Approximate Amenability of Fréchet Algebras.

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Abstract

The notion of approximate amenability was introduced by Ghahramani and Loy, in the hope that it would yield Banach algebras without bounded approximate identity which nonetheless had a form of amenability. So far, however, all known approximately amenable Banach algebras have bounded approximate identities. In this paper we define approximate amenability and contractibility of Fréchet algebras, and we prove the analogue of the result for Banach algebras that these properties are equivalent. We give examples of Fréchet algebras which are approximately contractible, but which do not have a bounded approximate identity. For a good many Fréchet algebras without b.a.i., we find either that the algebra is approximately amenable, or it is "obviously" not approximately amenable because it has continuous point derivations. So the situation for Fréchet algebras is quite close to what was hoped for Banach algebras.

Keywords: Frechet algebra, Banach algebra, amenability, contractibility, approximate identity, derivation

1 Introduction.

We first recall some of the basic facts about approximate contractibility in the Banach algebra setting.

Definition 1.1. Let A be an algebra, and let X be an A-bimodule. A linear map $D : A \to X$ is called a *derivation* if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A)\,.$$

For each $x \in X$, we define a map $ad_x : A \to X$ by

$$ad_x(a) = a \cdot x - x \cdot a \qquad (a \in A)\,.$$

Note that ad_x is a derivation. Derivations of this form are called *inner derivations*. Suppose that A is a Banach algebra, let X be a Banach A-bimodule, and let $D : A \to X$ be a continuous derivation. Then D is said to be *approximately inner* if there is a net $(x_\alpha) \subset X$ such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \qquad (a \in A)\,.$$

Hence D is approximately inner if it is in the closure of the set of inner derivations with respect to the strong operator topology on $\mathcal{B}(A)$.

Definition 1.2. A Banach algebra A is said to be *contractible* if, given any Banach A-bimodule X, every continuous derivation $D : A \to X$ is inner. The Banach algebra A is *approximately contractible* if every continuous derivation $D : A \to X$ into every Banach A-bimodule X is *approximately* inner.

Given a Banach A-bimodule X, the dual space X^* is a Banach A-bimodule in a natural way. See [3] for the definition of a dual module.

Definition 1.3. The Banach algebra A is *amenable* if, given any Banach A-bimodule X, every derivation $D : A \to X^*$ is inner. A is *approximately amenable* if every derivation $D : A \to X^*$ is approximately inner. For an account of amenability see [3].

There are several characterizations of approximate contractibility, which we list below. The equivalence of assertions (i), (ii) and (iv) can be found in [4]. The proofs of the equivalence of all of the statements are the same as those given later, for the Fréchet algebra case. Note that we will regard A as a subspace of A^{**} via the canonical embedding $\iota : A \to A^{**}$, and that we will denote $\iota(a)$ by $\widehat{a}$.

Theorem 1.4. *Let A be a Banach algebra. Then the following conditions are all equivalent:*

(i) A is approximately contractible

(ii) There exists a net $(d_\alpha)_{\alpha\in\Lambda} \subset A^\sharp \widehat{\otimes} A^\sharp$ such that

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \qquad (a \in A)$$

and such that

$$\pi(d_\alpha) = e \qquad (\alpha \in \Lambda)\,,$$

where $\pi : A^\sharp \widehat{\otimes} A^\sharp \to A$ is the diagonal map.

(iii) There exists a net $(D_\alpha)_{\alpha\in\Lambda} \subset (A^\sharp \widehat{\otimes} A^\sharp)^{**}$ such that

$$a \cdot D_\alpha - D_\alpha \cdot a \to 0 \qquad (a \in A)$$

and such that

$$\pi^{**}(D_\alpha) = \widehat{e} \qquad (\alpha \in \Lambda)$$

(iv) There exists nets $(d_\alpha) \subset A\widehat{\otimes}A$, (u_α), $(v_\alpha) \subset A$ *such that*

$$a \cdot d_\alpha - d_\alpha \cdot a + u_\alpha \otimes a - a \otimes v_\alpha \to 0 \qquad (a \in A)\,,$$

such that

$$au_\alpha - a, v_\alpha a - a \to 0 \qquad (a \in A)\,,$$

and such that

$$\pi(d_\alpha) = e \qquad (\alpha \in \Lambda)$$

(v) A is approximately amenable

2 Approximately Contractible Fréchet algebras.

We now formulate contractibility of Fréchet algebras. By a Fréchet algebra we mean a complete topological algebra, whose topology is given by an increasing countable family of submultiplicative semi-norms. When we say that a Fréchet space X is a Fréchet A-bimodule, we will mean that X is an algebraic bimodule of A such that the action on both sides is continuous. For the general theory see [5].

Definition 2.1. Let A be a Fréchet algebra. Then we say that A is *approximately contractible* if given any A-bimodule X, and any continuous derivation $D : A \to X$, there is a net $(x_\alpha) \subset X$ such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \qquad (a \in A)\,.$$

Any continuous derivation D with this property is said to be *approximately inner.*

We now show that the characterizations (i), (ii) and (iv) of Theorem 1.4 of approximate contractibility for the Banach algebra case also hold for Fréchet algebras. Conditions (iii) and (v) will be discussed later. We first need two lemmas, both of which follow in the same way as for the Banach algebra case, proved in [4]. Note that, given a unital Fréchet algebra A with identity e, a Fréchet A-bimodule is called *unit linked* if $e \cdot x = x$ for every x in X.

Lemma 2.2. *Let A be a unital Fréchet algebra. Then A is approximately contractible if and only if every continuous derivation into any unit linked Fréchet bimodule is approximately inner.*

Proof: The proof is very similar to that of Lemma 2.3 of [4]. Suppose that every continuous derivation into any unit linked Fréchet bimodule is approximately inner. Let X be a Fréchet A-bimodule, and let $D : A \to X$ be a continuous derivation. Let e be the unit of A, define $X_1 = e \cdot X \cdot e$, $X_2 = (1-e)\cdot X \cdot e$, $X_3 = e \cdot X \cdot (1-e)$, $X_4 = (1-e)\cdot X \cdot (1-e)$, and let $\pi_j X \to X_j$ be the associated projections. Then we have $X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$. For $j = 1, 2, 3, 4$, let $D_j = \pi_j \circ D$. Then $D = D_1 + D_2 + D_3 + D_4$. But we have that $D_2 = ad_{(-D_2 e)}$, $D_3 = ad_{D_3 e}$, and $D_4 = 0$. Hence D is the sum of an inner derivation and a derivation into the unit linked module X_1. We conclude that D must be approximately inner. ■

Lemma 2.3. *The Fréchet algebra A is approximately contractible if and only if $A^\sharp$ is approximately contractible.*

Proof: This is proved in exactly the same manner as in Proposition 2.4 of [4]. ■

Theorem 2.4. *Let A be a Fréchet algebra. Then the following conditions are all equivalent:*

(i) A is approximately contractible

(ii) There exists a net $(d_\alpha)_{\alpha \in \Lambda} \subset A^\sharp \widehat{\otimes} A^\sharp$ such that

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \qquad (a \in A)$$

and such that

$$\pi(d_\alpha) = e \qquad (\alpha \in \Lambda)\,,$$

where $\pi : A^\sharp \widehat{\otimes} A^\sharp \to A$ is the diagonal map.

(iii) There exists nets $(d_\alpha) \subset A \widehat{\otimes} A$, (u_α), $(v_\alpha) \subset A$ such that

$$a \cdot d_\alpha - d_\alpha \cdot a + u_\alpha \otimes a - a \otimes v_\alpha \to 0 \qquad (a \in A)\,,$$

such that

$$a u_\alpha - a, v_\alpha a - a \to 0 \qquad (a \in A)\,,$$

and such that

$$\pi(d_\alpha) = e \qquad (\alpha \in \Lambda)$$

Proof: $(i) \Rightarrow (ii)$: By Lemma 2.3 we have that $A^\sharp$ is approximately contractible. Let $u = e \otimes e \in A^\sharp \widehat{\otimes} A^\sharp$. Regarding $A^\sharp \widehat{\otimes} A^\sharp$ as a Fréchet $A^\sharp$-bimodule, we have that

$$ad_u(a) = a \cdot u - u \cdot a = a \otimes e - e \otimes a\,,$$

so that ad_u is a derivation from $A^\sharp$ into ker π, which is a Fréchet A-bimodule. Hence there is a net $(e_\alpha)_{\alpha\in\Lambda} \subset \ker \pi$ with

$$ad_u(a) = lim_\alpha ad_{e_\alpha}(a) \qquad (a \in A^\sharp)\,.$$

Let

$$d_\alpha = u - e_\alpha \qquad (\alpha \in \Lambda)\,.$$

Then

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \qquad (a \in A)$$

and

$$\pi(d_\alpha) = \pi(u) = e \qquad (\alpha \in \Lambda)\,,$$

as required.
$(ii) \Leftrightarrow (iii)$: this follows the same method as that used to prove Corollary 2.2 of [4].
$(ii) \Rightarrow (i)$: We prove that $A^\sharp$ is approximately contractible, which is sufficient by 2.3. Let X be a Fréchet A-bimodule and let $D : A^\sharp \to X$ be a continuous derivation. By 2.2 we may assume that X is unit linked. Let $(d_\alpha)_{\alpha\in\Lambda} \subset A^\sharp \widehat{\otimes} A^\sharp$ be as in (ii). Then, for each $n \in \mathbb{N}$, and each $\alpha \in \Lambda$, we can find a_n^α, b_n^α such that

$$d_\alpha = \sum_{n=1}^{\infty} a_n^\alpha \otimes b_n^\alpha \qquad (\alpha \in \Lambda)\,.$$

Now define

$$x_\alpha = \sum_{n=1}^{\infty} a_n^\alpha \cdot D(b_n^\alpha) \in X \qquad (\alpha \in \Lambda)\,.$$

Note that these elements are defined in the obvious way via the continuous linear extension of the map

$$a \otimes b \mapsto a \cdot D(b) \qquad (a, b \in A)\,.$$

Then, there exist nets $(h_1)_{\alpha\in\Lambda}$, $(h_2)_{\alpha\in\Lambda} \subset X$ with $h_1, h_2 \to 0$ such that

$$\begin{aligned}
a \cdot x_\alpha &= \sum_{n=1}^{\infty} aa_n^\alpha \cdot Db_n^\alpha \\
&= \sum_{n=1}^{\infty} a_n^\alpha \cdot D(b_n^\alpha a) + h_1(\alpha) \\
&= \sum_{n=1}^{\infty} (a_n^\alpha b_n^\alpha \cdot D(a) + a_n^\alpha \cdot Db_n^\alpha \cdot a) + h_1(\alpha) \\
&= \pi(d_\alpha) \cdot D(a) + x_\alpha \cdot a + h_1(\alpha) \\
&= D(a) + x_\alpha \cdot a + h_2(\alpha) + h_1(\alpha)\,,
\end{aligned}$$

and so

$$D(a) = lim_\alpha a \cdot x_\alpha - x_\alpha \cdot a \qquad (a \in A)\,,$$

as claimed. ■

We next discuss conditions (iii) and (v) of theorem 1.4. These both involve dual spaces, and so if we wish to find analogues for Fréchet spaces, we need to make clear what we mean by the dual space X^* of a Fréchet space X. Unless otherwise stated, our dual space X^* will be endowed with the *strong* topology, which means the topology of uniform convergence on bounded subsets of X. Note that this coincides with the usual norm topology if X is a normed space. Basic properties we will need are that X^* is a locally convex space, and X^{**} is a Fréchet space. Also, X can be continuously embedded in X^{**} via the usual injection $\iota : X \to X^{**}$, and $\iota(X)$ is weak*-dense in X^{**}. Also note that given a Fréchet A-bimodule X, X^* is a locally convex bimodule with continuous actions in the usual way.

Definition 2.5. The Fréchet algebra A is *approximately amenable* if given any A-bimodule X, every continuous derivation $D : A \to X$ is approximately inner.

Theorem 2.6. *Let A be a Fréchet algebra. Then the following are equivalent:*
(i) A is approximately contractible
(ii) There exists a net $(D_\alpha)_{\alpha\in\Lambda} \subset (A^\sharp\widehat{\otimes}A^\sharp)^{**}$ such that

$$a \cdot D_\alpha - D_\alpha \cdot a \to 0 \qquad (a \in A)$$

and such that

$$\pi^{**}(D_\alpha) = \widehat{e} \qquad (\alpha \in \Lambda)$$

(iii) A is approximately amenable

Proof: $(i) \Rightarrow (ii)$: This follows easily from condition (ii) of 1.4.
$(ii) \Rightarrow (i)$: For each $a \in A$ define

$$B_a = \{a \cdot d - d \cdot a : d \in A^\sharp\widehat{\otimes}A^\sharp, \pi(d) = e\}\,,$$

$$\Gamma_a = \{a \cdot D - D \cdot a : D \in (A^\sharp\widehat{\otimes}A^\sharp)^{**}, \pi^{**}(D) = \widehat{e}\}\,.$$

Now suppose that A is *not* approximately contractible. Then there is a finite set $S \subset A$ such that

$$0 \notin \bigcap_{a\in S} \overline{B_a}\,,$$

which is closed and convex. Hence we can find a continuous linear functional $\lambda \in A^*$ such that

$$\lambda(\bigcap_{a\in S} \overline{B_a}) = 1\,.$$

Therefore we have that

$$0 \notin \overline{\iota(\bigcap_{a\in S} \overline{B_a})}^{wk^*} \supset \bigcup_{a\in S} \overline{\iota(B_a)}^{wk^*}\,,$$

where we have used that ι is injective. But $\iota(B_a) = \Gamma_a \cap (A^\sharp \widehat{\otimes} A^\sharp)$, and so we have that $\overline{\iota(B_a)}^{wk^*} = \overline{\Gamma_a}^{wk^*}$. Therefore

$$0 \notin \bigcup_{a\in S} \overline{\Gamma_a}^{wk^*} \supset \bigcup_{a\in S} \overline{\Gamma_a}\,,$$

and this implies that A cannot satisfy condition (ii), as required.

$(ii) \Leftrightarrow (iii)$: This can be done along similar lines to the method employed for the Banach algebra case in [4]. ■

We next give a sufficient condition which is useful for finding examples of approximately contractible Fréchet algebras.

Lemma 2.7. *Let A be a Fréchet algebra given by an increasing sequence of seminorms $(p_n)_{n\in\mathbb{N}}$. Suppose that there exists a (possibly unbounded) approximate identity (u_α). Then A is approximately contractible provided the following conditions hold:*

(i) For each $n \in \mathbb{N}$ we have

$$p_n(au_\alpha - a)p_n(u_\alpha) \to 0 \qquad (a \in A)$$

(ii) For each $n \in \mathbb{N}$ we have

$$p_n(u_\alpha a - a)p_n(u_\alpha) \to 0 \qquad (a \in A)$$

(iii) There is a net $(d_\alpha) \subset A\widehat{\otimes}A$ such that

$$\pi(d_\alpha) = 2u_\alpha - u_\alpha^2$$

and such that

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \qquad (a \in A)\,.$$

Proof: Define

$$D_\alpha := u_\alpha \otimes u_\alpha + d_\alpha\,.$$

We clearly have that

$$\pi(D_\alpha) = 2u_\alpha\,,$$

so it is enough to show that

$$a \cdot D_\alpha - D_\alpha \cdot a + u_\alpha \otimes a - a \otimes u_\alpha \to 0 \qquad (a \in A)\,.$$

Let $\widehat{p_n}$ be the nth seminorm on $A\widehat{\otimes}A$, thus

$$\widehat{p_n}(x) = \inf\left\{\sum_{k=1}^{\infty} p_n(a_k)p_n(b_k) : x = \sum_{k=1}^{\infty} a_k \otimes b_k\right\} \qquad (x \in A\widehat{\otimes}A)\,.$$

Now

$$\begin{aligned} a \cdot D_\alpha - D_\alpha \cdot a + u_\alpha \otimes a - a \otimes u_\alpha &= (au_\alpha - a) \otimes u_\alpha + u_\alpha \otimes (u_\alpha a - a) \\ &\quad + a \cdot d_\alpha - d_\alpha \cdot a\,, \end{aligned}$$

so that

$$\begin{aligned} \widehat{p_n}(a \cdot D_\alpha - D_\alpha \cdot a + u_\alpha \otimes a - a \otimes u_\alpha) &\le p_n(au_\alpha - a)p_n(u_\alpha) + p_n(u_\alpha)p_n(u_\alpha a - a) \\ &\quad + \widehat{p_n}(a \cdot d_\alpha - d_\alpha \cdot a)\,, \end{aligned}$$

which tends to zero by conditions $(i), (ii)$ and (iii) above. ■

Obviously, for any bounded approximate identity (u_α), conditions (i) and (ii) of Lemma 2.7 will automatically hold. However, we have not been able to construct an example of a Banach algebra with an *unbounded* approximate identity (u_α) such that these conditions hold. But the situation is very different for Fréchet algebras.

3 Examples

In this section, we give some examples to show that the conditions given in Lemma 2.7 occur in certain well known Fréchet algebras which do not possess a bounded approximate identity. Pirkovskii [9] has developed the theory of amenable Fréchet algebras. In particular, he shows that any amenable Fréchet algebra must possess a *locally bounded* approximate identity, which means an approximate identity $(u_\alpha) \subset A$ such that

$$\sup_\alpha p_n(u_\alpha) < \infty \qquad (n \in \mathbb{N})\,.$$

A Fréchet algebra A has a locally bounded approximate identity if and only if for each Arens-Michael decomposition

$$A = \lim_{\longleftarrow} A_\lambda,$$

each A_λ has a bounded approximate identity.

The first two examples are of approximately contractible Fréchet algebras which are not amenable.

Example 3.1. Let $s \subset \mathbb{C}^{\mathbb{N}}$ be the vector space of all complex sequences which tend to zero faster than any polynomial, i.e let

$$s = \{(x_k)_{k\in\mathbb{N}} : k^n |x_k| \to 0 \ as \ k \to \infty, \text{ for all } n \in \mathbb{N}\}$$

This is known as the space of rapidly decreasing sequences (see [8] for a detailed discussion of this space). From now on, $(x_k)_{k\in\mathbb{N}}$ will be denoted $\mathbf{x}$. The vector space A becomes a commutative algebra when equipped with pointwise multiplication, and is a Fréchet algebra with respect to the norms $(p_n)_{n\in\mathbb{N}}$, given by

$$p_n(\mathbf{x}) = \sup\{k^n |x_k| : k > 0\} \qquad (n \in \mathbb{N}).$$

Let $(\mathbf{e_i})_{i\in\mathbb{N}}$ be the usual basis of $\mathbb{C}^{\mathbb{N}}$, so that

$$(\mathbf{e_i})_j = \delta_{ij} \qquad (i, j \in \mathbb{N}).$$

Then it is clear that

$$\mathbf{u_i} := \mathbf{e_1} + \mathbf{e_2} + \ldots + \mathbf{e_i}$$

is an unbounded sequential approximate identity for A. We also note that each $\mathbf{u_i}$ is an idempotent. Next, we define the elements $d_i \in A \otimes A$ by

$$d_i = \sum_{j=1}^{i} \mathbf{e_j} \otimes \mathbf{e_j} \qquad (i \in \mathbb{N}).$$

Then we have that

$$\pi(d_i) = \mathbf{u_i} = 2\mathbf{u_i} - \mathbf{u_i}^2 \qquad (i \in \mathbb{N}),$$

and that

$$\mathbf{x} \cdot d_i - d_i \cdot \mathbf{x} = 0 \qquad (i \in \mathbb{N}, \mathbf{x} \in A).$$

Hence the only thing we need to check is that condition (i) and (ii) of Lemma 2.7 are satisfied, i.e. that

$$p_n(\mathbf{x}\mathbf{u_i} - \mathbf{x}) p_n(\mathbf{u_i}) \to 0 \ as \ i \to \infty \qquad (n \in \mathbb{N}).$$

But we have that

$$\begin{aligned} p_n(\mathbf{x}\mathbf{u_i} - \mathbf{x})p_n(\mathbf{u_i}) &= i^n \cdot \sup\{k^n|x_k| : k > i\} \\ &\leq \sup\{k^{2n}|x_k| : k > i\}\,, \end{aligned}$$

and this must converge to zero as i tends to infinity because

$$k^{2n}|x_k| \to 0 \;\; as \;\; k \to \infty$$

by definition of A. Hence we have that A is approximately contractible by Lemma 2.7. However it is clear that A does not have a locally bounded approximate identity, and so it is not amenable.

Our second example is a continuous version of the first example.

Example 3.2. Let $A \subset C_0(\mathbb{R})$ be the collection of continuous complex-valued functions which tend to zero faster than any polynomial, i.e.

$$A = \{f \in C(\mathbb{R}) : |x|^n|f(x)| \to 0 \;\; as \;\; |x| \to \infty \text{ for all } n \in \mathbb{N}\}\,.$$

Then, with pointwise addition and multiplication, A is a Fréchet algebra with respect to the norms

$$p_n(f) = \sup\{(1 + |x|)^n|f(x)| : x \in \mathbb{R}\}\,.$$

To see that A satisfies the conditions of Lemma 2.7, we first choose an approximate identity. Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $0 \leq u \leq 1$, $u(x) = 1$ for $|x| = 1$ and $u(x) = 0$ for $|x| \geq 2$. Next, define the functions $u_k : \mathbb{R} \to \mathbb{R}$ by

$$u_k(x) = u\left(\frac{x}{k}\right) \qquad (k \in \mathbb{N}, x \in \mathbb{R})\,.$$

Then $(u_k)_{k\in\mathbb{N}}$ is an unbounded sequential approximate identity, as in the first example. Also, in a similar fashion to the last example we have (since u_k is supported on $[-2k, 2k]$),

$$p_n(u_kf - f)\cdot p_n(u_k) \leq \sup\{(1+|x|)^n|f(x)| : |x| \geq k\}\cdot(1+2k)^n \qquad (k \in \mathbb{N}, f \in A)\,,$$

which tends to zero as k tends to infinity, because

$$(1 + |x|)^{2n+1}|f(x)| \to 0 \;\; as \;\; n \to \infty \qquad (n \in \mathbb{N})\,,$$

by definition of the algebra A. Hence, we have that the u_k form an unbounded approximate identity satisfying conditions (i) and (ii) of Lemma 2.7. We only need to show that condition (iii) can be satisfied. Firstly, given $k \in \mathbb{N}$, define

$$\varepsilon_{r,k}(x) = e^{\frac{irx}{4k+1}} \qquad (r \in \mathbb{Z}, x \in \mathbb{R})\,,$$

and let

$$\delta_{r,k}(x) = \sqrt{2u_k(x) - u_k(x)^2} \cdot \varepsilon_{r,k}(x) \qquad (r \in \mathbb{Z}, x \in \mathbb{R}),$$

noting that $\delta_{r,k}$ is supported in $[-2k, 2k]$. Finally, define

$$d_{R,k} = \frac{1}{2R+1} \sum_{r=-R}^{R} \delta_{r,k} \otimes \delta_{-r,k} \qquad (R \in \mathbb{N}).$$

Note that

$$\pi(d_{R,k}) = 2u_k - u_k^2 .$$

Observe that if $f|_{[-2k,2k]}$ agrees on $[-2k, 2k]$ with a trigonometric function $\varepsilon_{s,k}$ for some $0 \leq s \leq R$ then

$$\begin{aligned} f \cdot d_{R,k} - d_{R,k} \cdot f &= \frac{1}{2R+1} \sum_{r=-R}^{R} (\delta_{r+s,k} \otimes \delta_{-r,k} - \delta_{r,k} \otimes \delta_{-r+s,k}) \\ &= \frac{1}{2R+1} \sum_{r=R+1}^{R+s} (\delta_{r,k} \otimes \delta_{s-r,k} - \delta_{s-r,k} \otimes \delta_{r,k}), \end{aligned}$$

and so we have

$$\begin{aligned} \widehat{p_n}(f \cdot d_{R,k} - d_{R,k} \cdot f) &\leq \frac{2s}{2R+1} \cdot \max_r p_n(\delta_{r,k})^2 \\ &\leq \frac{2s}{2R+1} (1+2k)^{2n} . \end{aligned}$$

We have exactly the same argument when $f|_{[-2k,2k]}$ agrees on $[-2k, 2k]$ with a trigonometric function $\varepsilon_{s,k}$ for some $-R \leq s < 0$. Likewise, if $f|_{[-2k,2k]}$ agrees on $[-2k, 2k]$ with a trigonometric polynomial

$$\sum_{s=-S}^{S} a_s \cdot \varepsilon_{s,k} ,$$

then

$$\widehat{p_n}(f \cdot d_{R,k} - d_{R,k} \cdot f) \leq \frac{2}{2R+1} (1+2k)^{2n} \cdot \sum_{s=-S}^{S} s|a_s| . \qquad (1)$$

But, for any $f \in A$, we clearly have that

$$\widehat{p_n}(f \cdot d_{R,k} - d_{R,k} \cdot f) \leq 2 \cdot (1+2k)^n \cdot \sup_{|t| \leq 2k} |f(t)| . \qquad (2)$$

Now, given a general $f \in A$, by the Stone-Weierstrass theorem, noting that the $\varepsilon_{r,k}$ have period $4k+1$, we can choose a trigonometric polynomial

$$g = \sum_{s=-S}^{S} a_s \cdot \varepsilon_{s,k}$$

such that

$$\sup_{|t|\leq 2k} |f(t) - g(t)| < \frac{\varepsilon}{4 \cdot (1+2k)^n}\,,$$

so that by (2) we have

$$\widehat{P_n}((f-g) \cdot d_{R,k} - d_{R,k} \cdot (f-g)) \leq \frac{\varepsilon}{2}\,.$$

Next, by (1), we can choose a large R so that

$$\widehat{p_n}(g \cdot d_{R,k} - d_{R,k} \cdot g) \leq \frac{\varepsilon}{2}\,.$$

Hence we will have that

$$\widehat{p_n}(f \cdot d_{R,k} - d_{R,k} \cdot f) \leq \varepsilon\,.$$

It is now clear that, for any for any finite set $F \subset A$, and any $\varepsilon > 0, k \in \mathbb{N}$, we can find an $R \in \mathbb{N}$ such that

$$\widehat{p_n}(f \cdot d_{R,k} - d_{R,k} \cdot f) \leq \varepsilon \qquad (f \in F)\,. \tag{3}$$

while

$$\pi(d_{R,k}) = 2u_k - u_k^2\,.$$

Finally, using (3), in a standard manner we can construct nets from the u_k, $d_{R,k}$ so that the conditions in Lemma 2.7 are satisfied. Hence, for the same reasons as the last example, A is approximately contractible but not amenable.

We next discuss the contractibility properties of some of the other common examples of Fréchet algebras. In particular, we show that there are some common Fréchet algebras which are not approximately contractible.

Example 3.3. Let S denote the Schwartz space, the algebra consisting of infinitely differentiable functions whose derivatives all tend to zero faster than any polynomial. In other words, S consists of all functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$|x|^p |f^{(m)}(x)| \to 0 \ \text{ as } \ |x| \to \infty \qquad (p, m \in \mathbb{N})\,.$$

We define seminorms p_n on S by

$$p_n(f) = \sup_{x \in \mathbb{R}} \left(|x|^n \sum_{k=0}^{n} \frac{|f^{(k)}(x)|}{k!} \right) .$$

With respect to these seminorms, S is a Fréchet algebra. However, S is *not* approximately amenable. To see this, first pick some $a \in \mathbb{R}$. Then

$$D : f \mapsto f'(a)$$

is a non-zero continuous point derivation with respect to the character

$$\phi : f \mapsto f(a) .$$

Since the inner derivations into the bimodule $\mathbb{C}$ are all zero, D is not approximately inner and A is not approximately contractible.

Example 3.4. In a similar manner to the above examples one can show that the algebra of continuous functions on $\mathbb{R}^d$ which vanish faster than any polynomial is approximately amenable, whereas the Schwartz space $S(\mathbb{R}^d)$ is not; neither are algebras A consisting of functions on $\mathbb{R}^d$ whose first k derivatives vanish faster than any polynomial $(d, k > 0)$. In all these cases, one either has obvious continuous point derivations, or the algebra turns out to be approximately amenable.

Example 3.5. Let $D \subset \mathbb{C}$ be the open unit disc, and let $A = \mathcal{O}(D)$ be the algebra of analytic functions $f : D \to \mathbb{C}$. Then A is a Fréchet algebra with respect to the system of seminorms p_n, given by

$$p_n(f) = \sup_{|x| \leq 1-1/n} |f(x)| .$$

However, as in the last example, A is not approximately amenable as the map

$$f \mapsto f'(0)$$

is a non-zero continuous point derivation.

We next discuss the approximate contractibility of certain Fréchet sequence algebras. In fact, as we shall see, Example 3.1 is a special case of the algebras discussed. As well as giving more examples of approximately contractible Fréchet algebras, these examples will give a class of Fréchet algebras which do not possess point derivations and yet are not approximately contractible. Also, these examples show that the existence of an approximate identity (u_α) does not ensure that conditions (i) and (ii) of Lemma 2.7 hold.

Definition 3.6. A matrix $A = (a_{j,k})_{j,k\in\mathbb{N}}$ of non-negative real numbers is called a *Köthe matrix* if it satisfies the following:
(i) For each $j \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ with $a_{j,k} > 0$.
(ii) $a_{j,k} \leq a_{j,k+1}$ for all $j, k \in \mathbb{N}$.
For each Köthe matrix A, we define

$$\lambda^1(A) = \left\{ \mathbf{x} \in \mathbb{C}^{\mathbb{N}} : \|\mathbf{x}\|_k := \sum_{j=1}^{\infty} |x_j| a_{j,k} < \infty \text{ for all } k \in \mathbb{N} \right\} .$$

For a detailed discussion of these spaces see [8].

Remark 3.7. Note that given a Köthe matrix A, $\lambda^1(A)$ is just the intersection of the weighted sequence spaces $l^1(\omega^{(n)})$, where $\omega_k^{(n)} = a_{k,n}$ Hence from now on we will write $A = (\omega_k^{(n)})_{n,k\in\mathbb{N}}$.

The following lemma is immediate.

Lemma 3.8. *Let A be a Köthe matrix such that $\omega_k^{(n)} \in \{0\} \cup [1, \infty)$ for all $n, k \in \mathbb{N}$. We equip $\lambda^1(A)$ with pointwise multiplication. Then, with respect to the semi-norms $\|\cdot\|_k$, $\lambda^1(A)$ is a Fréchet algebra.*

We first give a condition which ensures that $\lambda^1(A)$ is approximately contractible.

Proposition 3.9. *Let $\lambda^1(A)$ be as in Lemma 3.8, and suppose that for each $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ and $C > 0$ such that*

$$C\omega_k^{(N)} \geq k(\omega_k^{(n)})^2 \qquad (k \in \mathbb{N}) .$$

Then $\lambda^1(A)$ is approximately contractible.

Proof: Let $(\mathbf{e_i})_{i\in\mathbb{N}}$, $\mathbf{u_i}$ and d_i be as in Lemma 3.1. Then, in exactly the same way we have that

$$\pi(d_i) = \mathbf{u_i} = 2\mathbf{u_i} - \mathbf{u_i}^2 \qquad (i \in \mathbb{N}) ,$$

and that

$$\mathbf{x} \cdot d_i - d_i \cdot \mathbf{x} = 0 \qquad (i \in \mathbb{N}, \mathbf{x} \in A) .$$

Hence we only need to verify that

$$\|\mathbf{x}\mathbf{u_i} - \mathbf{x}\|_n \|\mathbf{u_i}\|_n \to 0 \text{ as } i \to \infty \qquad (n \in \mathbb{N}) .$$

But we have that

$$\begin{aligned} \|\mathbf{x}\mathbf{u_i} - \mathbf{x}\|_n \|\mathbf{u_i}\|_n &= \sum_{k=1}^{i} \omega_k^{(n)} \sum_{k=i+1}^{\infty} |x_k| \omega_k^{(n)} \\ &\leq i\omega_i^{(n)} \sum_{k=i+1}^{\infty} |x_k| \omega_k^{(n)} \\ &\leq \sum_{k=i+1}^{\infty} |x_k| k (\omega_k^{(n)})^2 \\ &\leq C \sum_{k=i+1}^{\infty} |x_k| \omega_k^{(N)} \\ &\to 0 \quad \text{as} \quad i \to \infty \end{aligned}$$

as required. ■

Remark 3.10. It can be seen (by Proposition 28.16 of [8]) that the space s as defined in example 3.1 is isomorphic to $\lambda^1(A)$ where $A = (\omega_k^{(n)})_{n,k\in\mathbb{N}}$ is given by $\omega_k^{(n)} = k^n$. Hence it is easy to see directly from proposition 3.9 that s is approximately contractible.

We finally give a condition on A which ensures that $\lambda^1(A)$ is *not* approximately contractible. The method of proof is essentially the same as that of Theorem 4.1 of [2].

Proposition 3.11. *Let $\lambda^1(A)$ be as in Lemma 3.8, and suppose that there exists $n \in \mathbb{N}$ such that*

$$\omega_1^{(n)} > 0, \qquad \omega_k^{(n)} \geq \omega_{k-1}^{(n)} \qquad (k \geq 2)\,. \tag{4}$$

Next, suppose that there exist $C, t, N > 0$ such that

$$\omega_{2k+1}^{(m)} \leq C\omega_k^{(n)} k^{1-t} \qquad (m \in \mathbb{N}, k \geq N)\,. \tag{5}$$

Then $\lambda^1(A)$ is not approximately contractible.

Proof: In order to gain a contradiction, assume that $\lambda^1(A)$ *is* approximately contractible. Now, $\lambda^1(A)$ is a commutative Fréchet algebra containing $c_{00}(\mathbb{N})$ as a dense subset, and so by exactly the same method as in Proposition 2.3 and Proposition 3.2 of [2], we see that there must exist a net $(F_\alpha) \subset c_{00} \otimes c_{00}$ such that

$$a \cdot F_\alpha - F_\alpha \cdot a + \pi(F_\alpha) \otimes a - a \otimes \pi(F_\alpha) \to 0 \qquad (a \in \lambda^1(A))$$

and such that

$$a - a\pi(F_\alpha) \to 0 \qquad (a \in \lambda^1(A))\,.$$

Let $\varepsilon > 0$ and let $S \subset \lambda^1(A)$ be a finite set. Then, in particular, we know that there exists $F \in c_{00} \otimes c_{00}$ such that

$$\|a \cdot F - F \cdot a + \pi(F) \otimes a - a \otimes \pi(F)\|_n < \varepsilon \qquad (a \in S)\,. \tag{6}$$

and such that

$$\|a - a\pi(F)\|_n < \varepsilon \qquad (a \in S)\,. \tag{7}$$

We will show that for a particular choice of ε, F and S this is impossible. Following the notation of [2] we will write

$$\Delta_a = a \cdot F - F \cdot a + \pi(F) \otimes a - a \otimes \pi(F)$$

and set

$$\mathbf{u} = \pi(F)\,.$$

Now, if $F \in c_{00} \otimes c_{00}$ then we can find $F_{i,j} \in \mathbb{C}$, $R \in \mathbb{N}$ such that

$$F = \sum_{i,j=1}^{R} F_{i,j} e_i \otimes e_j\,.$$

It is easy to see that

$$\|F\|_n = \sum_{i,j=1}^{R} |F_{i,j}| \omega_i^{(n)} \omega_j^{(n)}\,.$$

Now fix some positive decreasing $\mathbf{x} \in \lambda^1(A)$ and let $S = \{a, b\}$, where

$$a = \sum_{j=1}^{\infty} x_j e_{2j-1}, \qquad b = \sum_{j=1}^{\infty} x_j e_{2j}\,.$$

Note that a and b lie in $\lambda^1(A)$ by (5). We may suppose that $\epsilon < \frac{1}{2} x_1 \omega_1$. Then inequality (7) tells us that $(x_1 - u_1 x_1)\omega_1 < \frac{1}{2} x_1 \omega_1$, and so $u_1 > \frac{1}{2}$. Now, replacing F by $\frac{F}{u_1}$ and u by $\frac{u}{u_1}$, we obtain elements $F \in c_{00} \otimes c_{00}$, $u \in c_{00}$ such that

$$u_1 = 1 \ \text{ and } \ \|\Delta_a\| + \|\Delta_b\| < \epsilon\,.$$

Now, for $v \in \lambda^1(A)$ write

$$\Delta_v = \sum_{i,j=1}^{\infty} (\Delta_v)_{i,j} e_i \otimes e_j\,.$$

Then we get that

$$\begin{aligned}
(\Delta_a)_{2i-1,2j} &= x_i F_{2i-1,2j} - x_i u_{2j}\,, \\
(\Delta_b)_{2i-1,2j} &= -x_j F_{2i-1,2j} + x_j u_{2i-1} \\
(\Delta_a)_{2i,2j-1} &= -x_j F_{2i,2j-1} + x_j u_{2i}\,, \\
(\Delta_b)_{2i,2j-1} &= x_i F_{2i,2j-1} - x_i u_{2j-1}\,.
\end{aligned}$$

Hence, noting that $\mathbf{x}$ is positive and decreasing and that $\omega^{(n)}$ is increasing, we obtain that

$$\begin{aligned}
\|\Delta_a\| + \|\Delta_b\| &\geq \sum_{j=1}^{\infty} x_j \omega_{2j}^{(n)} \sum_{i=1}^{j} (|F_{2i-1,2j} - u_{2j}| + |u_{2i-1} - F_{2i-1,2j}|)\omega_{2i-1}^{(n)} \\
&+ \sum_{j=2}^{\infty} x_j \omega_{2j-1}^{(n)} \sum_{i=1}^{j-1} (|F_{2i,2j-1} - u_{2i}| + |u_{2j-1} - F_{2i,2j-1}|)\omega_{2i}^{(n)} \qquad (8) \\
&\geq \sum_{j=1}^{\infty} x_j \omega_j^{(n)} \sum_{i=1}^{j} |u_{2i-1} - u_{2j}|\omega_i^{(n)} + \sum_{j=2}^{\infty} x_j \omega_j^{(n)} \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}|\omega_i^{(n)}
\end{aligned}$$

where in the first term of (8) we have summed over values of i, j with $i \leq j$, and in the second term we have summed over i, j with $i \leq j-1$ and $j \geq 2$. Again, following the proof of Theorem 4.1 of [2], define

$$\Phi(\mathbf{x}, \mathbf{u}) = \sum_{j=1}^{\infty} x_j \omega_j^{(n)} \sum_{i=1}^{j} |u_{2i-1} - u_{2j}|\omega_i^{(n)} + \sum_{j=2}^{\infty} x_j \omega_j^{(n)} \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}|\omega_i^{(n)}\,,$$

so that $\|\Delta_a\| + \|\Delta_b\| \geq \Phi(\mathbf{x}, \mathbf{u})$, and set

$$\theta(\mathbf{x}) = \inf\{\Phi(\mathbf{x}, \mathbf{u}) : \mathbf{u} \in c_{00}, u_1 = 1\}\,.$$

If we can find $\mathbf{x} \in \lambda^1(A)$ such that $\theta(\mathbf{x}) > 0$ then we are done. Define $\mathbf{x} \in \mathbb{C}^{\mathbb{N}}$ by

$$x_j = \frac{1}{j(j+1)\omega_j^{(n)}} \qquad (j \in \mathbb{N})\,.$$

Then condition (5) ensures that $\mathbf{x}$ is in $\lambda^1(A)$. Also, note that we have the identity

$$\sum_{j=k+1}^{\infty} x_j \omega_j^{(n)} = k\omega_k^{(n)} x_k\,. \qquad (9)$$

It is clear that the value of $\Phi(\mathbf{x}, \mathbf{u})$ decreases if every value of u_i which is greater than one 1 is reduced to 1 and if every value which is greater than 0 is increased to 0. Hence we may assume that

$$0 \leq u_i \leq 1 \qquad (i \in \mathbb{N})\,.$$

For each $d \geq 2$, let

$$S_d = \{\mathbf{u} \in c_{00} : u_1 = 1, u_i \in [0,1] \ \ (i = 1, \dots, d), u_i = 0 \ \ (i > d)\}\,.$$

We show that if $\mathbf{u} \in S_d$ and $u_d > 0$, then the value of $\Phi(\mathbf{x}, \mathbf{u})$ is decreased by taking $u_d = 0$. To see this, first suppose that $d = 2k+1$ for some $k \in \mathbb{N}$. Then, by reducing u_d to 0 in the expression

$$\Phi(\mathbf{x}, \mathbf{u}) = \sum_{j=1}^{\infty} x_j \omega_j^{(n)} \sum_{i=1}^{j} |u_{2i-1} - u_{2j}| \omega_i^{(n)} + \sum_{j=2}^{\infty} x_j \omega_j^{(n)} \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}| \omega_i^{(n)}\,,$$

we increase the sum

$$\sum_{j=2}^{\infty} x_j \omega_j^{(n)} \sum_{i=1}^{j-1} |u_{2i} - u_{2j-1}| \omega_i^{(n)}$$

by at most

$$x_{k+1} \omega_{k+1}^{(n)} u_{2k+1} \left(\sum_{i=1}^{k} \omega_i^{(n)} \right)\,,$$

and we decrease the sum

$$\sum_{j=1}^{\infty} x_j \omega_j^{(n)} \sum_{i=1}^{j} |u_{2i-1} - u_{2j}| \omega_i^{(n)}$$

by exactly

$$\left(\sum_{j=k+1}^{\infty} x_j \omega_j^{(n)} \right) u_{2k+1} \omega_{k+1}^{(n)}\,.$$

Hence, using (9), we see that $\Phi(\mathbf{x}, \mathbf{u})$ is reduced by at least $u_{2k+1}\omega_{k+1}^{(n)}$ multiplied by

$$\begin{aligned}
\left(\sum_{j=k+1}^{\infty} x_j \omega_j^{(n)} - \left(\sum_{i=1}^{k} \omega_i^{(n)} \right) x_{k+1} \right) &\geq k\omega_k^{(n)} x_k - \left(\sum_{i=1}^{k} \omega_i^{(n)} \right) x_k \\
&= (k-1)\omega_k^{(n)} x_k - \left(\sum_{i=1}^{k-1} \omega_i^{(n)} \right) x_k \\
&\geq 0
\end{aligned}$$

because $\omega^{(n)}$ is increasing. In the same fashion, we deduce that when $d = 2k$, reducing u_d to 0 reduces $\Phi(\mathbf{x}, \mathbf{u})$ by at least $u_{2k}\omega_k$ multiplied by

$$(k-1)\omega_k^{(n)} x_k - \left(\sum_{i=1}^{k-1} \omega_i^{(n)} \right) x_k \geq 0\,,$$

as claimed. We deduce that

$$\theta(\mathbf{x}) = \Phi(\mathbf{x}, (1, 0, 0, 0, \ldots)) = \omega_1 \sum_{j=1}^{\infty} x_j \omega_j = \omega_1 > 0 \, ,$$

and so $\lambda^1(A)$ is not approximately contractible, as claimed. ■

Corollary 3.12. *Let $f : \mathbb{N} \to \mathbb{R}^+$. Let $A = (\omega^{(n)})_{n \in \mathbb{N}}$ be given by*

$$\omega_k^{(n)} = k^{f(n)} \qquad (k, n \in \mathbb{N}) \, .$$

Then $\lambda^1(A)$ is approximately contractible if and only if f is unbounded.

Proof: This follows directly from the conditions given in Proposition 3.9 and Proposition 3.11. ■

Remark 3.13. In particular, this shows that if f is bounded and does not attain its upper bound, then $\lambda^1(A)$ is a non-Banach Fréchet algebra which is not approximately contractible.

We finish with some open questions.

Question 3.14. *Does there exist an approximately contractible Banach algebra without a bounded approximate identity? (This was the question that originally prompted this research).*

Question 3.15. *Does there exist a Banach algebra A with an unbounded approximate identity (u_α) such that the conditions (i) and (ii) of Lemma 2.7:*

$$\|u_\alpha a - a\| \|u_\alpha\| \to 0 \qquad (a \in A) \, ,$$

$$\|a u_\alpha - a\| \|u_\alpha\| \to 0 \qquad (a \in A)$$

are satisfied?

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