Nahm's equations in geometry and Lie group theory

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Let G be a (compact) Lie group and $\mathfrak{g}$ its Lie algebra. The *Nahm equations* are

$$\begin{aligned} \dot{T}_1 &= [T_2, T_3] \\ \dot{T}_2 &= [T_3, T_1] \\ \dot{T}_3 &= [T_1, T_2], \end{aligned} \tag{1}$$

for a triple of $\mathfrak{g}$-valued maps $T_1, T_2, T_3 : (0, a) \to \mathfrak{g}$.

Imposing different *boundary conditions* gives many interesting manifolds: spaces of rational maps to flag manifolds, resolutions of Kleinian singularities, coadjoint orbits ... They all come with a *hyperkähler structure*, i.e. a Riemannian metric and a parallel (for the Levi-Civita connection) fibrewise action of quaternions.

The best way to see this is to introduce a fourth component $T_0 : (0, a) \to \mathfrak{g}$ and view spaces of solutions to Nahm's equations as hyperkähler quotients.

$$\mathcal{A} = \{T_0 + iT_1 + jT_2 + kT_3 : [0,a] \longmapsto \mathfrak{g} \otimes \mathbb{H}\}$$

On this space the natural L^2-metric is Kähler for three anti-commuting complex structures I_1, I_2, I_3 given by right multiplication by i, j and k. Thus we get three symplectic forms ω_1, ω_2, ω_3, which can be explicitly computed. E.g. $\omega_1 = \int_0^1 dT_0 \wedge dT_1 - dT_2 \wedge dT_3$. In other words, $\mathcal{A}$ is *hyperkähler*.

The gauge group $\mathcal{G}$ of maps $g : (0,a) \to G$ acts on $\mathcal{A}$ preserving the metric and the symplectic forms via:

$$g.(T_0,T_1,T_2,T_3) = (gT_0g^{-1} - \dot{g}g^{-1}, gT_1g^{-1}, gT_2g^{-1}, gT_3g^{-1}).$$

"Good" boundary conditions will be preserved by a subgroup $\mathcal{G}_0 \subset \mathcal{G}$, whose action will be free and Hamiltonian for all 3 symplectic forms, giving us *three* moment maps:

$$\begin{aligned} \mu_1 &= \dot{T}_1 + [T_0,T_1] - [T_2,T_3] \\ \mu_2 &= \dot{T}_2 + [T_0,T_2] - [T_3,T_1] \qquad (2) \\ \mu_3 &= \dot{T}_3 + [T_0,T_3] - [T_1,T_2]. \end{aligned}$$

We can perform the *hyperkähler reduction*, i.e. consider the quotient

$$\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)/\mathcal{G}_0$$

To see which manifold one gets one usually identifies the hyperkähler quotient with the complex symplectic quotient

$$\left(\mu_2 + \sqrt{-1}\mu_3\right)^{-1}(0)/\mathcal{G}_0^{\mathbb{C}},$$

i.e. solutions to the *complex Nahm equation* (Lax equation):

$$\frac{d}{ds}(T_2 + iT_3) = [T_2 + iT_3, T_0 - iT_1] \tag{3}$$

modulo *complex* gauge transformations.

Example [Kronheimer].

Consider all regular solutions on $[0,1]$ modulo $\{g(t); g(0) = g(1) = 1\}$. For any solution (T_0, T_1, T_2, T_3) we can find a unique $h : [0,1] \to G^{\mathbb{C}}$, $h(0) = 1$ and $h(T_0 - iT_1)h^{-1} - \dot{h}h^{-1} = 0$. This makes $T_2 + iT_3$ constant. The map

$$(T_0, T_1, T_2, T_3) \longmapsto \Big(T_2(0) + iT_3(0),\ \ h(1)\Big)$$

identifies this moduli space of solutions to Nahm's equations with

$$G^{\mathbb{C}} \times \overline{\mathfrak{g}^{\mathbb{C}}} \simeq T^{0,1}G \simeq T^*G^{\mathbb{C}}.$$

Adjoint orbits of complex reductive Lie groups

Adjoint orbits of a compact Lie group are Kähler manifolds: $G/C(T) \simeq G^{\mathbb{C}}/P$ (P-parabolic). Similarly, adjoint orbits of a complex reductive Lie group $G^{\mathbb{C}}$ are hyperkähler manifolds (Kronheimer, Biquard, Kovalev).

Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$ and let $\underline{\tau} = (\tau_1, \tau_2, \tau_3) \in \mathfrak{h}^3$. Consider also $\mathfrak{su}(2)$ with the basis $\{e_1, e_2, e_3\}$ and relations $[e_1, e_2] = -e_3, \ldots$ and let $\sigma : \mathfrak{su}(2) \to Z(\tau_1, \tau_2, \tau_3)$ be a homomorphism. Then

$$S_i(t) = \tau_i + \sigma(e_i)/(t+1) \qquad \text{for } i = 1,2,3 \qquad (4)$$

is a solution to Nahm's equations for all $t \geq 0$. One defines

$$\mathcal{M}_{\underline{\tau};\sigma} = \{\text{solutions asymptotic to } S_i(t) \text{ for } i = 1,2,3\}/\mathcal{G}_0,$$

where both "asymptotic" and $\mathcal{G}_0$ need to be defined carefully.

This is again a hyperkähler manifold. Its generic complex structure is that of an adjoint orbit, e.g. if $Z(\tau_2, \tau_3) = Z(\tau_1, \tau_2, \tau_3)$, then w.r.t. I_1 we get the orbit of:

$$(\underbrace{\tau_2 + i\tau_3}_{\text{semisimple}}) + (\underbrace{\sigma(e_2) + i\sigma(e_3)}_{\text{nilpotent}}).$$

This follows from the Lax pair description: the biholomorphism $\mathcal{M}_{\underline{\tau};\sigma} \rightarrow \textit{orbit}$ is

$$\big(T_0(t), T_1(t), T_2(t), T_3(t)\big) \mapsto T_2(0) + iT_3(0).$$

The complex-symplectic form $\omega_2 + i\omega_3$ is the Kostant-Kirillov-Souriau form of the complex orbit.

With respect to other complex structures $\mathcal{M}_{\underline{\tau};\sigma}$ is a bundle over a generalised flag manifold, e.g. if $\underline{\tau} = (\tau_1, 0, 0)$, then $\big(\mathcal{M}_{\underline{\tau};0}, I_1\big) \simeq T^*\big(G/C(\tau_1)\big)$.

Geometric significance?

The hyperkähler metric of $\mathcal{M}_{(\tau_1,0,0);0}$ is uniquely determined by the Kähler metric on G/K, $K = C(\tau_1)$. This begs the question: how to obtain this hyperkähler structure directly from the complex geometry of the flag manifold G/K? This is equivalent to asking what is the diffeomorphism between the $G^{\mathbb{C}}$-orbit $\mathcal{O}(\tau_1) \simeq G^{\mathbb{C}}/K^{\mathbb{C}}$ and $T^*(G/K)$, which gives two anti-commuting complex structures. A natural homogeneous fibration of $G^{\mathbb{C}}/K^{\mathbb{C}}$ over G/K?

The answer is unknown, except in the case when G/K is a Hermitian symmetric space (Biquard-Gauduchon): the fibration is the Mostow fibration, i.e. if $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{m}$ is the symmetric decomposition, then the fibre over $[1] \in G/K$ is the $K^{\mathbb{C}}$-orbit of e^{ip}, $p \in \mathfrak{m}$.

Application: the Kostant-Sekiguchi correspondence

The Kronheimer-Biquard-Kovalev metrics on complex adjoint orbits provide plenty of representation-theoretic information. Here is an example for real groups.

Let (G, K) be a compact symmetric pair, i.e. there exists an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \quad \text{with} \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \ \ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \tag{5}$$

Then $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{m}$ is also a Lie algebra and if G^* is the corresponding Lie group, then (G^*, K) is the dual symmetric pair.

Example 1. $G = SU(n)$ and $K = S(U(p) \times U(q))$ with $p + q = n$. $\mathfrak{m}$ consists of the two off-diagonal $p \times q$- and $q \times p$-blocks in the algebra of skew-hermitian matrices and $SU(n)^* = SU(p, q)$, i.e the group preserving an indefinite hermitian form.

Example 2. $G = SU(n)$ and $K = SO(n)$. Then $\mathfrak{m}$ is the space of imaginary symmetric matrices. It follows that $SU(n)^* = SL(n, \mathbb{R})$.

The properties (**??**) implies that the condition $T_0(s), T_1(s) \in \mathfrak{k}$ and $T_2(s), T_3(s) \in \mathfrak{m}$ is preserved by the Nahm flow. For any moduli space $\mathcal{M}$ of $\mathfrak{g}$-valued solutions to Nahm's equations, we can consider a related moduli space of $(\mathfrak{g}, \mathfrak{k})$-valued solutions:

$$\mathcal{M}^{(\mathfrak{g},\mathfrak{k})} = \mathcal{N}^{(\mathfrak{g},\mathfrak{k})}/\{g \in \mathcal{G}_0; g(s) \in K \text{ for all } s\}.$$

These spaces have been first considered by P. Saksida in relation to Hitchin systems. They actually generalise the usual moduli spaces of solutions to Nahm's equations: $\mathcal{M}$ can be thought of as $\mathcal{M}^{(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{g})}$.

The spaces $\mathcal{M}^{(\mathfrak{g},\mathfrak{k})}$ are of course no longer hyperkähler, but we do have a Lax pair for every (original) complex structure, e.g. for I_1 and I_3 we have:

$$\dot{T}_2+i\dot{T}_3 = [T_2+iT_3, T_0-iT_1], \quad \dot{T}_1+i\dot{T}_2 = [T_1+iT_2, T_0-iT_3].$$

Observe that $(T_2 + iT_3)(s) \in \mathfrak{m}^{\mathbb{C}}$ and $(T_1 + iT_2)(s) \in \mathfrak{g}^*$.

Consider now $\mathcal{M}^{(\mathfrak{g},\mathfrak{k})}_{\underline{\mathcal{I}};\sigma}$. Assume that I_1, I_3 are generic so that

$$(\mathcal{M}^{(\mathfrak{g},\mathfrak{k})}_{\underline{\mathcal{I}};\sigma}, I_1) \simeq \mathcal{O}_{v_1} \quad \text{with} \quad v_1 = \tau_2 + i\tau_3 + \sigma(e_2) + i\sigma(e_3)$$

$$(\mathcal{M}^{(\mathfrak{g},\mathfrak{k})}_{\underline{\mathcal{I}};\sigma}, I_3) \simeq \mathcal{O}_{v_3} \quad \text{with} \quad v_2 = \tau_1 + i\tau_2 + \sigma(e_1) + i\sigma(e_2).$$

We assume that the restricted moduli space $\mathcal{M}^{(\mathfrak{g},\mathfrak{k})}_{\underline{\mathcal{I}};\sigma}$ is non-empty, i.e. $\tau_1 \in \mathfrak{k}$ and $\tau_2, \tau_3 \in \mathfrak{m}$. Then, restricting the isomorphism *solutions* $\to$ *orbit* gives:

$$\phi_1 : \mathcal{M}^{(\mathfrak{g},\mathfrak{k})}_{\underline{\mathcal{I}};\sigma} \xrightarrow{\sim} \mathcal{O}_{v_1} \cap \mathfrak{m}^{\mathbb{C}} \quad \text{and} \quad \phi_3 : \mathcal{M}^{(\mathfrak{g},\mathfrak{k})}_{\underline{\mathcal{I}};\sigma} \xrightarrow{\sim} \mathcal{O}_{v_3} \cap \mathfrak{g}^*.$$

Thus we have a diffeomorphism

$$\mathcal{O}_{v_1} \cap \mathfrak{m}^{\mathbb{C}} \xrightarrow{\sim} \mathcal{O}_{v_3} \cap \mathfrak{g}^*, \qquad (6)$$

which provides a correspondence between $K^{\mathbb{C}}$-orbits in $\mathfrak{m}^{\mathbb{C}}$ and adjoint orbits of the dual group G^*. For nilpotent orbits, it is known as the *Kostant-Sekiguchi correspondence* and the diffeomorphism (**??**) as the *Vergne diffeomorphism*.

Spectral curve and the Kähler potential for orbits

From the integrable systems point of view, Nahm's equations are an example of the Lax equation with parameter. Namely, if, for $\zeta \in \mathbb{P}^1$, we set

$$\beta_\zeta = (T_2 + iT_3) + 2iT_1\zeta + (T_2 - iT_3)\zeta^2,$$

$$\alpha_\zeta = (T_0 - iT_1) + (T_2 - iT_3)\zeta,$$

then the Nahm equations are equivalent to

$$\dot{\beta}_\zeta = [\beta_\zeta, \alpha_\zeta] \quad \forall\zeta.$$

The spectrum of β_ζ is now an invariant of the Nahm flow and defines an algebraic curve

$$S = \{(\zeta, \eta);\ \det(\eta \cdot 1 - \beta_\zeta) = 0\} \subset T\mathbb{P}^1,$$

called the *spectral curve* of a solution.

The Krichever method identifies solutions to Nahm equations with a fixed spectral curve S with a linear flow on the Jacobian of S. For the manifolds $\mathcal{M}_{\underline{\tau},\sigma}$, the curve is never irreducible. Moreover, it is reduced only if G is $U(k)$ (or $SU(k)$) and the triple (τ_1, τ_2, τ_3) is regular, i.e. $Z(\tau_1, \tau_2, \tau_3) = \mathfrak{h}$. In this case, S is a union of k rational curves, and $\mathcal{M}_{\underline{\tau},0}$ can be identified with a principle G-bundle over a connected component of the real Jacobian of S.

Under some other (mild) restrictions, I have given (following ideas of Hitchin) a formula for the Kähler potential of the hyperkähler metric on $\mathcal{M}_{\underline{\tau},0}$ in terms of the theta function of S.

A more explicit formula for the Kähler potential is known in the case of complexified hermitian symmetric spaces (Biquard-Gauduchon) and small nilpotent orbits (A. Swann and collaborators).

Solutions with poles

Plenty of geometrically significant spaces can be obtained by allowing meromorphic solutions to Nahm's equations. One can show that such a solution must have simple poles, and its residues define a representation $\sigma : \mathfrak{su}(2) \to \mathfrak{g}$.

Example [$-$]. Let $\mathfrak{g}$ be of type A, D, E and consider solutions on $(0, +\infty)$, which at $+\infty$ behave like solutions in $\mathcal{M}_{\underline{\tau},0}$ and at 0 have poles, with fixed residues defining *the subprincipal representation*, i.e. $\sigma(e_2) + i\sigma(e_3)$ is a subregular nilpotent element. If $Z(\underline{\tau}) = \mathfrak{h}$ (i.e. τ_1, τ_2, τ_3 do not all lie on a Weyl wall), then the moduli space of such solutions is one of Kronheimer's ALE spaces, i.e. a hyperkähler metric on a resolution of a Kleinian singularity $\mathbb{C}^2/\Gamma$, $\Gamma \subset SU(2)$.

Example [– & Atiyah].

For any compact $\mathfrak{g}$ consider solutions as above but with residues at 0 defining *the principal representation*. The moduli space is a point. We can use gauge transformations to make $T_0 \equiv 0$, and so, for every regular (τ_1, τ_2, τ_3) we obtain a unique solution $\big(T_1(s), T_2(s), T_3(s)\big)$ with prescribed poles at 0 and such that $T_i(\infty)$ is conjugate to τ_i, $i = 1, 2, 3$. The centraliser of $\big(T_1(+\infty), T_2(+\infty), T_3(\infty)\big)$ is a Cartan subalgebra, and so we get a map

$$\{\text{regular triples in } \mathfrak{h}\} \to G/T.$$

This map is equivariant w.r.t. the action of the Weyl group and provides an answer to the so-called *Berry-Robbins problem*.

Hypercomplexification of manifolds

In this example, the Nahm equations take values in the Lie algebra of vector fields on a manifold.

The problem is as follows: given a manifold X with a linear connection ∇, does there exist a *hypercomplex structure* in the neighbourhood of the 0-section in TTX, which has certain natural properties. In particular:

- For any geodesic γ in X, the immersed submanifold $TT\gamma$ is a hypercomplex submanifold and the hypercomplex structure of $TT\gamma$ is that of an open subset of $\mathbb{H}$.

- X is totally geodesic w.r.t. the Obata connection and the restriction of the Obata connection to X is the unique *torsion-free* connection with the same geodesics as ∇.

In general, I proved the existence of such a hypercomplex structure for any real-analytic (X, ∇) using twistor space construction.

In the case, when the connection has null curvature (e.g. $X = G$ with the $+$- or $-$-connection), the problem (of finding real sections of the twistor space) reduces to solving Nahm's equations with values in $\Gamma(TX)$. One considers $TX \oplus TX \oplus TX$ (which a linear connection identifies canonically with TTX). A triple of tangent vectors at a point $x \in X$ is then extended to a triple of vector fields on X via the parallel transport (*curvature*$= 0$*!*). These vector fields are the initial values for Nahm equations, which we then solve on $[0, 1]$. Then we proceed analogously to Kronheimer's description of the twistor space of $T^*G^{\mathbb{C}}$.