-1} \alpha^{(r+1)(n+1-j_1)} \big(D S^{j+k} - \rho_r S^{j+k-n-1} \big) \right\|_N \\ &\quad \leq B^{2j_1-1} D^{-1} (\alpha^{-(n+1)-(r+1)j_1} + \alpha^{-(r+1)j_1}) \\ &\quad \leq B^{2j_1-1} D^{-1} (\alpha^{-(n+1)} + 1) \alpha^{-n(C+1)} \end{aligned}$$

since $r \leq C$ and $j_1 \leq n$. So our estimate (4.25) is established provided

$$D > (2KB^{2n-1}(\alpha^{-(n+1)} + 1)\alpha^{-n(C+1)})/\eta ,$$

perhaps our most impressive condition of "rapid increase" yet.

With the conclusion of Case 3 we have investigated $\| p_1(S) p_j(S) \|_N$ for every $j = 1, \ldots, N$, so we know that the operator norm of S with respect to $\| \cdot \|_N$ is at most 1. We have also finished checking (4.16); from now on we need only check (4.14), for values i, j both greater than n.

Proving (4.14)**, Case 4:** $i = r(n+1)$ **and** $j = s(n+1)$**.**

Here r and s range from 1 to $C+1$. Now (4.12) gives us

$$p_i(S) p_j(S) = D^{r+s} \alpha^{(r+s+2)(n+1)} S^{(r+s)(n+1)} .$$

If $r+s > C+1$ then $(r+s)(n+1) > N$ and $\| p_i(S)p_j(S) \|_N = 0$. If $r+s \leq C+1$ then (4.12) shows us

$$\| p_i(S)p_j(S) \|_N = \alpha^{n+1} < 1.$$

Proving (4.14)**, Case 5:** $i = r(n+1)$ **and** $j \in (s(n+1), s(n+1)+n]$**.**

Again, r and s range from 1 to $C+1$. Set $j_1 = j - s(n+1)$, and $u = (r+s+2)(n+1) - (s+1)j_1$. Now (4.12, 4.13) give

$$p_i(S)p_j(S) = D^{r+s-1}B^{2j_1-1}\alpha^u(DS^{i+j} - \rho_s S^{i+j-n-1}) . \tag{4.26}$$

Now it may be that $r+s \geq C+3$, in which case both $i+j$ and $i+j-n-1$ are bigger than N so $\| p_i(S)p_j(S) \|_N = 0$. It may be that $r+s = C+2$, in which case $i+j > N$ so that $\left\| S^{i+j} \right\|_N = 0$ and

$$\| p_i(S)p_j(S) \|_N = \rho_s D^{C+1}B^{2j_1-1}\alpha^{(C+4)(n+1)-(s+1)j_1} \left\| S^{i+j-n-1} \right\|_N ,$$

where $i+j-n-1 = j_1 + (C+1)(n+1)$. Now $\rho_{C+1} = 0$ so (4.13) (applied with $i = j_1 + (C+1)(n+1)$) gives

$$\left\| D^{C+1}B^{2j_1-1}\alpha^{(C+2)(n+1-j_1)}S^{j_1+(C+1)(n+1)} \right\|_N = 1 ,$$

whence

$$\| p_i(S)p_j(S) \|_N = \rho_s \alpha^{2(n+1)+(C+1-s)j_1} \leq \alpha^{2(n+1)} < 1 .$$

Finally we consider the usual case: $r+s \leq C+1$. Then since $i+j \in (r+s)(n+1), (r+s)(n+1)+n]$, with $i+j-(r+s)(n+1) = j-s(n+1) = j_1$, we have

$$p_{i+j}(S) = D^{r+s-1}B^{2j_1-1}\alpha^{(r+s+1)(n+1-j_1)}(DS^{i+j} - \rho_{r+s}S^{i+j-n-1}) . \tag{4.27}$$

Comparing (4.26) with (4.27) and noting that $u-(r+s+1)(n+1-j_1) = n+1+rj_1$, we have

$$p_i(S)p_j(S) - p_{i+j}(S)\alpha^{n+1+rj_1} = D^{r+s-1}B^{2j_1-1}\alpha^u(\rho_{r+s} - \rho_s)S^{i+j-n-1} ,$$

where $\rho_{r+s} - \rho_s = -r/C$ and $i+j-n-1 = (r+s-1)(n+1)+j_1$. Accordingly, since $N \geq i+j$,

$$\| p_i(S)p_j(S) \|_N \leq \alpha^{n+1+rj_1} + D^{r+s-1}B^{2j_1-1}rC^{-1}\alpha^u \left\| S^{(r+s-1)(n+1)+j_1} \right\|_N \tag{4.28}$$

Now (4.22) tells us that for all $t = 1, \ldots, C+1$ and $j_1 = 1, \ldots, n$, we have

$$\left\| D^t S^{t(n+1)+j_1} - \prod_{w=1}^{t} \rho_w S^{j_1} \right\|_N \leq B^{1-2j_1} \cdot \sum_{l=1}^{t} \prod_{w=l+1}^{t} \rho_w \alpha^{(l+1)(j_1-n-1)} ,$$

whence, since $\| S^{j_1} \|_N \leq \| S \|_N^{j_1} \leq 1$, and using (4.11),

$$\begin{aligned} \left\| D^t S^{t(n+1)+j_1} \right\|_N &\leq 1 + B^{1-2j_1} \cdot \sum_{l=1}^{t} \prod_{w=l+1}^{t} \rho_w \alpha^{(l+1)(j_1-n-1)} \\ &\leq 1 + B^{1-2j_1} A \alpha^{(t+1)(j_1-n-1)} \end{aligned} \tag{4.29}$$

since

$$\sum_{l=1}^{t} \alpha^{(l+1)(j_1-n-1)} \leq \Big(\sum_{i=0} \alpha^i\Big) \alpha^{(t+1)(j_1-n-1)} \leq \frac{\alpha^{(t+1)(j_1-n-1)}}{1-\alpha} = A\alpha^{(t+1)(j_1-n-1)} .$$

Putting $t = r+s-1$ we get

$$\left\| D^{r+s-1} B^{2j_1-1} S^{(r+s-1)(n+1)+j_1} \right\|_N \leq B^{2j_1-1} + A\alpha^{(r+s)(j_1-n-1)} .$$

Now $u = (r+s+2)(n=1) - (s+1)j_i \geq r(n+1)$, so that $\alpha^u < \alpha^r$. Therefore (4.28) may be estimated by

$$\begin{aligned} &\alpha^{n+1+rj_1} + rAC^{-1}\alpha^{2n+2+(r-1)j_1} + rC^{-1}\alpha^u B^{2j_1-1} \\ &\qquad\qquad \leq \alpha + C^{-1} r\alpha^r (A + B^{2j_1-1}) . \end{aligned}$$

With $\gamma(A) = \max\{r\alpha^r : r \geq 1\} = \big(-e \log \alpha\big)^{-1}$ we thus have

$$\| p_i(S) p_j(S) \|_N \leq \alpha + C^{-1}\gamma(A)(A + B^{2n}) < 1$$

provided C is chosen sufficiently large (once A and B have been chosen). This mild condition of rapid increase brings Case 5 to a close.

Proving (4.14)**, Case 6:** $i \in (r(n+1), r(n+1)+n]$ **and** $j \in (s(n+1), s(n+1)+n]$**.**

Two applications of (4.13) give us, with $i_1 = i - r(n+1)$, $j_1 = j - s(n+1)$, and $v = (r+s+2)(n+1) - (r+1)i_1 - (s+1)j_1$,

$$p_i(S)p_j(S) = D^{r+s-2} B^{2(i_1+j_1)-2} \alpha^v (DS^i - \rho_r S^{i-n-1})(DS^j - \rho_s S^{j-n-1}) . \tag{4.30}$$

SUBCASE 1: $i_1+j_1 \le n$, and $r+s \le C+1$.

Here

$$p_{i+j}(S) = D^{r+s-1}B^{2(i_1+j_1)-1}\alpha^{(r+s+1)(n+1-i_1-j_1)}(DS^{i+j} - \rho_{r+s}S^{i+j-n-1}), \tag{4.31}$$

and

$$p_{i+j-n-1}(S) = \\ D^{r+s-2}B^{2(i_1+j_1)-1}\alpha^{(r+s)(n+1-i_1-j_1)}(DS^{i+j-n-1} - \rho_{r+s-1}S^{i+j-2n-2}). \tag{4.32}$$

Comparing (4.30–4.32), and noting that $v-(r+s+1)(n+1-i_1-j_1) = rj_1+si_1+n+1$, we conclude that

$$\begin{aligned} &p_i(S)p_j(S) - B^{-1}\alpha^{n+1+si_1+rj_1}p_{i+j}(S) \\ &= D^{r+s-2}B^{2(i_1+j_1)-2}\alpha^v\big((\rho_{r+s}-\rho_r-\rho_s)DS^{i+j-n-1} + \rho_r\rho_s S^{i+j-2(n+1)}\big) \\ &= B^{-1}\alpha^{2n+2+(s-1)i_1+(r-1)j_1}p_{i+j-n-1}(S)(\rho_{r+s}-\rho_r-\rho_s) + \\ &\qquad \big(\rho_r\rho_s + \rho_{r+s-1}(\rho_{r+s}-\rho_r-\rho_s)\big)D^{r+s-2}B^{2(i_1+j_1)-2}\alpha^v S^{i+j-2(n+1)}. \end{aligned}$$

Now

$$\rho_{r+s}-\rho_r-\rho_s = 1 - \frac{r+s-1}{C} - 2 + \frac{r-1}{C} + \frac{s-1}{C} = -1 - \frac{1}{C},$$

so in particular this number has magnitude less than 2. Accordingly

$$\begin{aligned} &\| p_i(S)p_j(S) \|_N \\ &\le 3B^{-1} + \big(\rho_r\rho_s - \rho_{r+s-1}(1+C^{-1})\big)D^{r+s-2}B^{2(i_1+j_1)-2}\alpha^v \left\| S^{i+j-2(n+1)} \right\|_N. \end{aligned} \tag{4.33}$$

Now $i+j-2(n+1) = (r+s-2)(n+1)+j_1+i_1$, and putting $t = r+s-2$ in (4.29) and replacing j_1 with i_1+j_1, we get

$$D^{r+s-2}\left\| S^{i+j-2(n+1)} \right\|_N \le \prod_{w=1}^{r+s-2}\rho_w + AB^{1-2(i_1+j_1)}\alpha^{(r+s-1)(j_1+i_1-n-1)}.$$

(4.11) and(4.33) together give

$$\| p_i(S)p_j(S) \|_N \le 3B^{-1} + |\rho_r\rho_s - \rho_{r+s-1}(1+C^{-1})|M,$$

where

$$M = B^{-1}A\alpha^{3(n+1)+(s-2)i_1+(r-2)j_1} + B^{2(i_1+j_1)-2}e^{-(r+s-3)^2/2C}\,.$$

Once again,

$$\rho_r\rho_s - \rho_{r+s-1} = (1-\frac{r-1}{C})(1-\frac{s-1}{C}) - (1-\frac{r+s-2}{C}) = \frac{(r-1)(s-1)}{C^2}\,,$$

and $\rho_{r+s-1}/C \le 1/C$, so that $\|\,p_i(S)p_j(S)\,\|_N$ is at most

$$\frac{3}{B} + \big(\frac{1}{C} + \frac{(r-1)(s-1)}{C^2}\big)\big(\frac{A}{B}\alpha^{3(n+1)+(s-2)i_1+(r-2)j_1} + B^{2n-2}e^{-(r+s-3)^2/2C}\big)\,.$$

Writing $T = r+s-3$, we have that

$$\frac{(r-1)(s-1)}{C^2}\cdot e^{-(r+s-3)^2/2C} \le T^2C^{-2}e^{-T^2/2C} \le 2/eC\,.$$

So using the function γ defined in case 5, the previous expression is at most

$$\frac{3}{B} + \frac{A}{BC} + \frac{A}{BC^2}\gamma(A)^2 + B^{2n-2}(\frac{1}{C} + \frac{2}{eC}) < 1$$

for large B and C ($B > 8$ and $C > A(1+\gamma(A)^2)$ and $C > 12B^{2n-2}$ will do).

SUBCASE 2: $i_1 + j_1 \le n$ and $r + s = C + 2$.

In this case $\tau_N(S^{i+j}) = 0$, hence (4.30) gives us

$$\begin{aligned}&\|\,p_i(S)p_j(S)\,\|_N\\ &\quad\le D^{r+s-2}B^{2(i_1+j_1)-2}\alpha^v\big((\rho_r+\rho_s)\,\big\|\,DS^{i+j-n-1}\,\big\|_N + \rho_r\rho_s\,\big\|\,S^{i+j-2n-2}\,\big\|_N\big)\,.\end{aligned} \tag{4.34}$$

Now (4.11) and (4.29) with $t = C+1 = r+s-1$ give

$$\big\|\,D^{r+s-1}S^{i+j-n-1}\,\big\|_N \le B^{1-2i_1-2j_1}A\alpha^{(C+2)(i_1+j_1-n-1)} \tag{4.35}$$

With $t = C = r+s-2$ these give

$$\big\|\,D^{r+s-2}S^{i+j-2n-2}\,\big\|_N \le C^{-1} + B^{1-2i_1-2j_1}A\alpha^{(C+2)(i_1+j_1-n-1)}\,. \tag{4.36}$$

Substituting (4.35, 4.36) in (4.34) we get

$$\begin{aligned}&\|\,p_i(S)p_j(S)\,\|_N\\ &\qquad\le (\rho_r+\rho_s+\rho_r\rho_s)AB^{-1}\alpha^{2(n+1)+(s-2)i_1+(r-2)j_1} + \rho_r\rho_sC^{-1}B^{2i_1+2j_1-2}\\ &\qquad\le 3AB^{-1} + B^{2n-2}C^{-1} < 1\end{aligned} \tag{4.37}$$

provided $B > 6A$ and $C > 2B^{2n-2}$.

SUBCASE 3: $i_1 + j_1 \le n$ and $r + s = C + 3$.

Then $\tau_N(S^{i+j}) = \tau_N(S^{i+j-n-1}) = 0$. Hence, (4.30) gives

$$\| p_i(S)p_j(S) \|_N = \rho_r \rho_s D^{C+1} B^{2i_1+2j_1-2} \alpha^v \left\| S^{i+j-2n-2} \right\|_N .$$

Here $i+j-2n-2 \in ((C+1)(n+1), (C+1)(n+1)+n]$ and $i+j-(C+1)(n+1) = i_1+j_1$, so by (4.29)

$$\left\| D^{C+1} S^{i+j-2n-2} \right\|_N \le B^{1-2i_1-2j_1} A \alpha^{(C+2)(i_1+j_1-n-1)} .$$

Accordingly,

$$\| p_i(S)p_j(S) \|_N \le AB^{-1} \alpha^{3(n+1)+(s-2)i_1+(r-2)j_1} \rho_r \rho_s < AB^{-1} < 1 .$$

SUBCASE 4: $i_1 + j_1 \le n$ and $r + s \ge C + 4$.

But in this case, $\tau_N(p_i p_j) = 0$ and so $\| p_i(S)p_j(S) \|_N = 0$.

SUBCASE 5: $i_1 + j_1 > n$.

Then $i + j \ge (r + s + 1)(n + 1)$ and so for $\varepsilon = 0, 1, 2$ we have $\left\| S^{i+j-\varepsilon(n+1)} \right\|_N \le \left\| S^{(r+s+1-\varepsilon)(n+1)} \right\|_N$, for we already know that the operator norm of S is at most 1. For $1 \le t \le C + 1$, (4.12) gives us $\left\| S^{t(n+1)} \right\|_N = D^{-t} \alpha^{-(t+1)(n+1)}$; for $t > C + 1$ the seminorm value is of course zero. So certainly for any $t > 0$ we have $\left\| S^{t(n+1)} \right\|_N \le D^{-t} \alpha^{-(C+2)(n+1)}$. Putting this in (4.30) and bounding α^v in (4.30) by 1, we get

$$\begin{aligned} \| p_i(S)p_j(S) \|_N &\le D^{-1} B^{2i_1+2j_1-2} \alpha^{-(C+2)(n+1)} (1 + \rho_r + \rho_s + \rho_r \rho_s) \\ &\le 4D^{-1} B^{2i_1+2j_1-2} \alpha^{-(C+2)(n+1)} . \end{aligned} \tag{4.38}$$

We bring our proof to a close by noting that $i_1 + j_1 \le 2n$, and introducing our final condition of rapid increase, namely

$$D > 4B^{4n-2} \alpha^{-(C+2)(n+1)} . \tag{4.39}$$

Thus the polynomials $p_{n+1}, \dots, p_N$ defined in (4.12, 4.13) do indeed extend $p_1, \dots, p_n$ to a valid partial basis satisfying all the conditions (4.2, 4.3, 4.5). $\square$

With the proof of Lemma 4.1, Theorem 4.2 follows, and we have our example $\mathfrak{A}$.

5. Further properties of the algebra $\mathfrak{A}$.

Two questions arise concerning the example $\mathfrak{A}$ of §4. Is $\mathfrak{A}$ amenable, and is it different to the previously known examples of weakly amenable, radical, commutative Banach algebras? We now answer these questions. Subject to mild further conditions of "rapid increase", $\mathfrak{A}$ is an integral domain. Thus, by Corollary 3.5, $\mathfrak{A}$ cannot be amenable. Further, as remarked in the Introduction, the previous examples are of the form $I(E)/\overline{J(E)}$ for certain sets E not of synthesis in $L^1(\mathbb{R})$, and for such E there are ideals $\overline{J(E)} \subsetneqq I_1, I_2 \subsetneqq I(E)$ with $I_1 \cap I_2 = \overline{J(E)}$, [6, Theorem 3.1.3]. Thus $(I_1 + \overline{J(E)})(I_2 + \overline{J(E)}) = 0$ and $I(E)/\overline{J(E)}$ has divisors of zero.

That $\mathfrak{A}$ is an integral domain is also interesting in its own right, telling us, for example, that one can have weak amenability in a situation (integral domain, approximation property, compact multiplication) in which you cannot have amenability. The example also provides negative answers to questions raised by P.C. Curtis Jr. and the fourth author in [1], as well as settling (again in the negative) the conjecture of the fourth author that a prime, weakly amenable algebra is necessarily ultraprime, [40]. For by [25, Proposition 3.4], $\mathfrak{A}$, being commutative and prime and different from $\mathbb{C}$, cannot be ultraprime.

Because the proof that $\mathfrak{A}$ is an integral domain is a little long and technical, we include an outline of the argument at this point to help in understanding the motivation for what follows.

Sketch Proof of Theorem 5.1. The idea is to mimic the proof that l_1, with its usual convolution multiplication, is an integral domain. If we call the generator S, then this is done by taking two nonzero sequences $p(S)$, $q(S)$ and considering the nonzero terms of lowest degree that contribute to p and q. If these are λS^i and μS^j respectively, then $p(S)q(S)$ has a nonzero term $\lambda\mu S^{i+j}$.

In our algebra $\mathfrak{A}$ we have necessarily renormed the polynomials $p(S)$ in such a way that the sequence (S^i) no longer forms a Schauder basis for the completion $\mathfrak{A}$. So it is *not* true, in the usual sense, that the general element $x \in \mathfrak{A}$ is a "sequence" $p(S)$. However, there are norm 1 projections onto the linear span of the first n powers of S; only they are not the obvious ones (τ_n) that truncate the sequence $p(S)$ at the nth place, but rather the projections Q such that $Q(p_i)$ is p_i $(i \leq n)$ or zero $(i > n)$. For the p_i *are* a Schauder basis for $\mathfrak{A}$.

So we take a convenient sequence (Q_k) of these projections and nonzero elements $x, y \in \mathfrak{A}$, and seek to imitate the simplicity of the usual l_1 proof that $xy \neq 0$. We choose k such that $Q_{k-1}(x)$ and $Q_{k-1}(y)$ are both "large" and claim that $Q_{k-1}(xy)$ is "roughly" equal to $Q_{k-1}(x)Q_{k-1}(y)$ and therefore $xy \neq 0$ because $Q_{k-1}(x)Q_{k-1}(y)$ is nonzero for the usual reason.

Let us further sketch what we mean by "large" and "roughly" in the previous paragraph. In the notation to be developed below, Q_k is the projection onto $\operatorname{lin}\{S^i : i \leq n_k\}$, where the n_k are the same indices n involved in our recursive construction of valid partial bases. The idea that $Q_{k-1}(xy)$ is "roughly" equal to $Q_{k-1}(x)Q_{k-1}(y)$ is therefore absurd unless two things happen.

First, the part of $Q_{k-1}(x)Q_{k-1}(y)$ that has coefficients S^i for $i > n_{k-1}$ must have very small norm, because these S^i are not even in the image of Q_{k-1}. This is not a problem because the recursive construction explicitly provides that $\| S^{n+1} \|$ is very small indeed for selected values of n.

Second, it is not much use if what is left of $Q_{k-1}(x)Q_{k-1}(y)$ after truncation – namely the vector $\tau_{n_{k-1}}Q_{k-1}(x)Q_{k-1}(y)$ – is zero or nearly so. So $Q_{k-1}(x)$ and $Q_{k-1}(y)$ ought to have some "weight", that is, some biggish terms λS^i, in "small" values i; certainly for $i \leq n_{k-1}/2$, in fact we go for $i \leq n_{k-1}/3$. That is the motivation for the sliding truncation operator defined at (5.21).

Furthermore – and this is a little awkward – the assertion that $Q_{k-1}(xy)$ is "roughly" equal to $Q_{k-1}(x)Q_{k-1}(y)$ is still not quite true. Rather, we find that there is an "approximately multiplicative" linear functional z_k^* such that $Q_{k-1}(xy)$ really is "close" to $Q_{k-1}(x)Q_{k-1}(y) + z_k^*(x)Q_{k-1}(y) + z_k^*(y)Q_{k-1}(x)$.

How close is close? Well, applying $S^2 Q_{k-1}$ to (5.9) and using (5.11) to estimate the last term, one may obtain

$$\begin{aligned}\|S^2(Q_{k-1}(xy) - Q_{k-1}(Q_{k-1}(x)Q_{k-1}(y)) - z_k^*(x)Q_{k-1}(y) - z_k^*(y)Q_{k-1}(x))\| \\ \leq (4\eta_k + \sqrt{\eta_k}) \| x \| \| y \| \ .\end{aligned}$$

So provided $Q_{k-1}(Q_{k-1}(x)Q_{k-1}(y))$ is very close to $Q_{k-1}(x)Q_{k-1}(y))$, that should be "close enough".

Incidentally, the presence of the unwanted factor of S^2 in the above equation is one of the reasons why we go for truncating at $n_k/3$ rather than the more natural $n_k/2$ in (5.21).

Now z_k^* really is "approximately multiplicative", (see (5.12)). For a hypothetical x, y such that $xy = 0$ one must have $z_k^*(x)z_k^*(y)$ "approximately" equal to $z_k^*(xy) = 0$. Assuming that $z_k^*(x)$ is the smaller of $z_k^*(x)$ and $z_k^*(y)$, we end up with $0 = Q_{k-1}(xy)$ being "close" to $Q_{k-1}(x)Q_{k-1}(y) + z_k^*(y)Q_{k-1}(x)$. That is what is asserted in the key equation (5.20). But of course, in non-unital l_1 with its usual multiplication, the equation $x'y' + \lambda x' = 0$ is just as impossible for nonzero x', y' as the equation $x'y' = 0$. And that brings us to the end of the sketch proof; we now begin the real thing. □

The further conditions of "rapid increase" required are centred around the following: the constant η used in extending the valid partial basis $p_1, \ldots, p_{n_{i-1}}$ to $p_1, \ldots, p_{n_i}$ after the manner of Lemma 4.1 (remember that n_i is the i-th choice of n and n_{i+1} is the i-th choice of N), must go down faster than the value $\eta = 1/n_{i-1}$ that we chose in §4. To be sure, the value of η depends only on the original sequence $p_1, \ldots, p_{n_{i-1}}$, but it must be "small" – exactly how small, we shall see below. For the moment, instead of picking large constants $A < B < C < D$ depending on $p_1, \ldots, p_{n_{i-1}}$, as in (4.9) let us think of picking large constants $E < A < B < C < D$, using A, B, C and D as described in §4 to make the transition to the longer valid partial basis $p_1, \ldots, p_{n_i}$, but with $\eta = 1/E$.

The constants A, B, C, D and (now) $E = 1/\eta$ involved in the transition from the valid partial basis $p_1, \ldots, p_{n_{i-1}}$ to the valid partial basis $p_1, \ldots, p_{n_i}$ will be denoted A_i, B_i, C_i, D_i, E_i. Other familiar constants that need to be labelled for different i are $\alpha_i = 1 - 1/A_i$, $\eta_i = 1/E_i$, $K_i = \max\{|p_j| : 1 \leq j \leq n_{i-1}\}$ and $\rho_r^{(i)} = 1 - (r-1)/C_i$, (*cf* (4.10)) . It will also be convenient to have chosen the C_k to be evenly divisible by 3.

Theorem 5.1. *For* $1 < E_i < A_i < B_i < C_i < D_i$, $i \geq 1$ *increasing sufficiently rapidly the construction of Lemma 4.1 gives rise to an algebra* $\mathfrak{A}$ *which is an integral domain.*

Proof. Let $\mathfrak{A}$ be the algebra constructed from rapidly increasing sequences $1 < E_i < A_i < B_i < C_i < D_i$, $i \geq 1$ as in Theorem 4.2. We will need some preliminary constructions.

For every $k > 1$ we define a linear functional z_k^* on $\mathfrak{A}$, designed to be "approximately multiplicative". For $j > n_{k-1}$, say, $j \in (n_{p-1}, n_p]$ with $p \geq k$, Lemma 4.1 with $i = 1$ gives the existence of a constant λ_j such that

$$\| Sp_j(S) - \lambda_j S \|_{n_p} < \eta_p \leq \eta_k. \tag{5.1}$$

Those values λ_j that are nonzero are given explicitly in Case 1 of "Proving 4.12, 4.14" of §4. Define

$$z_k^*(p_j(S)) = \lambda_j = \begin{cases} \alpha_p^{(r+1)(n_{p-1}+1)} \cdot \prod_{s=1}^{r} \rho_s^{(p)} & j = r(n_{p-1}+1),\ 1 \le r \le C_p + 1, \\ & p \ge k \\ 0 & \text{otherwise} \end{cases}$$

Since $|\lambda_j| \le 1$, z_k^* extends to a continuous linear functional of norm at most one on $\mathfrak{A}$ (after all, the dual of l_1 is l_∞).

Now let $Q_k : \mathfrak{A} \to \mathfrak{A}$ be the projection such that

$$Q_k(p_i(S)) = \begin{cases} p_i(S) & i \le n_k \\ 0 & i > n_k \end{cases}$$

and set $P_k = I - Q_k$. Note that $\|Q_k\| = \|P_k\| = 1$.

We need to improve (5.1), which involves the truncated norm $\|\cdot\|_{n_p}$ (where $\|y\|_{n_p} = \big\|\tau_{n_p}(y)\big\|$) rather than $\|\cdot\|$ itself. The following lemma enables us to do this.

Lemma 5.2. *For all $p > 0, r \ge 0$ and $i, j \le n_p$ we have*

$$\big\|(I - \tau_{n_p})\big(S^r p_i(S) p_j(S)\big)\big\| \le \eta_{p+1}. \tag{5.2}$$

Equivalently, for all $r \ge 0$ and $x, y \in \operatorname{Im} Q_p$

$$\big\|(I - \tau_{n_p})\big(S^r xy\big)\big\| \le \eta_{p+1} \|x\| \cdot \|y\|. \tag{5.3}$$

Furthermore,

$$\big\|P_p\big(p_i(S) p_j(S)\big)\big\| \le \eta_{p+1}. \tag{5.4}$$

If the values λ_j are as above, then for every $i \le n_{k-1}$ and $j > n_{k-1}$, say, $j \in (n_{p-1}, n_p]$, we have

$$\|p_j(S) p_i(S) - \lambda_j p_i(S)\| < 2\eta_p \le 2\eta_k. \tag{5.5}$$

Finally, for every $x \in \operatorname{Im} P_{k-1},\ y \in \operatorname{Im} Q_{k-1}$ we have

$$\|xy - z_k^*(x) y\| \le 2\eta_k \|x\| \cdot \|y\|. \tag{5.6}$$

Proof. For (5.2), we note that $y = (I - \tau_{n_p})\big(S^r p_i(S)p_j(S)\big)$ is a vector with $|y| \leq |p_i| \cdot |p_j| \leq K_p^2$, and $y \in \mathrm{lin}\{S^l : l > n_p\}$. Consequently

$$\| y \| \leq K_p^2 \max\{\| S^l \| : l > n_p\} = K_p^2 \, \| S^{1+n_p} \| \leq K_p^2 \eta_{p+1}^{1+n_p} \tag{5.7}$$

because $\| S \| \leq 1$ and by (4.15). We may assume the perfectly reasonable "rapid increase" condition $K_p^2 \eta_{p+1}^{1+n_p} < \eta_{p+1}$ for every p, and thus we arrive at (5.2). Also (5.3) really is equivalent to (5.2) because the unit ball of $\mathrm{Im}\, Q_p$ is the absolutely convex hull of the $p_i(S)$, $i \leq n_p$.

For (5.4), note that $P_p \tau_{n_p} = 0$ so $P_p\big(p_i(S)p_j(S)\big) = P_p(I - \tau_{n_p})\big(p_i(S)p_j(S)\big)$, and then use (5.2) and the fact that $\| P_p \| = 1$.

For (5.5), note that (4.16) in Lemma 4.1 also gives that for some constants $\lambda_{i,j}$ we have

$$\| p_i(S)p_j(S) - \lambda_{i,j} p_i(S) \|_{n_p} < \eta_p$$

for every $i \leq n_{p-1}$ and $j \in (n_{p-1}, n_p]$. And our actual $\lambda_{i,j}$ that we use in "Proving 4.10, 4.12" depend on j but not on i, so that $\lambda_{i,j} = \lambda_j$. Hence, for every $i \leq n_{k-1}$ and $j > n_{k-1}$, $j \in (n_{p-1}, n_p]$, we have the equation

$$\| p_i(S)p_j(S) - \lambda_j p_i(S) \|_{n_p} = \big\| \tau_{n_p}\big(p_i(S)p_j(S)\big) - \lambda_j p_j(S) \big\| < \eta_p \leq \eta_k .$$

However, we can now estimate $\big\| (I - \tau_{n_p})\big(p_i(S)p_j(S)\big) \big\|$ using (5.2), and adding up the two estimates we have (5.5).

Lastly, the unit ball of ker Q_{k-1} is the closed absolutely convex hull of $\{p_i(S) : i > n_{k-1}\}$; and the unit ball of Im Q_{k-1} is the closed absolutely convex hull of $\{p_i(S) : i \leq n_{k-1}\}$. So (5.5) implies that for all $x \in \ker Q_{k-1}$, $y \in \mathrm{Im}\, Q_{k-1}$, we have

$$\|xy - z_k^*(x)y\| \leq 2\eta_k \|x\|\| \cdot \|y\| ,$$

which is (5.6). □

For general $x, y \in \mathfrak{A}$, (5.6) means that

$$\|P_{k-1}(x)Q_{k-1}(y) - z_k^*(x)Q_{k-1}(y)\| \leq 2\eta_k \|x\| \cdot \|y\|. \tag{5.8}$$

Adding in the analogous formula with x and y interchanged, and using the fact that

$$xy = (P_{k-1}(x) + Q_{k-1}(x)) \cdot (P_{k-1}(y) + Q_{k-1}(y)) ,$$

we have

$$\|xy - Q_{k-1}(x)Q_{k-1}(y) - z_k^*(x)Q_{k-1}(y) - z_k^*(y)Q_{k-1}(x) - P_{k-1}(x)P_{k-1}(y)\| \\ \leq 4\eta_k\|x\| \cdot \|y\| \tag{5.9}$$

It will be convenient to collect some crucial estimates in a lemma.

Lemma 5.3. *For* $u, v \in \ker Q_{k-1}$

$$|z_k^*(uv) - z_k^*(u)z_k^*(v)| \leq \sqrt{\eta_k}\|u\| \cdot \|v\| \tag{5.10}$$

$$\left\| S^2 Q_{k-1}(uv) \right\| \leq \sqrt{\eta_k}\|u\| \cdot \|v\| \tag{5.11}$$

provided (E_k) *increases sufficiently rapidly. Furthermore, for* $x, y \in \mathfrak{A}$,

$$|z_k^*(xy) - z_k^*(x)z_k^*(y)| \leq 6\sqrt{\eta_k}\|x\| \cdot \|y\| \tag{5.12}$$

Proof. Since $\|uS - z_k^*(u)S\| < 2\eta_k\|u\|$ and $\|vS - z_k^*(v)S\| < 2\eta_k\|v\|$,

$$\|uvS^2 - z_k^*(u)S^2v - z_k^*(v)S^2u + z_k^*(u)z_k^*(v)S^2\| < 4\eta_k^2\|u\| \cdot \|v\|,$$

with $\|S^2v - z_k^*(v)S^2\| < 2\eta_k\|v\|$ and $\|S^2u - z_k^*(u)S^2\| < 2\eta_k\|u\|$, so

$$\|uvS^2 - z_k^*(u)z_k^*(v)S^2\| < (4\eta_k^2 + 4\eta_k)\|u\| \cdot \|v\|.$$

We also have, taking $x = uv, y = S^2$ in (5.8),

$$\|uvS^2 - Q_{k-1}(uv)S^2 - z_k^*(uv)S^2\| < 2\eta_k\|u\| \cdot \|v\| ,$$

so

$$\|(z_k^*(uv) - z_k^*(u)z_k^*(v))S^2 + Q_{k-1}(uv)S^2\| < (4\eta_k^2 + 6\eta_k)\|u\| \cdot \|v\|. \tag{5.13}$$

Now observe that *both* terms inside the norm in (5.13) have to be small. For the norm on Im Q_{k-1} is controlled by the constant K_{k-1}; if $|z|$ denotes the "usual" l_1 norm of z (such that $|p(S)| = |p|$), then since $|p_i| \leq K_{k-1}$ for $i \leq n_{k-1}$ we have

$$\| z \| \geq |z|/K_{k-1}$$

for $z \in$ Im Q_{k-1}; yet because $\left\| S^i \right\| \leq 1$, certainly

$$\| z \| \leq |z|.$$

Since $S^2Q_{k-1}(uv) \in \operatorname{lin}\{S^3, S^4, \ldots, S^{n_{k-1}+2}\}$, $\tau_{n_{k-1}}\big(Q_{k-1}(uv)S^2\big) \in \operatorname{lin}\{S^3, S^4, \ldots, S^{n_{k-1}}\}$, accordingly we have

$$\begin{aligned}\|(z_k^*(uv)-z_k^*(u)z_k^*(v))S^2 + \tau_{n_{k-1}}\big(Q_{k-1}(uv)S^2\big)\| \\ \geq \big(\|(z_k^*(uv) - z_k^*(u)z_k^*(v))S^2\| + \|\tau_{n_{k-1}}\big(Q_{k-1}(uv)S^2\big)\|\big) \quad . \\ \geq \big(|z_k^*(uv) - z_k^*(u)z_k^*(v)| + \|\tau_{n_{k-1}}\big(S^2Q_{k-1}(uv)\big)\|\big)/K_{k-1}\end{aligned} \tag{5.14}$$

However, by Lemma 5.2 (with $r = 0, x = S^2$ and $y = Q_{k-1}(uv)$),

$$\big\|\,(I - \tau_{n_{k-1}})\big(S^2Q_{k-1}(uv)\big)\,\big\| \leq \eta_k \,\|\, Q_{k-1}(uv)\,\| \cdot \big\|\, S^2 \,\big\| \leq \eta_k \,\|\, u\,\| \cdot \|\, v\,\|\,. \tag{5.15}$$

In that case, (5.13, 5.14) tell us that

$$\begin{aligned}|z_k^*(uv) - z_k^*(u)z_k^*(v)|+\|\tau_{n_{k-1}}S^2Q_{k-1}(uv)\| \\ \leq K_{k-1}(4\eta_k^2 + 7\eta_k)\|u\| \cdot \|v\| \\ \leq \sqrt{\eta_k}\|u\| \cdot \|v\|,\end{aligned} \tag{5.16}$$

given another "rapid increase" condition. This establishes inequality (5.10). For (5.11) note that since $\|\, u\,\| \leq |u|$ for all u, we may add (5.15, 5.16) to get

$$\big\|\, S^2Q_{k-1}(uv)\,\big\| \leq K_{k-1}(4\eta_k^2 + 8\eta_k)\|u\| \cdot \|v\| \leq \sqrt{\eta_k}\|u\| \cdot \|v\| \tag{5.17}$$

Now let $x, y \in \mathfrak{A}$ have unit norm, and write $x = Q_{k-1}x + u = x' + u$, $y = Q_{k-1}y + v = y' + v$ where $u, v \in \ker\, Q_{k-1}$. Then since $z_k^* \circ Q_{k-1} = 0$,

$$z_k^*(xy) - z_k^*(x)z_k^*(y) = z_k^*(uv + uy' + vx' + x'y') - z_k^*(u)z_k^*(v), \tag{5.18}$$

where $\|\, uy' - z_k^*(u)y'\,\| \leq 2\eta_k$ and $\|\, vx' - z_k^*(v)x'\,\| \leq 2\eta_k$, so since $z_k^* \circ Q_{k-1} = 0$, $|z_k^*(uy' + vx')| \leq 4\eta_k$. Also, by Lemma 5.2, and noting that $z_k^* = z_k^* \circ P_{k-1}$,

$$|z_k^*(x'y')| \leq \|\, P_{k-1}(x'y')\,\| \leq \eta_k\,. \tag{5.19}$$

Substituting our estimates in (5.18) we have

$$|z_k^*(xy) - z_k^*(x)z_k^*(y)| \leq 5\eta_k + |z_k^*(uv) - z_k^*(u)z_k^*(v)| \leq 5\eta_k + \sqrt{\eta_k} \leq 6\sqrt{\eta_k},$$

by (5.10), and this establishes inequality (5.12). $\square$

So z_k^* is indeed "approximately multiplicative", as we claimed when defining it.

Now, returning to the proof of the Theorem 5.1, suppose that $x, y \in \mathfrak{A}$ have unit norm and $xy = 0$. Substituting this fact into (5.9), and using the notation of Lemma 5.3, we get

$$\|x'y' + z_k^*(u)y' + z_k^*(v)x' + uv\| \leq 4\eta_k,$$

and in (5.12) we get

$$|z_k^*(x)z_k^*(y)| = |z_k^*(u)z_k^*(v)| \leq 6\sqrt{\eta_k}$$

for every k. So either $|z_k^*(x)| \leq 3\eta_k^{1/4}$ or $|z_k^*(y)| \leq 3\eta_k^{1/4}$. Swapping x and y as necessary we may assume that $|z_k^*(x)| \leq 3\eta_k^{1/4}$ for infinitely many k. Therefore,

$$\begin{aligned} \|x'y' + z_k^*(v)x' + uv\| &= \|Q_{k-1}(x)Q_{k-1}(y) + z_k^*(y)Q_{k-1}x + P_{k-1}(x)P_{k-1}(y)\| \\ &\leq 4\eta_k + 3\eta_k^{1/4} \end{aligned}$$

for infinitely many k. Apply S^2Q_{k-1} to the vector in the above equation and we get

$$\|S^2Q_{k-1}(x'y' + z_k^*(v)x' + uv)\| \leq 4\eta_k + 3\eta_k^{1/4}$$

or by (5.17),

$$\|S^2Q_{k-1}(x'y' + z_k^*(v)x')\| \leq 4\eta_k + 3\eta_k^{1/4} + \sqrt{\eta_k}\,.$$

As in (5.19), $\|P_{k-1}(x'y')\| \leq \eta_k$, so

$$\begin{aligned} \|S^2(x'y' + z_k^*(v)x')\| &= \|S^2\big(Q_{k-1}(x)Q_{k-1}(y) + z_k^*(y)Q_{k-1}(x)\big)\| \\ &\leq 5\eta_k + 3\eta_k^{1/4} + \sqrt{\eta_k} \leq 9\eta_k^{1/4} \end{aligned} \tag{5.20}$$

for infinitely many k. The argument will be concluded by showing that (5.20) cannot hold for infinitely many k.

Recall that we have assumed that all C_k are divisible by 3. For $k > 1$ define a "sliding truncation" operator $T_k : \mathrm{Im}\ Q_k \to \mathrm{Im}\ Q_k$ as follows: if $i \in [r(n+1), r(n+1)+n]$, where $n = n_{k-1}$ and $0 \leq r \leq 1 + C = 1 + C_k$, then

$$T_k(S^i) = \begin{cases} (1 - 3r/C_k)S^i & r \leq C_k/3 \\ 0 & r \geq C_k/3 \end{cases} \tag{5.21}$$

One may then verify that if $i \in [r(n+1), r(n+1)+n]$, then writing $i_1 = i - r(n+1)$ and $\mu_r = \rho_r^{(k)} \cdot D_k^{r-1} \cdot B_k^{2i_1-1}\alpha_k^{(r+1)(n+1-i_1)}$, we have

$$T_k(p_i(S)) = \begin{cases} (1 - 3r/C_k)p_i(S) & r = 0 \text{ or } i_1 = 0,\ r \leq C_k/3 \\ (1 - 3r/C_k)p_i(S) - 3\mu_r S^{i-n-1}C_k^{-1} & i_1 > 0,\ 1 \leq r \leq C_k/3 \\ 0 & r > C_k/3 \end{cases}$$

The norm of the "error vector" $3\mu_r S^{i-n-1}C_k^{-1}$ is estimated by (4.20), which tells us (for $i_1 \leq n$)

$$\left\| D_k^{r-1} S^{(r-1)(n_{k-1}+1)+i_1} - \prod_{t=1}^{r-1} \rho_t S^{i_1} \right\| \leq A_k/B_k\,,$$

so in particular

$$\left\| D_k^{r-1} S^{(r-1)(n_{k-1}+1)+i_1} \right\| \leq 1 + A_k/B_k\,.$$

Hence

$$\| (3\mu_r/C_k)\cdot S^{i-n-1} \| \leq 3(1+A_k/B_k)B_k^{2i_1-1}/C_k \leq C_k^{-1/2},$$

subject to "rapid increase" again. Therefore, $\| T_k \| \leq 1 + C_k^{-1/2}$, or $\| T_k(p_i(S)) \| \leq 1 + C_k^{-1/2}$ for all i. Since the T_k are uniformly bounded, and since for each finite linear combination x of the S^i we have $T_kQ_k(x) = x$ for all but finitely many k, it follows that

$$T_kQ_k(x) \to x \qquad (x \in \mathfrak{A})\,.$$

Choose (hypothetical) unit norm x and y such that (5.20) is satisfied; that is,

$$\| S^2(Q_k(x)Q_k(y) + z_{k+1}^*(y)Q_k(x)) \| < 9\eta_{k+1}^{1/4}$$

infinitely often. Then even

$$\| \tau_{n_k}\big(S^2Q_k(x)Q_k(y)\big) + z_{k+1}^*(y)S^2Q_k(x) \| < 10\eta_{k+1}^{1/4} \tag{5.22}$$

infinitely often, since another application of Lemma 5.2 gives

$$\| (I-\tau_{n_k})\big(S^2Q_k(x)Q_k(y)\big) \| \leq \eta_{k+1}.$$

For some such $k > 1$ we have $\| T_kQ_k(x) \| > 1/2$ and $\| T_kQ_k(y) \| > 1/2$. Let $\tau[k]$ denote the truncation operator $\tau_{c_k'}$,where $c_k' = (1 + C_k/3)(1 + n_{k-1})$.

From (5.21) we have $|T_k(z)| \leq |\tau[k](z)|$ for all $z \in \operatorname{Im}\ Q_k$, so since $|\cdot| \geq \|\cdot\|$, $|\tau[k](Q_k(x))| \geq 1/2$ and $|\tau[k](Q_k(y))| \geq 1/2$. Now $\| Q_k(x) \|$, $\| Q_k(y) \| \leq 1$, so $|Q_k(x)| \leq K_k$, $|Q_k(y)| \leq K_k$.

The collection $\mathcal{G}$ of pairs of vectors (x', y') such that $x', y' \in \operatorname{lin}\{S^i, i \leq n_k\}$, $|x'|, |y'| \leq K_k$ and $|\tau[k](x')|$, $|\tau[k](y')| \geq 1/2$ is compact. For no such pair is it true that $\tau_{n_k}(S^2x'y') \in \operatorname{lin}(S^2x')$ (for $S^2x'y'$ has a nonzero coordinate in S^i with $i \leq 2(1 + C_k/3)(1 + n_{k-1}) + 2 < n_k$ so it is not possible that $\tau_{n_k}(S^2x'y') = 0$. On the other hand S^2x' has a nonzero coordinate at least one to the left of any in

$\tau_{n_k}(S^2x'y')$, for of course $y' \in \mathrm{lin}\{S, S^2, \ldots\}$. So it is not possible for $\tau_{n_k}(S^2x'y') = \delta \cdot S^2x'$ for any nonzero δ either). Thus

$$\beta(c'_k, n_k, K_k) = \inf\{|\tau_{n_k}(S^2x'y') + z| : z \in \mathrm{lin}(S^2x'), (x', y') \in \mathcal{G}\} > 0\,.$$

The final condition of rapid increase is to choose η_{k+1}, for each k, so small that

$$10\eta_{k+1}^{1/4} < \beta(c'_k, n_k, K_k)/K_k.$$

(this is a fairly serious condition of rapid increase on E_{k+1} because the unknown function β may be very small.) Then (referring back to (5.22)), we find that for k large enough that $\|\, T_kQ_k(x) \,\| > 1/2$ and $\|\, T_kQ_k(y) \,\| > 1/2$,

$$\|S^2(Q_k(x)Q_k(y) + z^*_{k+1}(y)Q_k(x))\| \geq |S^2(Q_k(x)Q_k(y) + z^*_{k+1}(y)Q_k(x))|/K_k$$

$$\geq d(S^2Q_k(x)Q_k(y), \mathrm{lin}(S^2Q_k(x)))/K_k \geq \beta(c'_k, n_k, K_k)/K_k > 10\eta_{k+1}^{1/4},$$

which means that contrary to hypothesis, (5.22) – and hence (5.20) – cannot happen for the fixed unit norm vectors x and y and infinitely many k. This contradiction stems from the assumption that $xy = 0$ with x, y nonzero. So $\mathfrak{A}$ is in fact an integral domain. □

Remark. As we have just shown, the example constructed here is not amenable. Since submission of this paper C.J. Read has built on the techniques here to construct a commutative, radical amenable Banach algebra [28]. This new example is not an integral domain, nor does it have compact multiplication. It is hoped that a variation on the construction of the present paper, using a modified version of (2.1) to give an approximate diagonal, might yield an example of a commutative, radical amenable algebra with compact multiplication and a bounded approximate identity of normalized powers.

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