

Matrices as rational maps

Roger Bielawski

February 21, 2008

- $M(n)$ - vector space (associative algebra, Lie algebra) of $n \times n$ matrices over \mathbb{C} .
- $GL(n)$ - Lie group of complex invertible matrices.
- $\exp : M(n) \rightarrow GL(n)$ - the exponential mapping

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

For $A \in M(n)$ and $m \leq n$, denote by $A_{(m)}$ the upper-left $m \times m$ minor of A , i.e.

$$A = [a_{ij}] \mapsto A_{(m)} = \begin{pmatrix} a_{11} & \dots & a_{1m} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m1} & \dots & a_{mm} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

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Define, for any $m \leq n-1$ and $j = 0, \dots, m-1$ an *action* of the additive group \mathbb{C} on $M(n)$ by:

$$z.A = \exp(zA_{(m)}^j) A \exp(-zA_{(m)}^j),$$

i.e. conjugate A by $\exp(zA_{(m)}^j)$.

Theorem (Kostant-Wallach 2004, Guillemin-Sternberg 1983, Gelfand-Zeitlin 1950)

The \mathbb{C} -actions for different (m, j) commute and define an action of the abelian group $\mathbb{C}^{n(n-1)/2}$ on $M(n)$.

What!?

Significance, interpretation, properties of this Gelfand-Zeitlin action.

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Hamiltonian mechanics and symplectic manifolds

Lagrangian formulation of the classical mechanics. One wants to study the evolution of some system, the space of which states is described by a (smooth) manifold X . The evolution is described by a time-dependent *Lagrangian* $L(x(t), \dot{x}(t), t)$, i.e. a function which depends on the state of the system $x(t) \in X$ and on the “velocity” $\dot{x} \in T_x X$ at a given time t .

The evolution of such a (holonomic) system occurs along paths $x(t) : [a, b] \rightarrow X$ which are extrema of the functional:

$$x(t) \longmapsto \int_a^b L(x(t), \dot{x}(t), t) dt.$$

If we just look at the local picture, i.e. a given state is described by n coordinates q_1, \dots, q_n , then the paths of evolution are given by the *Euler-Lagrange equations*:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n.$$

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An equivalent description was found by Hamilton, who replaced the velocities \dot{q}_i by momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and the Lagrangian by what is now known as *Hamiltonian*:

$$H(q_i, p_i, t) = \sum_{i=1}^n p_i \dot{q}_i - L.$$

(a small problem: need to solve for the \dot{q}_i).

The Euler-Lagrange equations are now equivalent to the *Hamilton equations*:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \dots, n. \quad (1)$$

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Taking a solution $x(t)$ with a given initial value $x(0)$ gives us an action of the additive group \mathbb{R} (perhaps not a global action). Thus we associate an \mathbb{R} -action to each smooth function $H: \mathbb{R}^{2n} \rightarrow \mathbb{R}$.

Example

- Take $n = 1$ and the Hamiltonian $H_1 = q$. Then (1) are $\dot{q} = 0$, $\dot{p} = -1$. Thus the action is $t.(p, q) = (p - t, q)$ - translations.
- Let $H_2 = \frac{1}{2}(p^2 + q^2)$. Then $\dot{q} = p$, $\dot{p} = -q$, so that $q(t) = \cos tq(0) + \sin tp(0)$, $p(t) = \cos tp(0) + \sin tq(0)$ - the standard circle action on \mathbb{R}^2 .
- Finally, let $H_3 = \frac{1}{2}q^2$. Then $\dot{q} = 0$, $\dot{p} = -q$, so that $t.(p, q) = (p - tq, q)$.

What we really do, is to associate, to a function H , a vector field

$$(q_i, p_i) \rightarrow \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i} \right).$$

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This whole picture generalises from \mathbb{R}^{2n} to any manifold Y equipped with a *symplectic form*, i.e. a 2-form ω (a smooth choice of a skew-symmetric bilinear form on each tangent space) which everywhere nondegenerate and *closed* (i.e. $d\omega = 0$).

For example, if $f, g : Y \rightarrow \mathbb{R}$ are two smooth functions, we can form a closed 2-form $df \wedge dg$ defined on a pair of tangent vectors $v, w \in T_m Y$ by

$$(df \wedge dg)(v, w) = df(v)dg(w) - dg(v)df(w).$$

$(df(v))$ - "directional derivative of f in the direction v ")

On \mathbb{R}^{2n} , we used the canonical form

$$dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n. \quad (2)$$

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Example (Coadjoint orbits)

Let G be a Lie group, \mathfrak{g} its Lie algebra, and \mathfrak{g}^* the dual of \mathfrak{g} (e.g. $G = GL(n, \mathbb{R})$, $\mathfrak{g} = M(n, \mathbb{R})$ - $n \times n$ real matrices, $\mathfrak{g}^* \simeq \mathfrak{g}$ via trace). G acts on \mathfrak{g}^* via the coadjoint action:

$$(g.f)(x) = f(g^{-1}xg)$$

Any orbit O of this action is a (homogeneous) symplectic manifold. At a point $f \in O$, any tangent vector is induced by an element $\rho \in \mathfrak{g}$. One sets

$$\omega(\check{\rho}_1, \check{\rho}_2) = f([\rho_1, \rho_2]).$$

Once we have a symplectic form, we can associate to any $H : Y \rightarrow \mathbb{R}$ a vector field X_H , by $dH(v) = \omega(v, X_H)$ for any tangent vector v . The flow $\dot{x}(t) = X_H(x(t))$ is then a generalisation of (1). Notice that H is invariant of the flow, since $dH(X_H) = \omega(X_H, X_H) = 0$.

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One says that two Hamiltonians H_1, H_2 (Poisson) commute if their flows (i.e. \mathbb{R} -actions) commute, i.e. $\omega(X_{H_1}, X_{H_2}) = 0$ at every point of Y .

Thus, if we have k commuting Hamiltonians H_1, \dots, H_k , then the resulting \mathbb{R}^k -action leaves invariant H_1, \dots, H_k . In particular, if the action is to have k -dimensional orbits, then $k \leq n$, where $\dim Y = 2n$. If $k = n$ and the orbits are generically n -dimensional, then we say that our Hamiltonian system is *completely integrable* (in the sense of Liouville).

We can then take special local coordinates

$q_1(x), \dots, q_n(x) = H_n(x)$ (action variables) and $(p_1, \dots, p_n)(x) = t \in \mathbb{R}^n$, such that $t \cdot x_0 = x$ (angle variables). The symplectic form becomes (2) and the Hamiltonian flow is now as in the first Example (with $H = q$), i.e. translations in the angle variables.

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Locally we can always find n commuting Hamiltonians, so that locally any symplectic form is of the form

$$dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$$

(compare with symmetric tensors, i.e. Riemannian metrics!). These are known as *Darboux coordinates*.

Locally, all completely integrable symplectic manifolds, on which the flows are complete ($t.x$ is defined for all $t \in \mathbb{R}^n$ and $x \in Y$), look the same. It is the singular orbits of the \mathbb{R}^n -action, which determine the topology of Y , its global geometry and interesting phenomena of the mechanical system.

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A symplectic form ω on a manifold Y induces an algebraic structure on $C^\infty(Y)$, namely the Poisson bracket

$$C^\infty(Y) \ni H_1, H_2 \longmapsto \{H_1, H_2\} = \omega(X_{H_1}, X_{H_2}) \in C^\infty(Y). \quad (3)$$

It is bilinear, skew-symmetric, satisfies the Jacobi identity and is a derivation of $C^\infty(Y)$ in each argument, i.e.

$$\{fg, h\} = f\{g, h\} + g\{f, h\}. \quad (4)$$

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$$C^\infty(Y) \ni H_1, H_2 \longmapsto \{H_1, H_2\} = \omega(X_{H_1}, X_{H_2}) \in C^\infty(Y). \quad (3)$$

It is bilinear, skew-symmetric, satisfies the Jacobi identity and is a derivation of $C^\infty(Y)$ in each argument, i.e.

$$\{fg, h\} = f\{g, h\} + g\{f, h\}. \quad (4)$$

We can abstract these properties and define a Poisson algebra

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The dual \mathfrak{g}^* of a Lie algebra is a Poisson manifold (but not a symplectic one). Namely, restrict a pair of functions $f, g \in C^\infty(\mathfrak{g}^*)$ to any coadjoint orbit and compute their (symplectic) Poisson bracket. On each orbit \mathcal{O} we get a function $\{f, g\}_{\mathcal{O}}$, which combine to give us a smooth function on \mathfrak{g}^* .

One should think of a Poisson manifold as an infinitesimal quantum deformation of the manifold itself - the Poisson bracket of functions (observables) is the quantum commutator up to first order.

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All this discussion holds equally well over \mathbb{C} , i.e. for complex manifolds, holomorphic symplectic forms, holomorphic Hamiltonians, complex Poisson algebras.

In particular, adjoint orbits of $GL(n, \mathbb{C})$ are (complex) symplectic manifolds. Note that the regular adjoint orbits have dimension $n^2 - n$, i.e. for a complete integrability we need $n(n-1)/2$ commuting Hamiltonians.

Theorem

The functions $A \mapsto \text{tr } A_{(m)}^j$, $m = 1, \dots, n$, $j = 1, \dots, m$, Poisson commute on $M(n)$ and define a completely integrable system on any adjoint orbit.

This is equivalent to our first Theorem. In other words, the above functions form a maximal commutative subalgebra of the Poisson algebra of algebraic functions on $M(n)$.

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and its proof:

Proof.

Since $f_i(A) = \operatorname{tr} A^i$, $i = 1, \dots, n$, is constant on any adjoint orbit, we have $\{f_i, g\} = 0$ for any g . Now replace $M(n)$ with $M(n-1)$. □

The orbit structure for the Gelfand-Zeitlin action on $M(n)$ is very complicated. No wonder, since it reflects the structure of adjoint orbits, representation theory of $GL(n, \mathbb{C})$, etc.

For example, consider the map $\Phi : M(n) \rightarrow \mathbb{C}^{n(n+1)/2}$, $A \mapsto (\text{tr } A_{(m)}^j)$, $m = 1, \dots, n, j = 1, \dots, m$. Any fibre is a union of finitely many orbits and any fibre contains an orbit of maximal dimension ($= n(n-1)/2$). But usually a fibre contains more than one orbit of maximal dimension.

Theorem (Guillemin-Sternberg, Kostant-Wallach)

If, for every $m \leq n$, $A_{(m)}$ has distinct eigenvalues, and the eigenvalues of $A_{(m)}$ are distinct from eigenvalues of $A_{(m-1)}$, then the fibre $\Phi^{-1}(\Phi(A))$ is connected (and $\simeq (\mathbb{C}^)^{n(n-1)/2}$).*

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We could also ask which matrices have a finite stabiliser.

The answer is that A has a finite stabiliser for the GZ-action if and only if A is *strongly regular*, i.e. each $A_{(m)}$ is regular in $M(m)$ and $Z(A_{(m-1)}) \cap Z(A_{(m)}) = 0$ for every $m = 1, \dots, n$, where Z denotes the centraliser.

One more property: the orbits are irreducible and constructible subsets of $M(n)$ (Kostant & Wallach).

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Rational maps into full flag manifolds

A sequence of subspaces of \mathbb{C}^{n+1} of the form

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_n \subset E_{n+1} = \mathbb{C}^{n+1},$$

with $\dim E_i = i$ is called a (full) *flag* of subspaces.

If we fix a basis $\{e_1, \dots, e_{n+1}\}$ of \mathbb{C}^{n+1} , then we have the *standard flag*:

$$E_0^+ = \{0\}, \quad E_1^+ = \langle e_1 \rangle, \dots, \quad E_{n+1}^+ = \mathbb{C}^{n+1}.$$

The group $GL(n+1, \mathbb{C})$ acts transitively on flags and the stabiliser of the standard flag is the subgroup B^+ of upper-triangular matrices. Thus, the manifold $F(n)$ of full flags is biholomorphic to the homogeneous space $GL(n+1, \mathbb{C})/B^+$. Using Gram-Schmidt orthogonalisation, one sees that $F(n)$ is diffeomorphic to $U(n+1)/T^{n+1}$.

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In particular, the codimension one cells X_1, \dots, X_n correspond to simple negative root spaces and the Poincaré duals of the homology classes of their closures freely generate $H^2(F(n), \mathbb{Z})$.

We view $F(n)$ as a compactification of N^- , and write $Z = F(n) - N^-$ for the "infinity", i.e. the complement of the open cell.

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On $F(1) \simeq \mathbb{C}P^1$, $N^- \simeq \mathbb{C}$ and $Z = \{\infty\}$. In terms of matrices a point $[p, q] \in \mathbb{C}P^1$ corresponds to the coset

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A topological invariant of such a map is its *degree* $\underline{k} = (k_1, \dots, k_n)$, where k_i is the intersection number of $f(\mathbb{P}^1)$ with \bar{X}_i .

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For general n , we denote by $\text{Rat}_{\underline{k}}(F(n))$ its subspace of maps of degree $\underline{k} = (k_1, \dots, k_n)$. The *poles* of such a rational map $f(z)$ are the points z , for which $f(z) \in Z$.

Let $z_1^i, \dots, z_{k_i}^i \in \mathbb{P}^1$ be the points which map to \bar{X}_i , then $f \in \text{Rat}_{\underline{k}}(F(n))$ defines n polynomials:

$$q_i(z) = \prod_{j=1}^{k_i} (z - z_j^i), \quad i = 1, \dots, n. \quad (5)$$

Because of the basing condition none of the z_j^i is ∞ , and, hence, each q_i has degree k_i .

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Consider $p(z)/q(z) \in \text{Rat}_k(\mathbb{CP}^1)$. Suppose that $q(z)$ has roots z_1, \dots, z_r with multiplicities m_1, \dots, m_r .

Then we can write uniquely

$$\frac{p(z)}{q(z)} = \frac{p_1(z - z_1)}{(z - z_1)^{m_1}} + \dots + \frac{p_r(z - z_r)}{(z - z_r)^{m_r}},$$

where $\deg p_i < m_i$ and $p_i(0) \neq 0$ for $i = 1, \dots, r$. Thus, to determine $p(z)$, we have to choose, at a pole of multiplicity r , an element of $\mathbb{C}^* \times \mathbb{C}^{r-1}$.

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Similar description for $\text{Rat}_k(F(n))$: this time the “multiplicity” of a pole z_0 is not a single integer but a sequence (r_1, \dots, r_n) , called the *type of a pole*, where r_i is the intersection number of $f(z)$ with \bar{X}_i at z_0 .

Again, for a pole of type $(0, \dots, 1, \dots, 0)$, a local principal part is an element of \mathbb{C}^* . Similarly, for $(0, \dots, m, \dots, 0)$, the local principal part is the same as for maps into \mathbb{CP}^1 . This is not surprising, since rational map $f: \mathbb{CP}^1 \rightarrow F(n)$ of degree $(0, \dots, m, \dots, 0)$ is represented by:

$$f(z) = \begin{pmatrix} * & * & * & \dots & \dots & * \\ 0 & * & * & \dots & \dots & * \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{p(z)}{q(z)} & * & \dots & * \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & * \end{pmatrix}.$$

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In general, local principal parts are much more complicated; e.g. for a pole of type $(1, 1)$ (for rational maps into $F(3)$), the local principal part is an element of $\mathbb{C}^* \times \{(u, w \in \mathbb{C}^2; uw = 1)\}$.

The description of rational maps in terms of poles and local principal parts leads to "additivity" of rational maps.

If $f, g : \mathbb{CP}^1 \rightarrow F(n)$ are two rational maps, the poles of which are disjoint, we can form a new rational map by taking the poles to be the union of poles of f and of g , and keeping the local principal part the same at every pole.

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For example, taking such an addition for the above “pure” rational maps gives

$$f(z) = \begin{pmatrix} * & * & * & \dots & * \\ \frac{p_1(z)}{q_1(z)} & * & * & \dots & * \\ 0 & \frac{p_2(z)}{q_2(z)} & * & \dots & * \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & \frac{p_n(z)}{q_n(z)} & * \end{pmatrix},$$

i.e. a generic rational map into $F(n)$ (not meeting codimension two Bruhat cells) can be identified with n rational maps $p_i(z)/q_i(z)$ into \mathbb{P}^1 .

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In particular $\dim \text{Rat}_{\underline{k}}(F(n)) = 2(k_1 + \dots + k_n)$.

A symplectic structure on $\text{Rat}_{\underline{k}}(F(n))$

There is a natural (complex) symplectic structure on each $\text{Rat}_{\underline{k}}(F(n))$. On the open dense subset, where all poles are distinct, and a rational map can be identified with n rational maps $p_i(z)/q_i(z)$, it is given by

$$\omega = \sum_{i=1}^n \sum_{j=1}^{k_i} \frac{dp_i(z_j^i)}{p_i(z_j^i)} \wedge dz_j^i, \quad (6)$$

where z_j^i are the poles. (The standard symplectic form on $\mathbb{C}^* \times \mathbb{C}$ many times). This extends to all of $\text{Rat}_{\underline{k}}(F(n))$. Observe that the coefficients of the $q_i(z)$ are globally defined commuting Hamiltonians. It is easy to show that the flows are global, so that we obtain an action of $A_{\underline{k}} \simeq \mathbb{C}^{|\underline{k}|}$, $|\underline{k}| = k_1 + \dots + k_n$, on $\text{Rat}_{\underline{k}}(F(n))$.

The orbit structure of this action is (relatively) easy to understand: it is enough to understand it for rational maps with one pole of a given type; the additivity of rational maps is compatible with the group action and gives the orbit structure on $\text{Rat}_{\underline{k}}(F(n))$.

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Matrices, Gelfand-Zeitlin action, and rational maps

- $M^{\text{reg}}(n)$ - the subset of regular matrices.
- S_n - the quotient of $\text{Rat}_{\underline{k}}(F(n))$, $\underline{k} = (1, 2, \dots, n)$, by the \mathbb{C}^n generated by coefficients of $q_n(z)$ (the n -th denominator).
 S_n is a complex Poisson manifold.

Theorem (- & Pidstrygach)

There exists a biholomorphism $\phi : M^{\text{reg}}(n) \rightarrow S_n$, equivariant with respect to the Gelfand-Zeitlin action on $M^{\text{reg}}(n)$ and the $A_{\underline{k}}$ -action on S_n . This action intertwines the Hamiltonians, i.e.

$$q_m(\phi(A))(z) = \det(z - A_{(m)}), \quad m = 1, \dots, n,$$

and induces an isomorphism of Poisson algebras of holomorphic functions.

Remark. If we fix q_n and $\det(z - A)$, we obtain a statement about a regular adjoint orbit.

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This theorem allows to obtain information about the orbit structure of the Gelfand-Zeitlin action. For example

Corollary (- & Pidstrygach)

Consider a subset $M_{\underline{\chi}}$ of $M(n)$ consisting of matrices A such that

$$\det(z - A_{(m)}) = \chi_m(z), \quad m = 1, \dots, n,$$

for fixed polynomials $\chi_1(z), \dots, \chi_n(z)$. Let z_1, \dots, z_r be distinct roots of χ_1, \dots, χ_n and let $t_i = \#\{j; \chi_j(z_i) = 0 = \chi_{j+1}(z_i)\}$. Finally, let $t = \sum_{i=1}^r t_i$.

Then $M_{\underline{\chi}}(n)$ contains exactly 2^t orbits of maximal dimension.

Remark. This result has been also obtained by M. Colarusso in his thesis (2007).

Another application: a simple proof of a theorem of Kostant-Wallach on the rational function field $F(n)$ of $M(n)$ (as a Poisson field).

Theorem (Kostant-Wallach)

There exists a Galois extension $E(n)$ of $F(n)$, which is isomorphic, as a Poisson field, to the rational function field of a classical phase space over a (Poisson central) function field, i.e.

$$E(n) = \mathbb{C} (u_1, \dots, u_{n(n-1)/2}, v_1, \dots, v_{n(n-1)/2}, w_1, \dots, w_n),$$

with Poisson relations

$$\{u_i, v_j\} = \delta_{ij}, \quad \{u_i, u_j\} = \{v_i, v_j\} = \{w_m, u_j\} = \{w_m, v_j\} = \{w_m, w_l\} = 0,$$

for all i, j, m, l .

(Commutative analogue of the Gelfand-Kirillov theorem)

Proof. $u_{k(k-1)/2+j} = z_j^k p_k(z_j^k)^{-1}$, $v_{k(k-1)/2+j} = p_k(z_j^k)$,

$k = 1, \dots, n-2$, $j = 1, \dots, k$, $w_m = z_m^n$. □

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