

MATH1050 : SOLUTIONS 6

1. (i) The ratio test gives

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{2^{n+1}(n+1)!/(n+1)^{n+1}}{2^n n!/n^n} = \frac{2(n+1)n^n}{(n+1)^{n+1}} \\ &= \frac{2}{(1+1/n)^n} \rightarrow 2/e < 1. \end{aligned}$$

So $\sum a_n$ is convergent.

(ii) $\frac{1}{\sqrt{n^2-1}} > \frac{1}{n}$, so since (QUOTE) $\sum 1/n$ diverges, the sum is divergent by the comparison test.

(iii) The sum is absolutely convergent by the integral test (for $\int_{x=1}^{\infty} dx/x^4 = [\frac{-1}{3x^3}]_1^{\infty} = 1/3$). Alternatively, it is absolutely convergent by comparison with the known convergent series $\sum_{n=1}^{\infty} 1/n^2$. Either way, absolutely convergent implies convergent, so the series converges.

(iv) $b_n = 1/\sqrt{n}$ is a decreasing sequence tending to zero, and we have the sum $\sum_{n=1}^{\infty} (-1)^n b_n$. It therefore converges by the alternating series test.

2. (i) Here $a_n = \frac{1}{3^n}$ so $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = 3$

When $x = 3$, the series is $\sum_0^{\infty} 1^n$ which is $D(u_n \not\rightarrow 0)$.

When $x = -3$, the series is $\sum_0^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \dots$, again D .

(ii) $R = \lim_{n \rightarrow \infty} \left| \frac{(2n)!}{n!} \frac{(n+1)!}{(2n+2)!} \right| = \lim_{n \rightarrow \infty} \frac{1}{2(2n+1)} = 0$.

Thus, the series is C for $x = 0$ only.

(iii) $R = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! + 1}{n! + 1} \right| = \lim_{n \rightarrow \infty} \frac{(n+1) + \frac{1}{n!}}{1 + \frac{1}{n!}} = \infty$.

So the series is absolutely convergent for every x .

(iv) $R = \lim_{n \rightarrow \infty} \left| \frac{2^n}{(n+1)} \cdot \frac{(n+2)}{2^{n+1}} \right| = \frac{1}{2}$.

When $x = \frac{1}{2}$, the series is $\sum \frac{1}{n+1}$, and this is D by comparison with $\sum \frac{1}{n}$.

When $x = -\frac{1}{2}$, the series is $\sum \frac{(-1)^n}{n+1}$ which is convergent by the alternating series test.

3. We have

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + R_4,$$

where $f(x) = \frac{1}{(x+1)^2}$. So, $f(0) = 1$, $f^{(1)}(x) = -2(x+1)^{-3}$, $f^{(1)}(0) = -2!$, $f^{(2)}(x) = (-2)(-3)(x+1)^{-4}$, $f^{(2)}(0) = 3!$, $f^{(3)}(x) = (-2)(-3)(-4)(x+1)^{-5}$, $f^{(3)}(0) = -4!$, $f^{(4)}(x) = (-2)(-3)(-4)(-5)(x+1)^{-6} = 5!(x+1)^{-6}$. Thus $f(x) = 1 - 2x + 3x^2 - 4x^3 + R_4$, where

$$R_4 = \frac{f^{(4)}(\lambda x)}{4!} x^4 = 5(\lambda x + 1)^{-6} x^4$$

for some λ between 0 and 1.

4. In this case

$$f(x) = f(9) + \frac{f^{(1)}(9)}{1!} (x-9) + \frac{f^{(2)}(9)}{2!} (x-9)^2 + R_3,$$

where $f(x) = x^{1/2}$. So $f(9) = 3$, $f^{(1)}(x) = \frac{1}{2}x^{-1/2}$, $f^{(1)}(9) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, $f^{(2)}(x) = \frac{1}{2}(-\frac{1}{2})x^{-3/2}$, $f^{(2)}(9) = -\frac{1}{4} \cdot \frac{1}{27} = -\frac{1}{108}$, $f^{(3)}(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^{-5/2} = \frac{3}{8} \cdot \frac{1}{x^{5/2}}$.

Thus

$$f(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + R_3.$$

(The students were exempted the rest of the question due to not having been told the exact formula for R_3 in the Taylor series case, rather than the Maclaurin series case).