The Strength of Some Martin–Löf Type Theories

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Abstract

One objective of this paper is the determination of the proof–theoretic strength of Martin–Löf's type theory with a universe and the type of well–founded trees. It is shown that this type system comprehends the consistency of a rather strong classical subsystem of second order arithmetic, namely the one with Δ^1_2 comprehension and bar induction. As Martin-Löf intended to formulate a system of constructive (intuitionistic) mathematics that has a sound philosophical basis, this yields a constructive consistency proof of a strong classical theory. Also the proof-theoretic strength of other inductive types like Aczel's type of iterative sets is investigated in various contexts.

Further, we study metamathematical relations between type theories and other frameworks for formalizing constructive mathematics, e.g. Aczel's set theories and theories of operations and classes as developed by Feferman.

0 Introduction

The intuitionistic theory of types as developed by Martin-Löf is intended to be a system for formalizing intuitionistic mathematics together with an informal semantics, called "meaning explanation", which enables him to justify the rules of his type theory by showing their validity with respect to that semantics. Martin-Löf's theory gives rise to a full scale philosophy of constructivism. It is a typed theory of constructions containing types which themselves depend on the constructions contained in previously constructed types. These dependent types enable one to express the general Cartesian product of the resulting families of types, as well as the disjoint union of such a family.

Using the Brouwer-Heyting semantics of the logical constants one sees how logical notions are obtained in this theory. This is done by interpreting propositions as types and proofs as constructions [ML 84]. In addition to ground types for the finite sets and the set of natural numbers the theory can be taken to contain types which play the role of successive universes or reflections over type construction rules previously introduced. This reflection process can be iterated into the transfinite in order to obtain successively more strongly closed universes known as Mahlo universes. Finally this theory can be taken to contain a type construction principle which, like the general Cartesian product and general disjoint union, is based on dependent families of types. It is the type of well-orderings or well-founded trees which can be constructed using a family of branching types indexed by a previously constructed type.

A number of results were known on the proof theoretic strength of Martin–Löf type theories. Many of the earliest results are presented in Beeson [Be 85]. For example, using an embedding argument it is shown there that intuitionistic type theory without universes has the complexity of

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†The author would like to thank the National Science Foundation of the USA for support by grant DMS–9203443.

true arithmetic. Aczel in [A 77] showed that intuitionistic type theory with one universe $\mathbf{ML_1}$, but without well-ordering types, has the same proof theoretic strength as the classical subsystem of analysis Σ^1_1–$\mathbf{AC}$. Palmgren later gave in [P 93] a lower bound for the system with well–ordering types $\mathbf{ML_1W}$ in terms of the fragment of second order arithmetic with Δ^1_2 comprehension, Δ^1_2– $\mathbf{CA}$. It was known by work of Feferman, Buchholz, Pohlers and Sieg (cf. Feferman's preface to [BFPS 81]) that Δ^1_2– $\mathbf{CA}$ is finitistically reducible to $\mathbf{ID}^{\mathbf{i}}_{<\varepsilon_0}(\mathcal{O})$, the intuitionistic theory of Church–Kleene constructive tree classes iterated any number $< \varepsilon_0$. Palmgren showed that $\mathbf{ID}^{\mathbf{i}}_{<\varepsilon_0}(\mathcal{O})$ is interpretable in $\mathbf{ML_1W}$. However, this result was not expected to be a sharp bound. Palmgren conjectured "that something similar to autonomous inductive definitions may be interpreted in $\mathbf{ML_1W}$", [P 93], p.92. In this paper we show that even the theory Δ^1_2– $\mathbf{CA}$ plus bar induction (Δ^1_2– $\mathbf{CA} + \mathbf{BI}$ for short) is finitistically reducible to $\mathbf{ML_1W}$. Δ^1_2– $\mathbf{CA} + \mathbf{BI}$ is perceptibly stronger than Δ^1_2– $\mathbf{CA}$. Δ^1_2– $\mathbf{CA} + \mathbf{BI}$ not only permits one to prove that Π^1_1 inductive definitions can be iterated along arbitrary well–orderings but (cf. [R 89]) also Δ^1_2 comprehension and Σ^1_2 dependent choices[1].

Key roles in determining a lower bound for the strength of $\mathbf{ML_1W}$ are played by Feferman's theory $\mathbf{T_0}$ of explicit mathematics, which is known to be proof–theoretically equivalent to Δ^1_2– $\mathbf{CA} + \mathbf{BI}$ (cf. [JP 82],[J 83]), and Aczel's version of constructive set theory $\mathbf{CZF}$ plus the regular extension axiom $\mathbf{REA}$ which Aczel interpreted in $\mathbf{ML_1W}$(cf [A 86]).

The paper is organized as follows. For the readers convenience and to make the paper more widely accessible, we give detailed descriptions of the system $\mathbf{T_0}$ in Sect. 1 and of $\mathbf{CZF}$ in Sect.2. For later use we also gather and prove elementary facts about the latter systems. The fruits are reaped in Sect. 3 where $\mathbf{T_0}$ is interpreted in $\mathbf{CZF} + \mathbf{REA}$.

Upper bounds for the proof–theoretic strengths of Martin–Löf type theories are obtained by using recursive realizability models following [Be 82]. Sect. 4 provides a modelling of the type theory $\mathbf{ML_1V}$, which Aczel used in [A 78] to interprete $\mathbf{CZF}$, in Kripke–Platek set theory $\mathbf{KP}$. As a result, $\mathbf{KP}$, $\mathbf{CZF}$ and $\mathbf{ML_1V}$ have the same strength.

Aczel's interpretation of $\mathbf{CZF} + \mathbf{REA}$ in $\mathbf{ML_1W}$ lends itself to a natural restriction of $\mathbf{ML_1W}$, called $\mathbf{ML_{1W}V}$, which has the type of iterated sets $\mathbf{V}$ but allows $\mathbf{W}$–formation only for families in the universe $\mathbf{U}$. The recursive realizability model for $\mathbf{ML_1W}$ is extended in Sect. 5 so as to accomodate the type $\mathbf{V}$ and restricted $\mathbf{W}$-formation. This time the modelling is carried out in the set theory $\mathbf{KPi}$ which is an extension of $\mathbf{KP}$ that axiomatizes a recursively inaccessible set universe. At the end of this section we conclude that the systems Δ^1_2– $\mathbf{CA} + \mathbf{BI}$, $\mathbf{CZF}$+$\mathbf{REA}$ and $\mathbf{ML_{1W}V}$ are of the same strength.

In connection with the system $\mathbf{ML_{1W}V}$ the question arises whether the type $\mathbf{V}$ really contributes to its strength. $\mathbf{V}$ appears to be crucial for the interpretation of $\mathbf{CZF}$+$\mathbf{REA}$ in type theory. To pursue this question we introduce a system of second order $\mathbf{IARI}$ that has unrestricted Replacement but comprehension only for arithmetic properties. In Sect. 6 $\mathbf{IARI}$ is interpreted in $\mathbf{ML_{1W}}$, that is $\mathbf{ML_{1W}V}$ without the type $\mathbf{V}$. In addition, it is shown that $\mathbf{IARI}$ suffices for the well–ordering proof of an ordinal notation system that was used for the analysis of $\mathbf{KPi}$, thereby yielding that $\mathbf{ML_{1W}}$ has the strength of Δ^1_2– $\mathbf{CA} + \mathbf{BI}$ too.

Finally, in Sect. 7, we investigate the strength of the full system $\mathbf{ML_1W}$, where one also has to deal with $\mathbf{W}$–types at large, i.e. those which do not belong to $\mathbf{U}$. We show that $\mathbf{ML_1W}$ can be modelled in a slight extension of $\mathbf{KPi}$. However, we make no attempt to determine the exact bound[2] as this theory seems to be of rather limited interest. More interesting are the theories

[1] These principles do not exhaust the strength of Δ^1_2– $\mathbf{CA} + \mathbf{BI}$ by far (cf. [R 89]).

[2] The exact strength of $\mathbf{ML_1W}$ is investigated in [S 93].

$\mathbf{ML}_n\mathbf{W}$ with n universes. The methods of this paper can also be applied to those theories so as to yield their proof–theoretic strength. As a rule of thumb, the tower of n universes in $\mathbf{ML}_n\mathbf{W}$ corresponds to n recursively inaccessible universes in classical Kripke–Platek set theory.

1 Some Background on Feferman's $\mathbf{T_0}$

1.1 Feferman's $\mathbf{T_0}$

$\mathbf{T_0}$ is the formal system for constructive mathematics due to Feferman. In the literature it has already been shown that within $\mathbf{T_0}$ one can define notions like arithmetic truth, the ramified analytic hierarchy and the constructive tree classes. Since $\mathbf{T_0}$ is one of the tools we use to estimate the strength of intuitionistic type theory with universes closed under well-ordering types, we include a brief discussion and summary.

The language of $\mathbf{T_0}$, $\mathfrak{L}(\mathbf{T_0})$, has two sorts of variables. The free and bound variables ($a, b, c, \dots$ and $x, y, z \dots$) are conceived to range over the whole constructive universe which comprises *operations* and *classifications* among other kinds of entities; while upper-case versions of these A, B, C, ... and X, Y, Z, ... are used to represent free and bound classification variables.

The intended objects of the constructive universe are in general infinite, seen classically. Since the underlying logic of $\mathbf{T_0}$ is intuitionistic the objects of this universe are *presented* in a finitary manner and, without loss of generality, by natural numbers. As far as algorithmic procedures on natural numbers are concerned there are a variety of mathematical formalizations each of which gives a way of presenting the derivations of algorithms using fixed rules for computing as finite objects. To be treated as objects these are then coded by natural numbers. Examples are Gödel numbers of formulae defining these functions in a formal theory, of formal derivations using schemes and codes of Turing machines. Yet another alternative, and the one adopted here, is Gödel numbers of application terms built up from variables and the two basic combinators $\mathbf{k}$ and $\mathbf{s}$ using a binary application operation to represent effectively computable number theoretic functions. This is the sense in which these objects are elements of the constructive universe. In a similar fashion one obtains natural numbers coding the formulae defining properties which give classifications as elements of the constructive universe. Which of these objects actually populate this universe will of course depend on the axioms of $\mathbf{T_0}$. To facilitate the formulation of these axioms, the language $\mathfrak{L}(\mathbf{T_0})$ is equipped with defined symbols for term application and strong or complete equality.

$\mathbf{N}$ is a classification constant taken to define the class of natural numbers. $\mathbf{0}$, $\mathbf{s_N}$ and $\mathbf{p_N}$ are operation constants whose intended interpretations are the natural number 0 and the successor and predecessor operations. Additional operation constants are $\mathbf{k}$, $\mathbf{s}$, $\mathbf{d}$, $\mathbf{p}$, $\mathbf{p_0}$ and $\mathbf{p_1}$ for the two basic combinators, definition by cases on $\mathbf{N}$, pairing and the corresponding two projections. Additional classification constants are generated using the axioms and the constants $\mathbf{j}$, $\mathbf{i}$ and $\mathbf{c_n}(\mathbf{n} < \omega)$ for *join*, *induction* and *comprehension.*

There is no arity associated with the various constants. The *terms* of $\mathbf{T_0}$ are just the variables and constants of the two sorts. The atomic formulae of $\mathbf{T_0}$ are built up using the terms and three primitive relation symbols $=$, App and ε as follows. If q, r, r_1, r_2 are terms, then $q = r$, $\mathrm{App}(q, r_1, r_2)$, and $q \,\varepsilon\, r$ (where r has to be a classification variable or constant) are atomic formulae. $\mathrm{App}(q, r_1, r_2)$ expresses that the operation q applied to r_1 yields the value r_2; $q \,\varepsilon\, r$ asserts[3] that q is in r or that q is classified under r.

[3]It should be pointed out that we use the symbol "ε" instead of "$\in$", the latter being reserved for the set–theoretic elementhood relation.

We write $t_1 t_2 \simeq t_3$ for $\mathrm{App}(t_1, t_2, t_3)$.

The set of formulae are then obtained from these using the propositional connectives and the two quantifiers of each sort.

In order to facilitate the formulation of the axioms, the language of $\mathbf{T_0}$ is expanded definitionally with the symbol $\simeq$ and the auxilliary notion of an *application term* is introduced. The set of application terms is given by two clauses:

1. all terms of $\mathbf{T_0}$ are application terms; and
2. if s and t are application terms, then (st) is an application term.

For s and t application terms, we have auxilliary, defined formulae of the form:

$$s \simeq t \quad := \quad \forall y(s \simeq y \leftrightarrow t \simeq y)$$

where the occurances of y are instances of a bound operation variable. We then define $s \simeq a$ (for a a free variable) inductively by:

$$s \simeq a \quad \text{is} \quad \begin{cases} s = a, & \text{if } s \text{ is a variable or a constant,} \\ \exists x, y[s_1 \simeq x \wedge s_2 \simeq y \wedge \mathrm{App}(x, y, a] & \text{if } s \text{ is an application term } (s_1 s_2) \end{cases}$$

Some abbreviations are $t_1 \ldots t_n$ for $((...(t_1 t_2)...)t_n)$; $t \downarrow$ for $\exists y(t \simeq y)$ and $\phi(t)$ for $\exists y(t \simeq y \wedge \phi(y))$.

Gödel numbers for formulae play a key role in the axioms introducing the classification constants $\mathbf{c_n}$. A formula is said to be *elementary* if it contains only free occurences of classification variables A (i.e., only as *parameters*), and even those free occurrences of A are restricted: A must occur only to the right of ε in atomic formulas. The Gödel number $\mathbf{c_n}$ above is is the Gödel number of an elementary formula. We assume that a standard Gödel numbering numbering has been chosen for $\mathfrak{L}(\mathbf{T_0})$; if ϕ is an elementary formula and $a, b_1, \ldots, b_m, A_1, \ldots, A_n$ is a list of variables which includes all parameters of ϕ, then $\{x : \phi(x, b_1, \ldots, b_n, A_1, \ldots, A_n)\}$ stands for $\mathbf{c_n}(\mathbf{b_1}, \ldots, \mathbf{b_n}, \mathbf{A_1}, \ldots, \mathbf{A_n})$; $\mathbf{n}$ is the code of the pair of Gödel numbers $\langle \ulcorner \phi \urcorner, \ulcorner (a, b_1, \ldots, b_m, A_1, \ldots, A_n) \urcorner \rangle$ and is called the 'index' of ϕ and the list of variables.

Some further conventions are useful. Systematic notation for *n-tuples* is introduced as follows: (t) is t, (s, t) is $\mathbf{p}st$, and $(t_1, \ldots, t_n)$ is defined by $((t_1, \ldots, t_n - 1), t_n)$. Finally, t' is written for the term $\mathbf{s_N}t$, and $\perp$ is the atomic formula $\mathbf{0} \simeq \mathbf{0}'$.

$\mathbf{T_0}$'s logic is intuitionistic two-sorted predicate logic with identity. Its non-logical axioms are:

I. Basic Axioms

1. $\forall X \exists x(X = x)$
2. $\mathrm{App}(a, b, c_1) \wedge \mathrm{App}(a, b, c_2) \rightarrow c_1 = c_2$

II. App Axioms

1. $(\mathbf{k}ab) \downarrow \wedge \mathbf{k}ab \simeq a$,
2. $(\mathbf{s}ab) \downarrow \wedge \mathbf{s}abc \simeq ac(bc)$,
3. $(\mathbf{p}a_1 a_2) \downarrow \wedge (\mathbf{p_1 a}) \wedge (\mathbf{p_2 a}) \downarrow \wedge \mathbf{p_i}(\mathbf{p a_1 a_2}) \simeq \mathbf{a_i}$ for $i = 0, 1$,
4. $(c_1 = c_2 \vee c_1 \neq c_2) \wedge (\mathbf{d}abc_1c_2) \downarrow \wedge (c_1 = c_2 \rightarrow \mathbf{d}abc_1c_2 \simeq a) \wedge (c_1 \neq c_2 \rightarrow \mathbf{d}abc_1c_2 \simeq b)$,
5. $a \,\varepsilon\, \mathbf{N} \wedge b \,\varepsilon\, \mathbf{N} \rightarrow [a' \downarrow \wedge \mathbf{p_N}(a') \simeq a \wedge \neg(a' = 0) \wedge (a' \simeq b' \rightarrow a \simeq b)]$.

III. Classification Axioms

Elementary Comprehension Axiom (ECA)

$\exists X[X \simeq \{x : \psi(x)\} \wedge \forall x(x \,\varepsilon\, X \leftrightarrow \psi(x))]$

for each elementary formula ψa, which may contain additional parameters.

Natural Numbers

(i) $\mathbf{0} \in \mathbf{N} \wedge \forall x(x \,\varepsilon\, \mathbf{N} \rightarrow x' \,\varepsilon\, \mathbf{N})$

(ii) $\phi(\mathbf{0}) \wedge \forall \mathbf{x}(\phi(\mathbf{x}) \rightarrow \phi(\mathbf{x}')) \rightarrow (\forall \mathbf{x} \,\varepsilon\, \mathbf{N})\phi(\mathbf{x})$ for each formula ϕ of $L(\mathbf{T_0})$.

Join (J)

$\forall x \exists Y fx \simeq Y \rightarrow \exists X[X \simeq \mathbf{j}(A, f) \wedge \forall z(z \,\varepsilon\, X \leftrightarrow \exists x \,\varepsilon\, A \exists y(z \simeq (x, y) \wedge y \,\varepsilon\, fx))]$

Inductive Generation (IG)

$$\exists X[X \simeq \mathbf{i}(A, B) \wedge \forall x \,\varepsilon\, A[\forall y[(y, x) \,\varepsilon\, B \rightarrow y \,\varepsilon\, X] \rightarrow x \,\varepsilon\, X]$$
$$\wedge\ \forall x \,\varepsilon\, A\, [\forall y\, ((y, x) \,\varepsilon\, B \rightarrow \phi(y)) \rightarrow \phi(x)] \rightarrow \forall x \,\varepsilon\, X\, \phi(x)]]$$

where ϕ is an arbitrary formula of $\mathbf{T_0}$.

1.2 A Finite Axiomitization of ECA

Later on in section 3 we are going to interprete $\mathbf{T_0}$ in intuitionistic set theory. For technical reasons which will become apparent then, we show that **ECA** is already captured by finitely many of its instances.

Definition 1.1 *For each elementary $\phi(x, y_1, \ldots, y_n, A_1, \ldots, A_m)$ write*

$$\{x : \phi(x, y_1, \ldots, y_n, A_1, \ldots, A_n)\} \textit{ for } c_\phi(y_1, \ldots, y_n, A_1, \ldots, A_m).$$

We may then make the following abbreviations:

$$\begin{array}{rll}
V & \textit{for} & \{x : x = x\}; \\
\emptyset & \textit{for} & \{x : \bot\}; \\
\{a_1, \ldots, a_n\} & \textit{for} & \{x : x = a_1 \vee \cdots \vee x = a_n\}; \\
A \Box B & \textit{for} & \{x : x \,\varepsilon\, A \Box x \,\varepsilon\, B\}, \quad (\Box \in \{\wedge, \vee, \rightarrow\}) \\
A^B & \textit{for} & \{f : x \,\varepsilon\, B \rightarrow fx \,\varepsilon\, A\}; \\
\mathit{Diag}_=(A, B, s, t) & \textit{for} & \{f : f \,\varepsilon\, A^B \ \wedge\ fs \simeq ft \wedge s, t \,\varepsilon\, B\} \\
\mathit{Diag}_{\mathrm{App}}(A, B, s, t, t') & \textit{for} & \{f : f \,\varepsilon\, A^B \ \wedge\ \mathit{App}(fs, ft, ft') \wedge s, t, t' \,\varepsilon\, B\} \\
\mathit{Proj}(A, B, C) & \textit{for} & \{f : f \,\varepsilon\, A^B \ \wedge\ \exists g \,\varepsilon\, C\, \forall x \,\varepsilon\, B(gx \simeq fx)\} \\
\mathit{Ext}(A, B, C, s) & \textit{for} & \{f : f \,\varepsilon\, A^B \ \wedge\ \forall z\, \exists g \,\varepsilon\, C[gs \simeq z \wedge \forall x \,\varepsilon\, B(gx \simeq fx)]\}; \\
f^{-1}\{a\} & \textit{for} & \{x : fx \simeq a\}.
\end{array}$$

In the next definition, to facilitate the writing, we will occasionally write $u \in \{x : \phi(x)\}$ instead of $\phi(u)$. Notice that we use the set–theoretic elementhood sign here.

Definition 1.2 *For the finite axiomatization of* **ECA** *we use the universal closures of the following formulae:*

$$
\begin{array}{ll}
(A1) & \exists X\,[X \simeq (A\Box B) \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow u\,\varepsilon\, A\Box u\,\varepsilon\, A)] \\
 & \quad \text{where } \Box \text{ is one of the connectives } \wedge, \vee, \rightarrow \\
 & \exists X\,[X \simeq \emptyset \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow u \in \emptyset)] \\
(A2) & \exists X\,[X \simeq f^{-1}\{a\} \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow fu \simeq a)] \\
(A3) & \exists X\,[X \simeq \{a\} \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow u = a)] \\
(A4) & \exists X\,[X \simeq \mathit{Diag}_{=}(A,B,a,b)) \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow u \in \mathit{Diag}_{=}(A,B,a,b))] \\
(A5) & \exists X\,[X \simeq \mathit{Diag}_{\mathrm{App}}(A,B,a,b,b') \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow u \in \mathit{Diag}_{\mathrm{App}}(A,B,a,b,b'))] \\
(A6) & \exists X\,[X \simeq \mathit{Proj}(A,B,C) \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow u \in \mathit{Proj}(A,B,C))] \\
(A7) & \exists X\,[X \simeq \mathit{Ext}(A,B,C,a) \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow u \in \mathit{Ext}(A,B,C,a))] \\
(A8) & \exists X\,[X \simeq A^B \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow u \in A^B)] \\
(A9) & \exists X\,[X \simeq \{u : \exists f(f\,\varepsilon\, A \wedge f0 \simeq u\} \;\wedge\; \forall u(u\,\varepsilon\, X \leftrightarrow \exists f(f\,\varepsilon\, A \wedge f0 \simeq u))]
\end{array}
$$

Let $\mathbf{T_0^f}$ arise from $\mathbf{T_0}$ by restricting the schema **ECA** to the finitely many instances from (A1),...,(A9) and only retaining those constants $\mathbf{c_n}$ that enter these axioms.

If $\psi(b)$ is an elementary formula of $\mathbf{T_0^f}$ and t_ψ is an application term of $\mathbf{T_0^f}$ such that

$$\mathbf{T_0^f} \vdash \exists X[X \simeq t \;\wedge\; \forall u\,(u\,\varepsilon\, X \leftrightarrow \psi(u))]$$

and the free variables of t_ψ are among the free variables of $\psi(\mathbf{0})$ we will say that $\{u : \psi(u)\}$ is a *classification (provably) in* $\mathbf{T_0^f}$.

Lemma 1.3 *If $\phi(a_0, \ldots, a_{n-1})$ is an elementary formula of $\mathbf{T_0^f}$ with free non–classification variables among $a_0, \ldots, a_{n-1}$, then we can construct an application term t_ϕ which witnesses that*

$$\{u : u\,\varepsilon\, A^{\{0,\ldots,n-1\}} \wedge \phi(u0, \ldots, u(n-1))]$$

is a classification in $\mathbf{T_0^f}$.

Proof: Write A^n for $A^{\{0,\ldots,n-1\}}$ and $\vec{a}$ for $a_0, \ldots, a_{n-1}$. Note that A^n is a classification in $\mathbf{T_0^f}$ using (A3), (A1), and (A8). We shall denote the term sought after by $\{f\,\varepsilon\, A^{\{0,\ldots,n-1\}} : \phi(f0, \ldots, f(n-1))\}]$. Set $\mathcal{H}_\phi := \{f\,\varepsilon\, A^n : \phi(f0, \ldots, f(n-1))\}$. We want to prove the lemma by induction on the generation of elementary formulas.

1. If $\phi(\vec{a})$ is $a_i = a_j$, then $\mathcal{H}_\phi = \mathrm{Diag}_{=}(A, \{0, \ldots, n-1\}, i, j)$. Since $\{0, \ldots, n-1\}$ is a classification using (A3) and (A1), so is $\mathcal{H}_\phi$ by (A4).

2. $\phi(\vec{a})$ is $a_i = c$, where c is a constant. Let $\psi(\vec{a})$ be the formula $a_i = a_n$. By (A3), (A2), and (A1), $C := \mathcal{H}_\psi \cap \{f\,\varepsilon\, A^{n+1} : fn \simeq c\}$ is a classification. Hence $\mathcal{H}_\phi$ is a classification by (A6) and (1) as $\mathcal{H}_\phi = \mathrm{Proj}(A, \{0, \ldots, n-1\}, C)$.

3. $\phi(\vec{a})$ is $c = d$ where c and d are both constants. Let $\psi(\vec{a})$ be $a_n = a_{n+1}$ and put

$$C := \mathcal{H}_\psi \cap \{f\,\varepsilon\, A^{n+1} : fn \simeq c\} \cap \{g\,\varepsilon\, A^{n+2} : g(n+1) \simeq d\}.$$

As C is a classification, so is $\mathcal{H}_\phi = \mathrm{Proj}(A, \{0, \ldots, n-1\}, C)$.

4. If $\phi(\vec{a})$ is $\mathrm{App}(a_i, a_j, a_k)$, then $\mathcal{H}_\phi = \mathrm{Diag}_{\mathrm{App}}(A, \{0, \ldots, n-1\}, i, j, k)$ is a classification by (A5).

5. Suppose $\phi(\vec{a})$ is $\mathrm{App}(f, s, t)$ where some of the terms f, s, t are constants. Define $\psi(a_0 \ldots, a_{n+2})$ by $\mathrm{App}(a_n, a_{n+1}, a_{n+2})$. Then $\mathcal{H}_\psi$ is a classification by (4). If for instance $\phi(\vec{a})$ is $\mathrm{App}(f, a_i, a_j)$ with f a constant, then $\mathcal{H}_\phi = \mathrm{Proj}(A, \{0, \ldots, n-1\}, C)$ with C being

$$\mathcal{H}_\psi \cap \{g \,\varepsilon\, A^{n+3} : gn \simeq f\} \cap \{g \,\varepsilon\, A^{n+3} : g(n+1) \simeq a_i\} \cap \{g \,\varepsilon\, A^{n+3} : g(n+2) \simeq a_j\}.$$

   Similarly, $\mathcal{H}_\phi$ can be shown to be a classification for the remaining possibilities.

6. If $\phi(\vec{a})$ is of the form $\phi_0(\vec{a}) \Box \phi_1(\vec{a})$, where $\Box$ is one of the connectives $\wedge, \vee, \rightarrow$, then the assertion follows inductively using (A1).

7. If $\phi(\vec{a})$ is $\exists z\, \psi(\vec{a}, z)$, then $\mathcal{H}_\theta = \mathrm{Proj}(A, \{0, \ldots, n-1\}, \mathcal{H})$ with $\theta(a_0, \ldots, a_n)$ being $\psi(\vec{a}, a_n)$. So $\mathcal{H}_\phi$ is a classification by (A6) and the inductive assumption.

8. If $\phi(\vec{a})$ is $\forall z\, \psi(\vec{a}, z)$, then $\mathcal{H}_\phi = \mathrm{Ext}(A, \{0, \ldots, n-1\}, \mathcal{H}_\theta, n)$ with $\theta(a_0, \ldots, a_n)$ being $\psi(\vec{a}, a_n)$. Thus $\mathcal{H}_\phi$ is a classification using (A7) and the inductive assumption.

9. If $\phi(\vec{a})$ is $a_i \,\varepsilon\, B$, then $\mathcal{H}_\phi = \{f \,\varepsilon\, A^n : fi \,\varepsilon\, B\} = \{f \,\varepsilon\, A^n : f \,\varepsilon\, B^{\{i\}}\} = A^n \cap B^{\{i\}}$. Thus $\mathcal{H}_\phi$ is a classification employing (A3), (A8), and (A1).

10. If $\phi(\vec{a})$ is $c \,\varepsilon\, B$ where c is a constant, then $\mathcal{H}_\phi = \mathrm{Proj}(A, \{0, \ldots, n-1\}, C)$ with C being $\mathcal{H}_\psi \cap \{g \,\varepsilon\, A^{n+1} : gn \simeq c\}$, where $\psi(a_0, \ldots a_n)$ is $a_n \,\varepsilon\, B$. $\Box$

Proposition 1.4 $\mathbf{T_0^f} \vdash \mathbf{ECA}$, *that is to say* $\{u : \psi(u)\}$ *is a classification in* $\mathbf{T_0^f}$ *for every elementary formula* ψ *of* $\mathbf{T_0^f}$.

Proof: Let $\phi(a_0, \ldots, a_{n-1})$ be elementary. Then

$$C := \{f \,\varepsilon\, A^n : \phi(f0, \ldots, f(n-1)) \wedge f1 \simeq a_1 \wedge \cdots f(n-1) \simeq a_{n-1}\}$$

is a classification using Lemma 1.3, (A1), (A2), (A3). Now

$$\{x : \phi(x, a_1, \ldots, a_{n-1})\} = \{x : \exists f \,\varepsilon\, C(f0 \simeq x)\},$$

and hence $\{x : \phi(x, a_1, \ldots, a_{n-1})\}$ is a classification by (A9). $\Box$

To conclude from the previous result that $\mathbf{T_0}$ can be interpreted in $\mathbf{T_0^f}$, we need the following lemma.

Lemma 1.5 (cf. [FS 81]) *(***Abstraction Lemma***) For each application term* $t(\mathfrak{c}_1, \ldots, \mathfrak{c}_n)$ *of* $\mathbf{T_0^f}$ *there is a new application term* t^* *such that the parameters of* t^* *are among the parameters of* $t(\mathfrak{c}_1, \ldots, \mathfrak{c}_n)$ *minus* $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$ *and such that*

$$\mathbf{T_0^f} \vdash t^* \downarrow \;\wedge\; t^*(\mathfrak{c}_1, \ldots, \mathfrak{c}_n) \simeq \mathfrak{t}.$$

$\lambda(\mathfrak{c}_1, \ldots, \mathfrak{c}_n).\mathfrak{t}$ *is written for* t^*.

Proof: To begin with, we show this for one parameter. Define $\lambda\mathfrak{c}.t(\mathfrak{c})$ by the complexity of the term t as follows. $\lambda\mathfrak{c}.\mathfrak{c}$ is $\mathbf{skk}$; $\lambda\mathfrak{c}.t$ is $\mathbf{k}t$ for t a constant or a variable other than $\mathfrak{c}$; $\lambda\mathfrak{c}.t_0t_1$ is $\mathbf{s}(\lambda\mathfrak{c}.t_0)(\lambda\mathfrak{c}.t_1)$.

For two parameters $\mathfrak{c}_1, \mathfrak{c}_2$, define $\lambda(\mathfrak{c}_1, \mathfrak{c}_2).t(\mathfrak{c}_1, \mathfrak{c}_2) \ := \ \lambda\mathfrak{c}.t(\mathbf{p_0}(\mathfrak{c}), \mathbf{p_1}(\mathfrak{c}))$; etc.

Notice that if the list $\mathfrak{c}_1, \dots, \mathfrak{c}_\mathfrak{n}$ comprises all the parameters of t, then $\lambda(\mathfrak{c}_1, \dots, \mathfrak{c}_\mathfrak{n}).\mathfrak{t}$ is a closed application term without any classification variables. $\Box$

For later use we state a consequence of the Abstraction Lemma which can be derived in the same way as for the λ–calculus (cf. [F 75],[Be 85],VI2.7).

Corollary 1.6 (Recursion Theorem)

$$\forall f \exists g \forall x_1 \dots \forall x_n \, g(x_1, \dots, x_n) \simeq f(g, x_1, \dots, x_n).$$

Corollary 1.7 $\mathbf{T_0}$ *can be interpreted in* $\mathbf{T_0^f}$.

Proof: The interpretation consists in successively replacing all those comprehension constants $\mathbf{c}_\phi$ which do not belong to $\mathbf{T_0^f}$ with a suitable closed application term $\mathbf{c}_\phi^\diamond$ which arises from abstracting an application term as provided by Proposition 1.4.

The construction of $\mathbf{c}_\phi^\diamond$ proceeds along the generation of $\mathbf{c}_\phi$. So assume that $\mathbf{c}_\phi$ is the index of an elementary formula $\phi(b, a_1, \dots, a_n, B_1, \dots, B_m)$ and a list of variables $b, a_1, \dots, a_n, B_1, \dots, B_m$ that includes all parameters of ϕ. Assume that we have already translated $\phi(b, a_1, \dots, a_n, B_1, \dots, B_m)$ into an elementary formula

$$\phi(b, a_1, \dots, a_n, B_1, \dots, B_m)^\diamond$$

of $\mathbf{T_0^f}$. Then let $t(a_1, \dots, a_n, B_1, \dots, B_m)$ be the term constructed in Proposition 1.4 which witnesses that $\{u : \phi(u, a_1, \dots, a_n, B_1, \dots, B_m)^\diamond\}$ is a classification in $\mathbf{T_0^f}$.

Let $\mathbf{c}_\phi^\diamond \ := \ \lambda(\mathbf{a_1}, \dots, \mathbf{a_n}, \mathbf{B_1}, \dots, \mathbf{B_m}).\mathbf{t}(\mathbf{a_1}, \dots, \mathbf{a_n}, \mathbf{B_1}, \dots, \mathbf{B_m})$. If we now replace $\mathbf{c}_\phi$ in an elementary formula with $\mathbf{c}_\phi^\diamond$, then the result[4] will also be an elementary formula. The faithfulness of this interpretation is immediate by 1.4 and 1.5 $\Box$

In the sequel we shall identify $\mathbf{T_0}$ with $\mathbf{T_0^f}$.

2 Constructive set theories

2.1 The System CZF

In this subsection we will summarize the language and axioms for Aczel's constructive set theory or **CZF**. The language of **CZF** is the first order language of ZF whose only non-logical symbol is $\in$. The logic of **CZF** is intuitionistic first order logic with equality. Its non-logical axioms are *extensionality*, *pairing*, *union*, and *infinity* in their usual forms. **CZF** has additionally axiom schemata which we will now proceed to summarize. Each is followed with some comments on the intuition involved and questions of constructivity.

[4] More precisely, that which is obtained after rewriting the result as a formula of $\mathbf{T_0}$.

Set Induction

$$\forall x[\forall y \in x\phi(y) \to \phi(x)] \to \forall x\phi(x)$$

for all formulae ϕ. Set induction replaces primarily the foundation axiom and corresponds to the principle of $\in$-induction which is a theorem of **ZF**. It should be compared, as far as constructivity goes, with the induction principle associated with any generalized inductive definition. Examples are the inductive definitions giving the higher number classes.

Bounded Separation

$$\forall a\exists b\forall x[x \in b \leftrightarrow x \in a \wedge \phi(x)]$$

for all *bounded* formulae ϕ, where ϕ is said to be bounded if all quantifiers in ϕ are bounded, i.e., having constructed the set a as well as those giving explicit bounds on all of the quantifiers in ϕ, we can construct the set of all elements of a which have the property ϕ.

Strong Collection

$$\forall a[\forall x \in a\exists y\phi(x,y) \to \exists b\phi'(a,b)]$$

for all formulae $\phi(x,y)$, where $\phi'(a,b)$ is the formula

$$\forall x \in a\,\exists y \in b\,\phi(x,y) \wedge \forall y \in b\,\exists x \in a\,\phi(x,y)$$

This axiom allows one to construct a set corresponding exactly to the image of a previously defined relation between sets one has already constructed.

Subset Collection

$$\forall a\forall a'\exists c\forall u(\forall x \in a\,\exists y \in a'\,\phi(x,y) \to \exists b \in c\,\phi'(a,b))$$

for all formulae $\phi(x,y)$. Note that u may occur free in $\phi(x,y)$, i.e., this is subset collection *with parameters.*

Dependent Choices (DC)

$$\forall x(\theta(x) \to \exists y(\theta(y) \wedge \phi(x,y))) \to \forall x(\theta(x) \to \exists z\psi(x,z))$$

for all formulae $\theta(x)$ and $\phi(x,y)$, where $\psi(x,z)$ is the formula which asserts that z is a function with domain the natural numbers such that $z(0) = x$ and for every natural number n, $\theta(z(n)) \wedge \phi(z(n), z(n+1))$ holds.

The formula in the language of **CZF** defining the property of a set A that it is *regular* states that A is transitive, and for every $a \in A$ and set $R \subseteq a \times A$ if $\forall x \in a\,\exists y\,(\langle x,y\rangle \in R)$, then there is a set $b \in A$ such that

$$\forall x \in a\,\exists y \in b\,(\langle x,y\rangle \in R)\ \wedge\ \forall y \in b\,\exists x \in a\,(\langle x,y\rangle \in R).$$

In particular, if $R:\ a \to A$ is a function, then the image of R is an an element of A. Let $\mathrm{Reg}(A)$ denote this assertion. With this auxilliary definition we can state:

Regular Extension Axiom (REA)

$$\forall x \exists y [x \subseteq y \wedge \mathrm{Reg}(y)]$$

Aczel in [A 86] interpreted **CZF** plus the regular extensions axiom in the intuitionistic theory of types due to Martin-Löf using a single universe closed under the type of well-orderings. Alternatively he showed that, by introducing rules in type theory for the collection of *iterative sets* (in anology with Russell's ramified theory of types), one can model the sets and formulae of the language of **CZF** using the types in this extension of intuitionistic type theory.

2.2 Some Properties of CZF

The mathematical power of **CZF** resides in the possibility of defining class functions by $\in$–recursion and the fact that the scheme of Bounded Separation allows for an extension with provable class functions occurring in otherwise bounded formulae.

We shall use the informal notion of class as in classical set theory.

Proposition 2.1 *(Definition of Recursion in* **CZF***) Let G be a total $(n+2)$–ary class function of* **CZF***, i.e.* **CZF** $\vdash \forall \vec{x} y z \exists! u\, G(\vec{x}, y, z) = u$*. Then there is a total $(n+1)$–ary class function F of* **CZF** *such that*[5]

$$\mathbf{CZF} \vdash \forall \vec{x} y [F(\vec{x}, y) = G(\vec{x}, y, (F(\vec{x}, z) | z \in y))].$$

Proof: Let $\Phi(f, \vec{x})$ be the formula

$$[f \text{ is a function}] \wedge [\mathrm{dom}(f) \text{ is transitive}] \wedge [\forall y \in \mathrm{dom}(f)\, (f(y) = G(\vec{x}, y, f \upharpoonright y))].$$

Set $\psi(\vec{x}, y, f) \;\; := \;\; [\Phi(f, \vec{x}) \wedge y \in \mathrm{dom}(f)]$. $\square$

Claim $\quad \mathbf{CZF} \vdash \forall \vec{x}, y \exists! f \psi(\vec{x}, y, f)$.

Proof of Claim: By $\in$–induction on y. Suppose $\forall u \in y\, \exists g\, \psi(\vec{x}, u, g)$. By Strong Collection we find a set A such that $\forall u \in y\, \exists g \in A\, \psi(\vec{x}, u, g)$ and $\forall g \in A \exists u \in y\, \psi(\vec{x}, u, g)$. Let $f_0 = \bigcup \{g : g \in A\}$. By our general assumption there exists a u_0 such that $G(\vec{x}, y, (f_0(u) | u \in y)) = u_0$. Set $f = f_0 \cup \{\langle y, u_0 \rangle\}$. Since for all $g \in A$, $\mathrm{dom}(g)$ is transitive we have that $\mathrm{dom}(f_0)$ is transitive. If $u \in y$, then $u \in \mathrm{dom}(f_0)$. Thus $\mathrm{dom}(f)$ is transitive and $y \in \mathrm{dom}(f)$. We have to show that f is a function. But it is readily shown that if $g_0, g_1 \in A$, then $\forall x \in \mathrm{dom}(g_0) \cap \mathrm{dom}(g_1) [g_0(x) = g_1(x)]$. Therefore f is a function. This also shows that $\forall w \in \mathrm{dom}(f) [f(w) = G(\vec{x}, w, f \upharpoonright w)]$, confirming the claim (using $\in$–induction).

Now define F by

$$F(\vec{x}, y) = w \;\; := \;\; \exists f [\psi(\vec{x}, y, f) \wedge f(y) = w]. \quad \square$$

Corollary 2.2 *There is a class function* **TC** *such that*

$$\mathbf{CZF} \vdash \forall a [\mathbf{TC}(a) = a \cup \bigcup \{\mathbf{TC}(x) : x \in a\}].$$

[5] $(F(\vec{x}, z) | z \in y) \;\; := \;\; \{\langle z, F(\vec{x}, z) \rangle : z \in y\}$

Proposition 2.3 *(Definition by* $\mathbf{TC}$*–Recursion) Under the assumptions of Proposition 2.1 there is an* $(n+1)$*–ary class function* F *of* $\mathbf{CZF}$ *such that*

$$\mathbf{CZF} \vdash \forall \vec{x} y[F(\vec{x}, y) = G(\vec{x}, y, (F(\vec{x}, z)|z \in \mathbf{TC}(y)))].$$

Proof: Let $\theta(f, \vec{x}, y)$ be the formula

$$[f \text{ is a function}] \wedge [\mathrm{dom}(f) = \mathbf{TC}(y)] \wedge [\forall u \in \mathrm{dom}(f)[f(u) = G(\vec{x}, u, f \restriction \mathbf{TC}(u))]].$$

Prove by $\in$–induction that $\forall y \exists! f\, \theta(f, \vec{x}, y)$. Suppose $\forall u \in y\, \exists! g\, \theta(g, \vec{x}, u)$. By Strong Collection we find a set A such that $\forall u \in y\, \exists g \in A\, \theta(g, \vec{x}, u)$ and $\forall g \in A\, \exists u \in y\, \theta(g, \vec{x}, u)$. By assumption, we also have

$$\forall u \in y \exists! z \exists g \exists a[\theta(g, \vec{x}, u) \wedge G(\vec{x}, u, g) = a \ \wedge\ z = \langle u, a\rangle].$$

Again by Strong Collection there is a function h such that $\mathrm{dom}(h) = y$ and

$$\forall u \in y\, \exists g\, [\theta(g, \vec{x}, u) \wedge G(\vec{x}, u, g) = h(u)]\,.$$

Now let $f = (\bigcup\{g : g \in A\}) \cup h$. Then $\theta(f, \vec{x}, y)$. $\square$

The next result was proved by Myhill (cf. [My 75]) for the theory **CST**.

Proposition 2.4 *(Extension by Defined Function Symbols) Suppose* $\mathbf{CZF} \vdash \forall \vec{x}\, \exists! y\, \Phi(\vec{x}, y)$*. Let* $\mathbf{CZF}'$ *be obtained by adjoining a function symbol* F_Φ *to the language*[6]*, extending the schemata to the enriched language, and adding the axiom* $\forall \vec{x}\, \Phi(\vec{x}, F_\Phi(\vec{x}\,))$*. Then* $\mathbf{CZF}'$ *is conservative over* $\mathbf{CZF}$*.*

Proof: We define the following translation * for formulas of $\mathbf{CZF}'$:

$$\begin{aligned} \phi^* &\equiv \phi \text{ if } F_\Phi \text{ does not occur in } \phi \\ (F_\Phi(\vec{x}\,) = y)^* &\equiv \Phi(\vec{x}, y) \end{aligned}$$

If ϕ is of the form $t = x$ with $t \equiv G(t_1, \ldots, t_k)$ such that one of the terms $t_1, \ldots, t_k$ is not a variable, then let

$$(t = x)^* \ \equiv\ \exists x_1 \ldots \exists x_k\, [(t_1 = x_1)^* \wedge \cdots \wedge (t_k = x_k)^* \wedge (G(x_1, \ldots, x_k) = x)^*]\,.$$

The latter provides a definition of $(t = x)^*$ by induction on t. If either t or s contains F_Φ, then let

$$\begin{aligned} (t \in s)^* &\equiv \exists x \exists y[(t = x)^* \wedge (s = y)^* \wedge x \in y], \\ (t = s)^* &\equiv \exists x \exists y[(t = x)^* \wedge (s = y)^* \wedge x = y], \\ (\neg\phi)^* &\equiv \neg\phi^* \\ (\phi_0 \Box \phi_1)^* &\equiv \phi_0^* \Box \phi_1^*, \quad \text{if } \Box \text{ is } \wedge, \vee, \text{ or } \rightarrow \\ (\exists x \phi)^* &\equiv \exists x \phi^* \\ (\forall x \phi)^* &\equiv \forall \phi^*. \end{aligned}$$

Let $\mathbf{CZF}_0$ be the restriction of $\mathbf{CZF}'$, where F_Φ is not allowed to occur in the Bounded Separation Scheme. Then it is obvious that $\mathbf{CZF}_0 \vdash \phi$ implies $\mathbf{CZF} \vdash \phi^*$. So it remains to show that $\mathbf{CZF}'$ proves the same theorems as $\mathbf{CZF}_0$. In actuality, we have to prove $\mathbf{CZF}_0 \vdash \exists x \forall y\, [y \in x \leftrightarrow y \in a \wedge \phi(a)]$ for any bounded formula ϕ of $\mathbf{CZF}'$. We proceed by induction on ϕ.

[6] This changes the notion of bounded formula.

1. $\phi(y) \equiv t(y) \in s(y)$. Now

$$\mathbf{CZF}_0 \vdash \forall y \in a \exists! z[(z = t(y)) \wedge \forall y \in a \exists! u(u = s(y))].$$

Using Replacement we find functions f and g such that

$$\mathrm{dom}(f) = \mathrm{dom}(g) = a \text{ and } \forall y \in a\, [f(y) = t(y) \wedge g(y) = s(y)]\,.$$

Therefore $\{y \in a : \phi(y)\} = \{y \in a : f(y) \in g(y)\}$ exists by Bounded Separation in $\mathbf{CZF}_0$.

2. $\phi(y) \equiv t(y) = s(y)$. Similar.

3. $\phi(y) \equiv \phi_0(y) \Box \phi_1(y)$, where $\Box$ is any of $\wedge, \vee, \rightarrow$. This is immediate by induction hypothesis.

4. $\phi(y) \equiv \forall u \in t(y)\ \phi_0(u, y)$. We find a function f such that $\mathrm{dom}(f) = a$ and $\forall y \in a\, f(y) = t(y)$. Inductively, for all $b \in a$, $\{u \in \bigcup \mathrm{ran}(f) : \phi_0(u, b)\}$ is a set. Hence there is a function g with $\mathrm{dom}(g) = a$ and $\forall b \in a\, g(b) = \{u \in \bigcup \mathrm{ran}(f) : \phi_0(u, b)\}$. Then $\{y \in a : \phi(y)\} = \{y \in a : \forall u \in f(y)(u \in g(y))\}$.

5. $\phi(y) \equiv \exists u \in t(y)\, \phi_0(u, y)$. With f and g as above, $\{y \in a : \phi(y)\} = \{y \in a : \exists u \in f(y)(u \in g(y))\}$. $\Box$

Remark 2.5 *The proof of Proposition 2.4 shows that the process of adding defined function symbols to* **CZF** *can be iterated. So if e.g.* $\mathbf{CZF}' \vdash \forall \vec{x} \exists y\, \psi(\vec{x}, y)$, *then also* $\mathbf{CZF}' + \{\forall \vec{x} \exists y\, \psi(\vec{x}, F_\psi(\vec{x}))\}$ *will be conservative over* **CZF**.

3 A Lower Bound for the Strength of $\mathbf{ML_1W}$

Employing an interpretation of intuitionistic theories of iterated strictly positive inductive definitions in Martin–Löf's type theory, Palmgren showed in [P 93] that $\mathbf{ML_1W}$ has at least the strength of $\mathbf{\Delta^1_2}$– **CA**. In this section we will establish that $\mathbf{ML_1W}$ is much stronger than $\mathbf{\Delta^1_2}$– **CA** in that we interpret Feferman's theory $\mathbf{T_0}$ in $\mathbf{ML_1W}$. By [FS 81], [JP 82], and [J 83] it is known that $\mathbf{T_0}$ has the same strength as Δ^1_2-comprehension plus bar induction. However, rather than modelling $\mathbf{T_0}$ directly in $\mathbf{ML_1W}$, our line of attack will be to model $\mathbf{T_0}$ in Aczel's constructive set theory $\mathbf{CZF} + \mathbf{REA}$, for which Aczel in [A 86] provided an interpretation in $\mathbf{ML_1W}$. Roughly, our interpretation of $\mathbf{T_0}$ in $\mathbf{CZF} + \mathbf{REA}$ proceeds by showing that Feferman's set–theoretical interpretation of $\mathbf{T_0}$ (cf. [F 75]) can be carried out in $\mathbf{CZF} + \mathbf{REA}$.

3.1 The Interpretation of $\mathbf{T_0}$ in $\mathbf{CZF} + \mathbf{REA}$

For later use we show that, provably in $\mathbf{CZF} + \mathbf{REA}$ the well-founded part of a binary set relation is a set.

For any class Φ the class X is said to be Φ–*closed* if $A \subseteq X$ implies $a \in X$ for every pair $\langle a, A\rangle \in \Phi$.

Lemma 3.1 (**CZF**) (Aczel, cf. [A 82],4.2) *For any class* Φ *there is a smallest* Φ*–closed class* $\mathbf{I}(\Phi)$.

Proof: Call a set relation G *good* if whenever $\langle x, y\rangle \in G$ there is a set A such that $\langle y, A\rangle \in \Phi$ and

$$\forall u \in A\, \exists v \in x\, \langle v, u\rangle \in G.$$

Call a set Φ–*generated* if it is in the range of some good relation. To see that the class of Φ–generated sets is Φ–closed, let A be a set of Φ–generated sets, where $\langle a, A\rangle \in \Phi$. Then

$$\forall y \in A\, \exists G\, [G \text{ is good} \wedge \exists x\, (\langle x, y\rangle \in G)].$$

By Strong Collection there is a set C of good sets such that

$$\forall y \in A\, \exists G \in C\, \exists x\, (\langle x, y\rangle \in G).$$

Letting $G_0 = \bigcup C \cup \{\langle b, a\rangle\}$, where $b = \{u : \exists y\, \langle u, y\rangle \in \bigcup C\}$, G_0 is good and $\langle b, a\rangle \in G_0$. Thus a is Φ–generated. Whence $\mathbf{I}(\Phi)$ is Φ–closed. Now if X is another Φ–closed class and G is good, then by set induction on x it follows $\langle x, y\rangle \in G \rightarrow y \in X$, so that $\mathbf{I}(\Phi) \subseteq X$. $\square$

Lemma 3.2 (Aczel, [A 86], Corollary 5.3) *For R a set let $R_i = \{j : \langle j, i\rangle \in R\}$. Let $\mathbf{WF(A,R)}$ be the smallest class X such that for $i \in A$*

$$R_i \subseteq X \textit{ implies } i \in X.$$

Then $\mathbf{WF(A,R)} = \mathbf{I}(\Phi)$ where $\Phi = \{\langle i, R_i\rangle : i \in A\}$. $\mathbf{CZF} + \mathbf{REA}$ proves that, for sets A and R, $\mathbf{WF(A,R)}$ is a set.

Proof: We argue in $\mathbf{CZF} + \mathbf{REA}$. For each set x let

$$\Gamma(x) \quad = \quad \{\imath : i \in A \wedge R_i \subseteq x\}. \tag{1}$$

Note that $\Gamma(x)$ is a set by Bounded Separation. Let X be the smallest class such that if

$$\forall y \in a\, \exists z \in b\, \langle y, z\rangle \in X \;\wedge\; \forall z \in b\, \exists y \in a\, \langle y, z\rangle \in X$$

then

$$\langle a, \Gamma(\bigcup b)\rangle \in X.$$

This definition can be couched in terms of an inductive definition falling under the scope of Lemma 3.1. Using $\in$–induction it is easily verified that for each set a there is a unique set y such that $\langle a, y\rangle \in X$, and if this unique y is written Γ^a then

$$\Gamma^a \quad = \quad \Gamma(\bigcup\{\Gamma^y : y \in a\}). \tag{2}$$

Next we want to show

$$\mathbf{I}\,(\Phi) \quad = \quad \bigcup\{\Gamma^u : u \in \mathbf{V}\}. \tag{3}$$

First note that if $\Gamma^y \subseteq \mathbf{I}\,(\Phi)$ for all $y \in a$, then $\bigcup\{\Gamma^y : y \in a\} \subseteq \mathbf{I}\,(\Phi)$, whence $\Gamma^a \subseteq \mathbf{I}\,(\Phi)$ since $\mathbf{I}\,(\Phi)$ is Φ–closed. Thus by set induction, $\Gamma^a \subseteq \mathbf{I}\,(\Phi)$ for all sets a, confirming "$\subseteq$" of (3). For the converse inclusion it suffices to show that $\bigcup\{\Gamma^u : u \in \mathbf{V}\}$ is Φ–closed. So suppose $x \subseteq \bigcup\{\Gamma^u : u \in \mathbf{V}\}$. Then

$$\forall y \in x\, \exists a\, (y \in \Gamma^a).$$

Employing Strong Collection there is a set b such that

$$\forall y \in x\ \exists a \in b\ (y \in \Gamma^a).$$

Therefore $x \subseteq \bigcup\{\Gamma^u : u \in b\}$ and hence $\Gamma(x) \subseteq \Gamma^b \subseteq \bigcup\{\Gamma^u : u \in \mathbf{V}\}$.

Finally we must show that $\mathbf{I}\,(\Phi)$ is a set. By the Regular Extension Axiom we can pick a regular set B such that $\forall i \in A\ R_i \in B$. Let

$$I \;=\; \bigcup\{\Gamma^u : u \in B\}.$$

Strong Collection and Union guarantee that I is a set. By (3) we have $I \subseteq \mathbf{I}\,(\Phi)$. So it remains to show that I is Φ–closed, since then $\mathbf{I}\,(\Phi) \subseteq I$. So assume $R_i \subseteq I$ for some $i \in A$. Then $\forall j \in R_i\ \exists u \in B\ j \in \Gamma^u$. By the regularity of B there is $b \in B$ such that $\forall j \in R_i\ \exists u \in b\ j \in \Gamma^u$. It follows that $R_i \subseteq \bigcup\{\Gamma^u : u \in b\}$, yielding, by (2), that $i \in \Gamma^b \subseteq I$. $\square$

Corollary 3.3 $\mathbf{CZF} + \mathbf{REA}$ *minus Subset Collection proves that, for sets A and R,* $\mathbf{WF(A,R)}$ *is a set.*

Proof: This is because nowhere in the proofs of 3.1 and 3.2 we had to use Subset Collection. $\square$

Since $\mathbf{I}\,(\Phi)$ is the smallest Φ–closed class we get the following induction principle for $\mathbf{WF(A,R)}$.

Corollary 3.4 ($\mathbf{CZF} + \mathbf{REA}$) *For an arbitrary formula of set theory $\psi(x)$ and sets A, R,*

$$\forall i \in A[\forall j \in R_i\, \psi(j) \to \psi(i)] \to \forall i \in \mathbf{WF(A,R)}\, \psi(i),$$

where $R_i = \{j \in A : \langle j, i\rangle \in R\}$. $\square$

For the interpretation of $\mathbf{T_0}$ it is convenient to work in a definitional extension of $\mathbf{CZF}$ which we denote again by $\mathbf{CZF}$. Drawing on Proposition 2.4, we may assume that $\mathbf{CZF}$ has a constant ω for the unique set whose existence is asserted in the axiom of infinity[7]. By the same token we may assume that $\mathbf{CZF}$ has a function symbol $\mathcal{S}$ for the successor function given by $\mathcal{S}(x) = x \cup \{x\}$, and a constant 0 denoting the empty set. $\mathbf{CZF}$ proves ω–induction (cf. [TD 88], Ch. 11, 8.5)

$$\phi(0) \wedge \forall x\,[\phi(x) \to \phi(\mathcal{S}(x))] \to \forall x \in \omega\, \phi(x).$$

The Peano Axioms for zero and successor can be shown in $\mathbf{CZF}$ to hold for 0 and $\mathcal{S}$ on ω. Employing Proposition 2.1 and Proposition 2.4 we may assume that $\mathbf{CZF}$ has as many function symbols for primitive recursive functions as we like and their defining axioms being among the axioms of $\mathbf{CZF}$. In particular we will assume that all those primitive recursive functions (and their defining axioms) required for introducing Kleene's T–predicate and the pertinent evaluation function U are part of $\mathbf{CZF}$. As usual we define

$$\{x\}(y) \simeq z \quad := \quad x, y, z \in \omega \wedge \exists v \in \omega[T(x, y, v) \wedge U(v) = z].$$

[7]Uniqueness is proved in [TD 88] Ch. 11, 8.3.

We shall assume that $\mathbf{CZF}$ has a constant $\bar{n}$ for each (meta) natural number n which names the n^{th} numeral, i.e. the n^{th} element of ω. We are going to translate $\mathrm{App}(x,y,z)$ as $\{x\}(y) \simeq z$. Particular numerals $\hat{\mathbf{s_N}}, \hat{\mathbf{p_N}}, \hat{\mathbf{k}}, \hat{\mathbf{s}}, \hat{\mathbf{d}}, \hat{\mathbf{p_0}}, \hat{\mathbf{p_1}}$ can be determined to satisfy (provably in $\mathbf{CZF}$) the translation of the applicative axioms pertaining to the constants $\mathbf{s_N}, \mathbf{p_N}, \mathbf{k}, \mathbf{s}, \mathbf{d}, \mathbf{p_0}, \mathbf{p_1}$ of $\mathbf{T_0}$. $\mathbf{N}$ is translated as $\hat{\mathbf{N}} := \omega$. Select $\hat{\mathbf{c_n}}, \hat{\mathbf{j}}, \hat{\mathbf{i}}$ in such a way so that

$$\mathbf{CZF} \vdash \forall x, y \in \omega \left[\hat{\mathbf{c_n}}(x) \simeq (\bar{1}, \bar{n}, x) \wedge \{\hat{\mathbf{j}}\}(x,y) \simeq (\bar{2}, x, y) \wedge \{\hat{\mathbf{i}}\}(x,y) \simeq (\bar{3}, x, y) \right],$$

where $(x,y) := \{\{\hat{\mathbf{p}}\}(x)\}(y)$ and $(x,y,z) := ((x,y),z)$. By Proposition 1.4, $\mathbf{ECA}$ can be finitely axiomatized, say via the instances

$$\forall A, B, C\, \forall x, y, z\, \exists X \left[\mathbf{c}_{\phi_\mathbf{i}}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{A},\mathbf{B},\mathbf{C}) \simeq \mathbf{X} \wedge \forall \mathbf{u} (\mathbf{u} \,\varepsilon\, \mathbf{X} \leftrightarrow \phi_\mathbf{i}(\mathbf{u},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{A},\mathbf{B},\mathbf{C})) \right],$$

where $1 \le i \le M$ for some M.

We are now in a position to define a class model for $\mathbf{T_0}$ in $\mathbf{CZF} + \mathbf{REA}$. Since in constructive set theory the ordinals are not at all as well behaved as they are classically, we cannot define this model by recursion along the ordinals as in [F 75], Theorem 1.1. Instead, we shall define it by recursion on the transitive closure along all sets, thereby employing Proposition 2.3.

Definition 3.5 *For each set s, sets $\mathbf{E_s}$ and $\mathbf{CL}_s$ are defined by $\mathbf{TC}$–recursion as follows:*

$$\begin{aligned}
\mathbf{CL}_{<s} &= \bigcup \{\mathbf{CL}_u : u \in \mathbf{TC}(s)\} \\
\mathbf{E_{<s}} &= \bigcup \{\mathbf{E_u} : \mathbf{u} \in \mathbf{TC(s)}\} \\
\mathbf{CL}_s &= \mathbf{CL}_{<s} \cup \{\hat{\mathbf{N}}\} \cup \\
&\quad \{\{\hat{\mathbf{c}}_{\phi_i}\}(x,y,z,a,b,c) : a,b,c \in \mathbf{CL}_{<s};\, x,y,z \in \omega;\, 1 \le i \le M\} \cup \\
&\quad \left\{\{\hat{\mathbf{j}}\}(a,f) : a \in \mathbf{CL}_{<s} \wedge f \in \omega \wedge \forall x \in \omega[\langle x,a\rangle \in \mathbf{E_{<s}} \to \{\mathbf{f}\}(\mathbf{x}) \in \mathbf{CL_{<s}}]\right\} \cup \\
&\quad \left\{\{\hat{\mathbf{i}}\}(a,b) : a,b \in \mathbf{CL}_{<s}\right\};
\end{aligned}$$

$\langle u,a\rangle \in \mathbf{E_s}$ iff either of the following holds:

1. *$\langle u,a\rangle \in \mathbf{E_{<s}}$;*
2. *$a = \hat{\mathbf{N}} \wedge u \in \omega$;*
3. *$a \simeq \{\hat{\mathbf{c}}_{\phi_i}\}(x,y,z,b,c,d)$ with $b,c,d \in \mathbf{CL}_{<s}$, $x,y,z \in \omega$ and $\phi_i(u,x,y,z,b,c,d)$ is true under the interpretation which takes $v \,\varepsilon\, w$ to mean $\langle v,w\rangle \in \mathbf{E_{<s}}$ and interprets App and the constants of $\mathbf{T_0}$ as stated above;*
4. *$a \simeq \{\hat{\mathbf{j}}\}(b,f)$ with $b \in \mathbf{CL_{<s}}$, $f \in \omega$ such that $\forall x \in \omega[\langle x,b\rangle \in \mathbf{E_{<s}} \to \{\mathbf{f}\}(\mathbf{x}) \in \mathbf{CL_{<s}}]$ and $u = (u_0,u_1)$ satisfies $\langle u_1, \{f\}(u_o)\rangle \in \mathbf{E_{<s}}$;*
5. *$a \simeq \hat{\mathbf{i}}(b,c)$ with $b,c \in \mathbf{CL_{<s}}$ and $u \in \mathbf{WF}(b^{<s}, c^{<s})$, where $b^{<s} = \{x : \langle x,b\rangle \in \mathbf{E_{<s}}\}$ and $c^{<s} = \{\langle x,y\rangle : \langle (x,y),c\rangle \in \mathbf{E_{<s}}\}$.*

We have to verify that the preceding definition falls under the scope of Proposition 2.3. But this follows once we see that the passage from $(\langle \mathbf{CL}_u, \mathbf{E_u}\rangle | \mathbf{u} \in \mathbf{s})$ to $\langle \mathbf{CL}_s, \mathbf{E_s}\rangle$ determines a class function of $\mathbf{CZF} + \mathbf{REA}$. It is obvious that $\mathbf{CL}_s$ is a set again. Regarding $\mathbf{E_s}$ this follows from the fact that $\mathbf{E_s}$ can be obtained by Bounded Separation from $(\langle \mathbf{CL}_u, \mathbf{E_u}\rangle | \mathbf{u} \in \mathbf{s})$ and $\{\mathbf{WF}(b^{<s}, c^{<s}) : b, c \in \mathbf{CL}_s\}$, the latter being a set by Lemma 3.2 and Strong Collection.

To complete the interpretation of $\mathbf{T_0}$ in $\mathbf{CZF} + \mathbf{REA}$ we are going to let classification variables range over $\mathbf{CL} = \bigcup\{\mathbf{CL}_u : u \in V\}$ and translate $x \,\varepsilon\, s$ as $\langle x, s\rangle \in \mathbf{E}$, where $\mathbf{E} = \bigcup\{\mathbf{E_u} : \mathbf{u} \in \mathbf{V}\}$. We shall write $u\mathbf{E_s x}$ and $u\mathbf{E_{<s} x}$ instead of $\langle u, x\rangle \in \mathbf{E_s}$ and $\langle u, x\rangle \in \mathbf{E_{<s}}$, respectively. Verifying that the classification axioms of $\mathbf{T_0}$ are preserved under the above interpretation requires a little lemma to the extent that all $\mathbf{E}$–elements of a member x of $\mathbf{CL}$ are added at the same time as x is declared to be in $\mathbf{CL}$.

Lemma 3.6 *If $a \in \mathbf{CL}_s \cap \mathbf{CL}_t$, then $\{u : u\mathbf{E_s a}\} = \{\mathbf{u} : \mathbf{uE_t a}\}$.*

The proof of 3.6 requires another lemma.

Lemma 3.7 *Let $e \in \mathbf{CL}_z$. Then*

1. $e \simeq \{\hat{\mathbf{c}}_{\phi_i}\}(x, y, z, a, b, c) \to a, b, c \in \mathbf{CL}_{<z}$,
2. $e \simeq \{\hat{\mathbf{j}}\}(a, f) \to a \in \mathbf{CL}_{<z}$,
3. $e \simeq \{\hat{\mathbf{i}}\}(a, b) \to a, b, \in \mathbf{CL}_{<z}$.

Proof: Use $\in$-induction on z. $\Box$

Proof of 3.6: We use $\in$-induction on s. Set $\mathbf{E_z(a)} = \{\mathbf{u} : \mathbf{uE_z a}\}$. If $a \in \mathbf{CL}_{<s}$, then $a \in \mathbf{CL}_z$ for some $z \in s$ and inductively $\mathbf{E_z(a)} = \mathbf{E_s(a)}$ and $\mathbf{E_z(a)} = \mathbf{E_t(a)}$, thus $\mathbf{E_s(a)} = \mathbf{E_t(a)}$. If $a \simeq \{\hat{\mathbf{i}}\}(c, d)$ with $c, d \in \mathbf{CL}_{<s}$, then $c, d \in \mathbf{CL}_{<t}$ by 3.7. Hence $\mathbf{E_{<s}(c)} = \mathbf{E_{<t}(c)}$ and $\mathbf{E_{<s}(d)} = \mathbf{E_{<t}(d)}$, using the inductive assumption. Thus $\mathbf{E_s(a)} = \mathbf{E_t(a)}$ by 2.3(5). The argument proceeds along the same way in the remaining cases. $\Box$

Corollary 3.8 *If $a \in \mathbf{CL}_s$, then $\{u : u\mathbf{E_s a}\} = \{\mathbf{u} : \mathbf{uEa}\}$.* $\Box$

In view of 3.8 and 3.5(3) it is clear that the interpretation renders the elementary comprehension axioms true. As to Join, suppose $a \in \mathbf{CL}$ and $\forall x[xEa \to \{f\}(x) \in \mathbf{CL}]$. Then there exists s so that $a \in \mathbf{CL}_s$ and $\forall x \in \mathbf{E_s(a)}\, \exists \mathbf{z}\, [\{\mathbf{f}\}(\mathbf{x}) \in \mathbf{CL_z}]$. Using Strong Collection we find a t such that $\forall x \in \mathbf{E_s(a)}\, \exists \mathbf{z} \in \mathbf{t}\, [\{\mathbf{f}\}(\mathbf{x}) \in \mathbf{CL_z}]$, and hence $\forall x \in \mathbf{E_s(a)}\, \{\mathbf{f}\}(\mathbf{x}) \in \mathbf{CL_{<t}}$. For $s' = \{s, t\}$ it follows $\forall x[x\mathbf{E_{<s'} a} \to \{\mathbf{f}\}(\mathbf{x}) \in \mathbf{CL_{<s'}}]$ using 3.6. Hence $\{\hat{\mathbf{j}}\}(a, f) \in \mathbf{CL}_{s'}$ and

$$\forall u \left[u\mathbf{E_{s'}}\{\hat{\mathbf{j}}\}(\mathbf{a}, \mathbf{f}) \iff \exists \mathbf{u_0} \exists \mathbf{u_1} [\mathbf{u} \simeq (\mathbf{u_0}, \mathbf{u_1}) \wedge \mathbf{u_1 E_{<s'}}\{\mathbf{f}\}(\mathbf{u_0})]\right].$$

Employing 3.6, this yields

$$\forall u \left[u\mathbf{E}\{\hat{\mathbf{j}}\}(\mathbf{a}, \mathbf{f}) \iff \exists \mathbf{u_0} \exists \mathbf{u_1} (\mathbf{u} \simeq (\mathbf{u_0}, \mathbf{u_1}) \wedge \mathbf{u_1 E}\{\mathbf{f}\}(\mathbf{u_0}))\right],$$

confirming the faithfulness of our translation with regard to Join. For Inductive Generation suppose $a, b \in \mathbf{CL}$. Then $a, b \in \mathbf{CL}_{<s}$ for some s. With $a^{<s} = \{x : x\mathbf{E_{<s} a}\}$ and $b^{<s} = \{\langle x, y\rangle : (x, y)\mathbf{E_{<s} b}\}$ we obtain

$$\forall u \left[u\mathbf{E_{<s}}\{\hat{\mathbf{i}}\}(\mathbf{a}, \mathbf{b}) \iff \mathbf{u} \in \mathbf{WF}(\mathbf{a}^{<s}, \mathbf{b}^{<s})\right],$$

according to 3.5.5. Therefore, using 3.6 and 3.8, we have

$$\forall u \left[u\mathbf{E}\{\hat{\mathbf{i}}\}(\mathbf{a},\mathbf{b}) \iff \mathbf{u} \in \mathbf{WF}(\mathbf{a^E},\mathbf{b^E}) \right],$$

where $a^E = \{x : x\mathbf{Ea}\}$ and $b^E = \{\langle x,y\rangle : (x,y)\mathbf{Eb}\}$. Consequently, by 3.4, we come to see that the interpretation also preserves the axiom of Inductive Generation.

Theorem 3.9 $\mathbf{T_0}$ *can be interpreted in* $\mathbf{CZF} + \mathbf{REA}$. $\square$

4 Interpretation of $\mathbf{ML_1V}$ in KP

$\mathbf{ML_1V}$ is the extension of $\mathbf{ML_1}$ with Aczel's set of iterative sets $\mathbf{V}$ (cf. [A 78]). The rules pertaining to $\mathbf{V}$ are:[8]

$$(\mathbf{V}\text{–formation}) \quad \mathbf{V} \text{ set}$$

$$(\mathbf{V}\text{–introduction}) \quad \frac{A \in \mathbf{U} \quad f \in A \to \mathbf{V}}{\mathbf{sup}(A,f) \in \mathbf{V}}$$

$$(\mathbf{V}\text{–elimination}) \quad \frac{c \in \mathbf{V} \quad \begin{array}{c} [A \in \mathbf{U}, f \in A \to \mathbf{V}] \\ [z \in (\mathbf{\Pi} v \in A) C(f(v))] \\ d(A,f,z) \in C(\mathbf{sup}(A,f)) \end{array}}{\mathbf{T_V(c,(A,f,z)d) \in C(c)}}$$

$$(\mathbf{V}\text{–equality}) \quad \frac{B \in \mathbf{U} \quad g \in B \to \mathbf{V} \quad \begin{array}{c} [A \in \mathbf{U}, f \in A \to \mathbf{V}] \\ [z \in (\mathbf{\Pi} v \in A) C(f(v))] \\ d(A,f,z) \in C(\mathbf{sup}(A,f)) \end{array}}{\mathbf{T_V(sup(B,g),(A,f,z)d) \ = \ d(B,g,(\boldsymbol{\lambda} v)T_V(g(v),(A,f,z)d)) \in C(sup(B,g))}}.$$

KP stands for Kripke–Platek set theory which we shall assume to contain the axiom of infinity. As usual in set theory we identify the natural numbers with the finite ordinals, i.e. $\mathbb{N} := \omega$. To simplify the treatment we will assume that **KP** has names for all (meta) natural numbers. Let $\bar{n}$ be the constant designating the n^{th} natural number. We also assume that **KP** has function symbols for addition and multiplication on $\mathbb{N}$ as well as for a primitive recursive bijective pairing function $\langle,\rangle_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$ and its primitive recursive inverses $(\,)_0, (\,)_1$, that satisfy $(\langle n,m\rangle_{\mathbb{N}})_0 = n$ and $(\langle n,m\rangle_{\mathbb{N}})_1 = m$. **KP** should also have a symbol T for Kleene's T-predicate and the result extracting function U. Let $\{e\}(n) \simeq k$ be a shorthand for $\exists m[T(e,n,m) \wedge U(m) = k]$. Further, let $\langle n,m,k\rangle_{\mathbb{N}} := \langle\langle n,m\rangle_{\mathbb{N}},k\rangle_{\mathbb{N}}$, $\langle n,m,k,l\rangle_{\mathbb{N}} := \langle\langle n,m,k\rangle_{\mathbb{N}},l\rangle_{\mathbb{N}}$, etc. We use e,d,f,n,m,l,k,s,t,j,i as metavariables for natural numbers. Occasionally, " $\Rightarrow$ " is used for "then" and " $\Leftrightarrow$ " for "iff". Let

$$\begin{aligned}
\pi(n,m) &= \langle 0, \langle n,m\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
\sigma(n,m) &= \langle 1, \langle n,m\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
\mathrm{pl}(n,m) &= \langle 2, \langle n,m\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
i(n,m,k) &= \langle 3, \langle n,m,k\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
\hat{N} &= \langle 4, 0\rangle_{\mathbb{N}} \\
\hat{N}_k &= \langle 4, k+1\rangle_{\mathbb{N}} \\
\sup(n,m) &= \langle 5, \langle n,,\rangle_{\mathbb{N}} m\rangle_{\mathbb{N}}
\end{aligned}$$

[8] To increase readability and intelligibility, we will use the formulation á la Russel for type theories with universes (cf. [ML 84]).

Definition 4.1 *By Σ recursion (cf. [Ba 75],I.6.4) along the ordinals we define simultaneously seven relations $R_1^\alpha, \ldots, R_7^\alpha$ on $\mathbb{N}$. To increase intelligibility, we shall write*

$$\begin{array}{lll}
\mathbb{U}_\alpha \models n \ \mathbf{set} & \textit{for} & R_1^\alpha(n) \\
\mathbb{U}_\alpha \models n \in m & \textit{for} & R_2^\alpha(n,m) \\
\mathbb{U}_\alpha \models n = m \in k & \textit{for} & R_3^\alpha(n,m,k) \\
\mathbb{U}_\alpha \models n = m & \textit{for} & R_4^\alpha(n,m) \\
\mathbb{U}_\alpha \models \text{“}f \text{ is a family of types over } k\text{”} & \textit{for} & R_5^\alpha(f,k) \\
\mathbb{V}_\alpha \models k \ \mathbf{set} & \textit{for} & R_6^\alpha(k) \\
\mathbb{V}_\alpha \models n = m & \textit{for} & R_7^\alpha(n,m)
\end{array}$$

If $\alpha = 0$, then $R_1^0 = \cdots = R_7^0 = \emptyset$. If α is a limit, then $R_i^\alpha = \bigcup_{\beta<\alpha} R_i^\beta$ for $i = 1, \ldots, 7$. The clauses in the definition for successors are as follows:

(i)

$$\begin{array}{lll}
\mathbb{U}_{\alpha+1} \models \langle 4, j\rangle \ \ \mathbf{set} & \textit{iff} & j \in \mathbb{N} \\
\mathbb{U}_{\alpha+1} \models m \in \langle 4, j\rangle & \textit{iff} & j = 0 \ \vee \ m+1 < j \\
\mathbb{U}_{\alpha+1} \models n = m \in \langle 4, j\rangle & \textit{iff} & n = m \ \wedge \ \mathbb{U}_{\alpha+1} \models n \in \langle 4, j\rangle.
\end{array}$$

*(ii) If $\mathbb{U}_\alpha \models k$ **set**, $\forall j[\mathbb{U}_\alpha \models j \in k \Rightarrow \mathbb{U}_\alpha \models \{e\}(j)$ **set**$]$ and $\forall j, i[\mathbb{U}_\alpha \models i = j \in k \Rightarrow \mathbb{U}_\alpha \models \{e\}(i) = \{e\}(j)]$, then*

$$\mathbb{U}_{\alpha+1} \models \text{“}e \text{ is a family of types over } k\text{”}.$$

*(iii) If $\mathbb{U}_\alpha \models$ “e is a family of types over k”, then $\mathbb{U}_{\alpha+1} \models \pi(k,e)$ **set**, $\mathbb{U}_{\alpha+1} \models \sigma(k,e)$ **set** and*

$$\begin{array}{lll}
\mathbb{U}_{\alpha+1} \models n \in \pi(k,e) & \textit{iff} & \forall i\, (\mathbb{U}_\alpha \models i \in k \ \Rightarrow \ \mathbb{U}_\alpha \models \{n\}(i) \in \{e\}(i)) \ \textit{and} \\
& & \forall i, j\, [\mathbb{U}_\alpha \models i = j \in k \ \Rightarrow \ \mathbb{U}_\alpha \models \{n\}(i) = \{n\}(j) \in \{e\}(i)] \\[1ex]
\mathbb{U}_{\alpha+1} \models n = m \in \pi(k,e) & \textit{iff} & \mathbb{U}_{\alpha+1} \models n \in \pi(k,e) \ \textit{and} \ \mathbb{U}_{\alpha+1} \models m \in \pi(k,e) \ \textit{and} \\
& & \forall j\, [\mathbb{U}_\alpha \models j \in k \ \Rightarrow \ \mathbb{U}_\alpha \models \{n\}(j) = \{m\}(j) \in \{e\}(j)] \\[1ex]
\mathbb{U}_{\alpha+1} \models n \in \sigma(k,e) & \textit{iff} & \mathbb{U}_\alpha \models (n)_0 \in k \ \textit{and} \ \mathbb{U}_\alpha \models (n)_1 \in \{e\}((n)_0) \\[1ex]
\mathbb{U}_{\alpha+1} \models n = m \in \sigma(k,e) & \textit{iff} & \mathbb{U}_{\alpha+1} \models n \in \sigma(k,e) \ \textit{and} \ \mathbb{U}_{\alpha+1} \models m \in \sigma(k,e) \ \textit{and} \\
& & \mathbb{U}_\alpha \models (n)_0 = (m)_0 \in k \ \textit{and} \ \mathbb{U}_\alpha \models (n)_1 = (m)_1 \in \{e\}((n)_0).
\end{array}$$

*(iv) If $\mathbb{U}_\alpha \models n$ **set** and $\mathbb{U}_\alpha \models m$ **set**, then $\mathbb{U}_{\alpha+1} \models pl(n,m)$ **set** and*

$$\begin{array}{lll}
\mathbb{U}_{\alpha+1} \models i \in pl(n,m) & \textit{iff} & [(i)_0 = 0 \ \textit{and} \ \mathbb{U}_\alpha \models (i)_1 \in n] \ \textit{or} \\
& & [(i)_0 = 1 \ \textit{and} \ \mathbb{U}_\alpha \models (i)_1 \in m] \\[1ex]
\mathbb{U}_{\alpha+1} \models i = j \in pl(n,m) & \textit{iff} & [(i)_0 = (j)_0 = 0 \ \textit{and} \ \mathbb{U}_\alpha \models (i)_1 = (j)_1 \in n] \ \textit{or} \\
& & [(i)_0 = (j)_0 = 1 \ \textit{and} \ \mathbb{U}_\alpha \models (i)_1 = (j)_1 \in m].
\end{array}$$

*(v) If $\mathbb{U}_\alpha \models n$ **set**, then $\mathbb{U}_{\alpha+1} \models i(n,m,k)$ **set** and*

$$\begin{array}{lll}
\mathbb{U}_{\alpha+1} \models s \in i(n,m,k) & \textit{iff} & s = 0 \ \textit{and} \ \mathbb{U}_\alpha \models m = k \in n, \\
\mathbb{U}_{\alpha+1} \models s = s' \in i(n,m,k) & \textit{iff} & s = s' = 0 \ \textit{and} \ \mathbb{U}_\alpha \models m = k \in n.
\end{array}$$

(vi) $\mathbb{U}_{\alpha+1} \models e = f$ *iff* $\mathbb{U}_\alpha \models e$ **set**, $\mathbb{U}_\alpha \models f$ **set**, $\forall s(\mathbb{U}_\alpha \models s \in e \Leftrightarrow \mathbb{U}_\alpha \models s \in f)$ *and* $\forall s,t(\mathbb{U}_\alpha \models s = t \in e \Leftrightarrow \mathbb{U}_\alpha \models s = t \in f)$.

(vii) *If* $\mathbb{U}_\alpha \models k$ **set**, $\forall s[\mathbb{U}_\alpha \models s \in k \Rightarrow \mathbb{V}_\alpha \models \{f\}(s)$ **set**$]$ *and* $\forall s,t\,[\mathbb{U}_\alpha \models s = t \in k \Rightarrow \mathbb{V}_\alpha \models \{f\}(s) = \{f\}(t)]$, *then* $\mathbb{V}_{\alpha+1} \models \sup(k,f)$ **set**.

(viii) $\mathbb{V}_{\alpha+1} \models \sup(k,f) = \sup(n,e)$ *iff* $\mathbb{V}_{\alpha+1} \models \sup(k,f)$ **set**, $\mathbb{V}_{\alpha+1} \models \sup(n,e)$ **set**, $\mathbb{U}_\alpha \models k = n$ *and* $\forall s\,[\mathbb{U}_\alpha \models s \in k \rightarrow \mathbb{V}_\alpha \models \{f\}(s) = \{e\}(s)]$.

Lemma 4.2 *If* $\alpha \leq \beta$ *and* $\mathbb{U}_\alpha \models k$ **set**, *then the following hold true:*

1. $\mathbb{U}_\beta \models k$ **set**;
2. $\forall n(\mathbb{U}_\alpha \models n \in k \Leftrightarrow \mathbb{U}_\beta \models n \in k)$;
3. $\forall n,m(\mathbb{U}_\alpha \models n = m \in k \Leftrightarrow \mathbb{U}_\beta \models n = m \in k)$;
4. *If* $\mathbb{U}_\alpha \models n = m$, *then* $\mathbb{U}_\beta \models n = m$;
5. *If* $\mathbb{U}_\alpha \models$ *"f is a family of types over n", then* $\mathbb{U}_\beta \models$ *"f is a family of types over n";*
6. *If* $\mathbb{V}_\alpha \models n$ **set**, *then* $\mathbb{V}_\beta \models n$ **set**;
7. *If* $\mathbb{V}_\alpha \models n = m$, *then* $\mathbb{V}_\beta \models n = m$.

Proof: One must prove all the assertions simultaneously by induction on β. Trivial, but lengthy. □

Definition 4.3 *The previous lemma justifies the next definition.*

$$\begin{aligned}
\mathbb{U} \models k \ \ \mathbf{set} &:= \exists\alpha\, \mathbb{U}_\alpha \models k \ \ \mathbf{set} \\
\mathbb{V} \models k \ \ \mathbf{set} &:= \exists\alpha\, \mathbb{V}_\alpha \models k \ \ \mathbf{set} \\
\mathbb{U} \models n = m &:= \exists\alpha\, \mathbb{U}_\alpha \models n = m \\
\mathbb{V} \models n = m &:= \exists\alpha\, \mathbb{V}_\alpha \models n = m \\
\mathbb{U} \models \text{“}f \textit{ is a family of types over } n\text{”} &:= \exists\alpha\, \mathbb{U}_\alpha \models \text{“}f \textit{ is a family of types over } n\text{”}.
\end{aligned}$$

Proposition 4.4 *The clauses (i)-(viii) of Definition 4.1 hold true with* $\mathbb{U}_\alpha$ *and* $\mathbb{U}_{\alpha+1}$ *being replaced with* $\mathbb{U}$ *as well as* $\mathbb{V}_\alpha$ *and* $\mathbb{V}_{\alpha+1}$ *being replaced with* $\mathbb{V}$.

Proof: This is an outgrowth of 4.2. As an example we shall verify (ii). The proofs for the other clauses being similar. So suppose $\mathbb{U} \models k$ **set**, $\forall j \in \mathbb{N}[\mathbb{U} \models j \in k \Rightarrow \mathbb{U} \models \{e\}(j)$ **set**$]$, and $\forall j,i \in \mathbb{N}\big[\mathbb{U} \models i = j \in k \Rightarrow \mathbb{U} \models \{e\}(i) = \{e\}(j)\big]$. We are to show $\mathbb{U} \models$ *"f is a family of types over n"*. Pick α such that $\mathbb{U}_\alpha \models k$ **set**. By 4.2 we obtain $\forall j \in \mathbb{N}[\mathbb{U} \models j \in k \Leftrightarrow \mathbb{U}_\alpha \models j \in k]$ and $\forall i,j \in \mathbb{N}[\mathbb{U} \models i = j \in k \Leftrightarrow \mathbb{U}_\alpha \models i = j \in k]$. Therefore

$$\begin{gathered}
\forall j \in \mathbb{N}\exists\beta[\mathbb{U}_\alpha \models j \in k \Rightarrow \mathbb{U}_\beta \models \{e\}(j) \ \ \mathbf{set}], \\
\forall j,i \in \mathbb{N}\exists\beta[\mathbb{U}_\alpha \models i = j \in k \Rightarrow \mathbb{U}_\beta \models \{e\}(i) = \{e\}(j)].
\end{gathered}$$

Thus, using Σ–Reflection (cf. [Ba 75],I.4.3), there exists a $\delta > \alpha$ such that

$$\begin{gathered}
\forall j \in \mathbb{N}\exists\beta < \delta[\mathbb{U}_\alpha \models j \in k \Rightarrow \mathbb{U}_\beta \models \{e\}(j) \ \ \mathbf{set}], \\
\forall j,i \in \mathbb{N}\exists\beta < \delta[\mathbb{U}_\alpha \models i = j \in k \Rightarrow \mathbb{U}_\beta \models \{e\}(i) = \{e\}(j)].
\end{gathered}$$

Employing 4.2 it follows $\mathbb{U}_\delta \models k$ **set** and

$$\forall j \in \mathbb{N}[\mathbb{U}_\delta \models j \in k \;\Rightarrow\; \mathbb{U}_\delta \models \{e\}(j) \;\; \mathbf{set}] \quad \text{and}$$
$$\forall j, i \in \mathbb{N}[\mathbb{U}_\delta \models i = j \in k \;\Rightarrow\; \mathbb{U}_\delta \models \{e\}(i) = \{e\}(j)].$$

Hence $\mathbb{U}_{\delta+1} \models$ *"e is a family of types over k"* according to 4.1(ii). Thus $\mathbb{U} \models$ *"e is a family of types over k."* $\square$

In order to define its interpretation in **KP**, we need a detailed account of the syntax of $\mathbf{ML_1V}$. Here we will follow [Be 85], Ch. XI; however, for the readers convenience, we shall recall most of the definitions.

If B is any expression, and $x_1, \ldots, x_n$ are variables, we form the expression $(x_1, \ldots, x_n)B$. The symbol $\stackrel{\triangle}{\equiv}$ will be used for the relation on expressions satisfying

$$((x_1, \ldots, x_n)B)(x_1, \ldots, x_n) \stackrel{\triangle}{\equiv} B$$

and $A \stackrel{\triangle}{\equiv} C$ for expressions A and C which differ only in the renaming of bound variables (cf. [Be 85],XI6).

Definition 4.5 *(cf. [Be 85],XI20.3) The* constants *of* $\mathbf{ML_1V}$ *are:* $\mathbf{\Pi}, \mathbf{\Sigma}, \mathbf{I}, +, \mathbf{N}, \mathbf{0}, \mathbf{s_N}$,$\mathbf{r}$,$\boldsymbol{\lambda}$, $\mathbf{ap}$, $\mathbf{E}$, $\mathbf{i}$,$\mathbf{j}$,$\mathbf{D}$, $\mathbf{J}$,$\mathbf{R}$,$\mathbf{T_V}$,$\mathbf{U}$,$\mathbf{V}$ *and for each natural number m,* $\mathbf{N_m}$ *and* $\mathbf{R_m}$*. The* terms *are generated by:*

1. *Every constant and variable is a term;*

2. *If t and s are terms, then $t(s)$ and (t, s) are terms;*

3. *If t is a term, then $(x_1, \ldots, x_n)t$ is a term, where the x_i are variables.*

Free and bound occurrences of variables in terms are defined as usual, letting abstraction, i.e. the formation of $(x_1, \ldots, x_n)t$ bind the variables $x_1, \ldots, x_n$. We now would like to assign to every term t of $\mathbf{ML_1W}$ a corresponding term $\hat{t}$ of the theory **KP** by replacing the abstract application of $\mathbf{ML_1W}$ with recursive application. However, the technicality here is that recursive application cannot be expressed on the level of terms within **KP**. Therefore, we first assign to every term t of $\mathbf{ML_1W}$ an application term t^* of the theory **EON** of [Be 85],VI.2 or equivalently of the theory **APP** as described in [TD 88],Ch.9,Sect.3. It is then a straightforward matter to translate a formula of the form $t^* \in X$ into a legitimate formula of **KP**.

Definition 4.6 *The theory* **EON** *consists of* $\mathbf{T_0}$ *without classification variables and without the constants* $\mathbf{c_n}, \mathbf{i}, \mathbf{j}$*. In particular,* **EON** *has no axioms* **ECA, (IG), (J)***.*

Note that t he Abstraction Lemma 1.5 and the Recursion Theorem 1.6 are also theorems of **EON**. Both devices will be used in the next definition.

Definition 4.7 *We now assign to each term t of* $\mathbf{ML_1V}$ *an application term t^* of the theory* **EON***. Occurrences of λ in the definition of t^* denote the λ-operator introduced in 1.5. We shall write (x, y) for $\mathbf{p}xy$ and, inductively, $x_1, \ldots, x_{k+1}$ for $\mathbf{p}(x_1, \ldots, x_k)x_{k+1}$. For constants* $\mathbf{c}$ *we define* $\mathbf{c^*}$ *by:*

$$\begin{aligned}
\mathbf{0}^* &\text{ is } \mathbf{0}\\
\mathbf{\Pi}^* &\text{ is } \lambda x.\lambda y.(0,x,y)\\
\mathbf{\Sigma}^* &\text{ is } \lambda x.\lambda y.(1,x,y)\\
+^* &\text{ is } \lambda x.\lambda y.(2,x,y)\\
\mathbf{I}^* &\text{ is } \lambda x.\lambda y.(3,x,y,z)\\
\mathbf{N}^* &\text{ is } (4,0)\\
\mathbf{N_k}^* &\text{ is } (4,k+1)\\
\mathbf{U}^* &\text{ is } (6,0)\\
\mathbf{V}^* &\text{ is } (7,0)\\
\mathbf{s_N}^* &\text{ is } \mathbf{s_N}\\
\mathbf{r}^* &\text{ is } \mathbf{0}\\
\boldsymbol{\lambda}^* &\text{ is } \lambda x.x \quad \text{(i.e. } \mathbf{skk}\text{)}\\
\mathbf{ap}^* &\text{ is } \lambda x.\lambda y.xy\\
\mathbf{sup}^* &\text{ is } \lambda x.\lambda y.(5,x,y)\\
\mathbf{E}^* &\text{ is } \lambda x.\lambda y.y(\mathbf{p_0}x,\mathbf{p_1}x)\\
\mathbf{i}^* &\text{ is } \lambda x.(0,x)\\
\mathbf{j}^* &\text{ is } \lambda x.(1,x)\\
\mathbf{D}^* &\text{ is } \lambda x.\lambda y.\lambda z(0,\mathbf{p_0}x,y(\mathbf{p_0}x),z(\mathbf{p_1}x))\\
\mathbf{J}^* &\text{ is } \lambda x.\lambda y.y
\end{aligned}$$

$\mathbf{R_k}^*$ is $\lambda m.\lambda x_0\cdots\lambda x_{k-1}.e_k(m,x_0,\dots,x_k-1)$, where e_k is chosen so that **EON** proves

$$e_k(m,x_0,\dots,x_k-1)\simeq\begin{cases} x_m & \text{if } m<k\\ \Omega & \text{otherwise,}\end{cases}$$

where $\Omega := \lambda x.xx(\lambda x.xx)$ (signifying undefinedness). $\mathbf{R}^*$ is an application term of **EON** introduced by the Recursion Theorem (cf. 1.6) to satisfy (provably in **EON**) $\mathbf{R}^*(a,b,0)=a$ and $\mathbf{R}^*(a,b,\mathbf{s_N x})\simeq\mathbf{b}(\mathbf{x},\mathbf{R}^*(\mathbf{a},\mathbf{b},\mathbf{x}))$. $\mathbf{T_V^*}$ is a term introduced by the recursion theorem of **EON** to satisfy

$$\mathbf{T_V^*}(\mathbf{sup}^*(\mathbf{a},\mathbf{b}),\lambda\mathbf{x}.\lambda\mathbf{y}.\lambda\mathbf{z}.\mathbf{e}(\mathbf{x},\mathbf{y},\mathbf{z}))\simeq\mathbf{e}(\mathbf{a},\mathbf{b},(\lambda\mathbf{x}.\mathbf{x})(\lambda\mathbf{v}.\mathbf{T_V^*}(\mathbf{ap}^*(\mathbf{b},\mathbf{v})),\lambda\mathbf{x}.\lambda\mathbf{y}.\lambda\mathbf{z}.\mathbf{e}(\mathbf{x},\mathbf{y},\mathbf{z}))).$$

For complex terms of $\mathbf{ML_1V}$ we define:

$$\begin{aligned}
((x_1,\dots,x_n)t)^* &\text{ is } \lambda x_1\cdots\lambda x_n.t^*;\\
(t(s))^* &\text{ is } t^*s^*;\\
(t,s)^* &\text{ is } \mathbf{p}t^*s^*.
\end{aligned}$$

Definition 4.8 *Rather than translating application terms of* **EON** *into the set–theoretic language of* **KP**, *we define the translation of expressions of the form $t\simeq u$, where t is an application term of* **EON** *and u is a variable. First, we select indices $\bar{\mathbf{k}},\bar{\mathbf{s}},\bar{\mathbf{p}},\bar{\mathbf{p}}_\mathbf{0},\bar{\mathbf{p}}_\mathbf{1}$, $\bar{\mathbf{s}}_\mathbf{N}$, $\bar{\mathbf{p}}_\mathbf{N}$ and $\bar{\mathbf{d}}$ for partial recursive functions so that:*

$$
\begin{array}{rcl}
\{\{\bar{\mathbf{k}}\}(n)\}(m) & \simeq & n \\
\{\{\{\bar{\mathbf{s}}\}(n)\}(m)\}(l) & \simeq & \{\{n\}(l)\}(\{m\}(l)) \\
\{\{\bar{\mathbf{p}}\}(n)\}(m) & \simeq & 2^n \cdot 3^m = \langle n, m\rangle_{\mathbb{N}} \\
\{\bar{\mathbf{p}}_\mathbf{0}\}(n) & \simeq & (n)_0 \\
\{\bar{\mathbf{p}}_\mathbf{1}\}(n) & \simeq & (n)_1 \\
\{\bar{\mathbf{s}}_\mathbf{N}\}(n) & \simeq & n+1 \\
\{\bar{\mathbf{p}}_\mathbf{N}\}(0) & \simeq & 0 \\
\{\bar{\mathbf{p}}_\mathbf{N}\}(n+1) & \simeq & n \\
\{\{\{\{\bar{\mathbf{d}}\}(n)\}(m)\}(x)\}(y) & \simeq & \begin{cases} x & \textit{if } n = m \\ y & \textit{if } n \neq m \end{cases}
\end{array}
$$

The definition of $(t \simeq u)^\wedge$ *proceeds along the way that* t *was built up.*

$$
\begin{array}{lll}
(v \simeq u)^\wedge & \textit{is} & v = u \wedge v \in \omega \quad \textit{if } v \textit{ is a variable;} \\
(0 \simeq u)^\wedge & \textit{is} & 0 = u; \\
(\boldsymbol{c} \simeq u)^\wedge & \textit{is} & \bar{\mathbf{c}} = u \textit{ if } \boldsymbol{c} \textit{ is one of the constants } \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p_0}, \mathbf{p_1}, \mathbf{s_N}, \mathbf{p_N}, \mathbf{d}.
\end{array}
$$

A complex application term of **EON** *has the form ts.Let*

$$(ts \simeq u)^\wedge := \exists x, y \in \omega[(t \simeq x)^\wedge \ \wedge \ (s \simeq y)^\wedge \ \wedge \ \{x\}(y) \simeq u].$$

EON *is interpreted in* **KP** *by setting*

$$
\begin{array}{rcl}
(App(s,t,r))^\diamond & := & \exists u \in \omega\, [(st \simeq u)^\wedge \ \wedge \ (r \simeq u)^\wedge] \\
(s = t)^\diamond & := & \exists u \in \omega\, [(s \simeq u)^\wedge \ \wedge \ (t \simeq u)^\wedge] \\
(s \in \mathbf{N})^\diamond & := & \exists u \in \omega\, [(s \simeq u)^\wedge] \\
(\phi \Box \psi)^\diamond & := & \phi^\diamond \Box \psi^\diamond \quad (\Box \in \{\wedge, \vee, \rightarrow\}) \\
(\forall x \phi(x))^\diamond & := & \forall x \in \omega\, \phi(x)^\diamond.
\end{array}
$$

Definition 4.9 *The* set terms *of* $\mathbf{ML_1V}$ *are defined inductively by*

1. $\mathbf{N}$ *and* $\mathbf{N_k}$ *are set terms (for each integer* k*);*
2. *If* A *and* B *are set terms, so is* $(A+B)$*;*
3. *IF* $B(x)$ *and* A *are set terms, and* x *is not free in* A *or in* B*, then* $\mathbf{\Pi}(A,B)$ *and* $\mathbf{\Sigma}(A,B)$ *are set terms;*
4. *If* A *is a set term and* t, s *are any terms of* $\mathbf{ML_1V}$*, then* $\mathbf{I}(A,s,t)$ *is a set term;*
5. $\mathbf{U}$ *and* $\mathbf{V}$ *are set terms;*
6. *If* A *is a set term and* $B \stackrel{\triangle}{\equiv} A$*, then* B *is a set term.*

Definition 4.10 *(Interpretation of* $\mathbf{ML_1V}$ *in* **KP***) By induction on the complexity of the set term* A *we shall assign to each judgement* Φ *of* $\mathbf{ML_1V}$ *of the form* $u \in A$ *or* $u = v \in A$ *(*u, v *variables) a formula* $(\Phi)^\wedge$ *of* **KP** *with the same free variables.* $(ux \in A)^\wedge$ *and* $(ux = uy \in A)^\wedge$ *will be used as shorthand for* $\exists z \in \omega[\{u\}(x) \simeq z \ \wedge \ (z \in A)^\wedge]$ *and* $\exists z \in \omega[\{u\}(x) \simeq z \ \wedge \ \{u\}(y) \simeq z \ \wedge \ (z \in A)^\wedge]$*, respectively. Likewise,* $(u \in A(vx))^\wedge$ *abbreviates* $\exists z \in \omega[\{v\}(x) \simeq z \wedge (u \in A(z))^\wedge]$*, etc. The clauses in the definition are as follows:*

$$
\begin{array}{rcl}
(u \in \mathbf{\Pi}(A,B))^\wedge & \text{is} & \forall x \in \omega[(x \in A)^\wedge \to (ux \in B(x))^\wedge] \wedge \\
& & \forall x, y \in \omega[(x = y \in A)^\wedge \to (ux = uy \in B(x))^\wedge] \\
(u = v \in \mathbf{\Pi}(A,B))^\wedge & \text{is} & \forall x \in \omega[(x \in A)^\wedge \to (ux = vx \in B(x))^\wedge] \wedge \\
& & (u \in \mathbf{\Pi}(A,B))^\wedge \wedge (v \in \mathbf{\Pi}(A,B))^\wedge \\
(u \in \mathbf{\Sigma}(A,B))^\wedge & \text{is} & (\bar{\mathbf{p}}_\mathbf{0} u \in A)^\wedge \wedge (\bar{\mathbf{p}}_\mathbf{1} u \in B(\bar{\mathbf{p}}_\mathbf{0} u))^\wedge \\
(u = v \in \mathbf{\Sigma}(A,B))^\wedge & \text{is} & (\bar{\mathbf{p}}_\mathbf{0} u = \bar{\mathbf{p}}_\mathbf{0} v \in A)^\wedge \wedge (\bar{\mathbf{p}}_\mathbf{1} u = \bar{\mathbf{p}}_\mathbf{1} v \in B(\bar{\mathbf{p}}_\mathbf{0} u))^\wedge \\
(u \in (A+B))^\wedge & \text{is} & [\bar{\mathbf{p}}_\mathbf{0} u = 0 \wedge (\bar{\mathbf{p}}_\mathbf{1} u \in A)^\wedge] \vee [\bar{\mathbf{p}}_\mathbf{0} u = 1 \wedge (\bar{\mathbf{p}}_\mathbf{1} u \in B)^\wedge] \\
(u = v \in (A+B))^\wedge & \text{is} & [\bar{\mathbf{p}}_\mathbf{0} u = 0 \wedge \bar{\mathbf{p}}_\mathbf{0} v = 0 \wedge (\bar{\mathbf{p}}_\mathbf{1} u = \bar{\mathbf{p}}_\mathbf{1} v \in A)^\wedge] \vee \\
& & [\bar{\mathbf{p}}_\mathbf{0} u = 1 \wedge \bar{\mathbf{p}}_\mathbf{0} v = 1 \wedge (\bar{\mathbf{p}}_\mathbf{1} u = \bar{\mathbf{p}}_\mathbf{1} v \in B)^\wedge] \\
(u \in \mathbf{I}(A,b,c))^\wedge & \text{is} & u = 0 \wedge (b = c \in A)^\wedge \\
(u = v \in \mathbf{I}(A,b,c))^\wedge & \text{is} & u = 0 \wedge v = 0 \wedge (b = c \in A)^\wedge \\
(u \in \mathbf{N})^\wedge & \text{is} & u \in \omega \\
(u = v \in \mathbf{N})^\wedge & \text{is} & u = v \wedge u \in \omega \\
(u \in \mathbf{N_k})^\wedge & \text{is} & u \in \omega \wedge u \in \bar{k} \\
(u = v \in \mathbf{N_k})^\wedge & \text{is} & u \in \omega \wedge u = v \wedge u \in \bar{k} \\
(u \in \mathbf{U})^\wedge & \text{is} & \mathbb{U} \models u \ \ \mathbf{set} \\
(u = v \in \mathbf{U})^\wedge & \text{is} & \mathbb{U} \models u = v \\
(u \in \mathbf{V})^\wedge & \text{is} & \mathbb{V} \models u \ \ \mathbf{set} \\
(u = v \in \mathbf{V})^\wedge & \text{is} & \mathbb{V} \models u = v
\end{array}
$$

If s and t are arbitrary terms of $\mathbf{ML_1V}$ and A is a set term of $\mathbf{ML_1V}$, we set:

$$
\begin{array}{rcl}
(t \in A)^\wedge & \text{is} & \exists u \in \omega[(t^* \simeq u)^\wedge \wedge (u \in A)^\wedge], \\
(s = t \in A)^\wedge & \text{is} & \exists u, v \in \omega[(s^* \simeq u)^\wedge \wedge (t^* \simeq v)^\wedge \wedge (u = v \in A)^\wedge].
\end{array}
$$

For set terms A and B we define $(A = B)^\wedge$ by

$$\forall u \in \omega[(u \in A)^\wedge \leftrightarrow (u \in B)^\wedge] \wedge \forall u, v \in \omega[(u = v \in A)^\wedge \leftrightarrow (u = v \in B)^\wedge].$$

Theorem 4.11 (Soundness of the Interpretation of $\mathbf{ML_1V}$ in $\mathbf{KP}$.) *If Φ is a judgement of $\mathbf{ML_1V}$ not of the form "A set," then $\mathbf{KP} \vdash (\Phi)^\wedge$.*

Proof: First note that if an expression of the form A set, $s \in A$, $s = t \in A$, or $A = B$ appears in a derivation of $\mathbf{ML_1V}$, then A is a set term in the sense of Definition 4.9, as is readily seen by induction on derivations in $\mathbf{ML_1V}$.[9] This ensures that any judgement of $\mathbf{ML_1V}$ gets translated under $^\wedge$. Secondly, it should be clear that the above interpretation replaces the abstract application of $\mathbf{ML_1V}$ by recursive application in a faithful way, i.e. the equations which the rules of $\mathbf{ML_1V}$ prescribe for the constants of $\mathbf{ML_1V}$ are satisfied by their translations. Since $\mathbf{ML_1V}$ derivations may involve hypothetical judgements, 4.11 has to be stated in a more general form so as to be able to carry out the proof by induction on derivations in $\mathbf{ML_1V}$. For the details we refer to

[9] Incidentally, there are more set terms than those that appear in such derivations.

[Be 85],XI.20.3.1, where this is done for the interpretation of $\mathbf{ML_0}$ in **EON**. Proposition 4.4 yields that the particular rules for **U** are sound with respect to the interpretation $^\wedge$. The soundness of **V**–introduction is implied by 4.4(vii). Next we shall verify the soundness of **V**–elimination.

To facilitate the writing we shall write $\psi(t)$ in place of $\exists u[(t \simeq u)^\wedge \wedge \psi(u)]$, when t is a term of **EON** and $\psi(u)$ a set-theoretic formula. Suppose $\mathbb{V} \models c_0$ **set** and , for all $a, f, g \in \omega$, if

$$\begin{aligned}
&\mathbb{U} \models a \ \ \mathbf{set}, && (4)\\
&\forall n \in \omega[\,\mathbb{U} \models n \in a \rightarrow \mathbb{V} \models \{f\}(n) \ \ \mathbf{set}],\\
&\forall n, m \in \omega[\,\mathbb{U} \models n = m \in a \rightarrow \mathbb{V} \models \{f\}(n) = \{f\}(m)],\\
&\forall n \in \omega[\,\mathbb{U} \models n \in a \rightarrow \phi(\{g\}(n), \{f\}(n))],\\
&\text{then}\\
&\phi\big(d^*(a, f, g), \mathbf{sup}^*(a, f)\big),
\end{aligned}$$

where $d(x_1, x_2, x_3)$[10] is a term of $\mathbf{ML_1V}$ and $\phi(u, v)$ an arbitrary set-theoretic formula. We want to show $\phi\big((\mathbf{T_V}(y, (x_1, x_2, x_3)d(x_1, x_2, x_3)))^*, y\big)[c_0/y]$, that is to say

$$\phi\big(\mathbf{T_V^*}(\mathbf{c_0}, \lambda\mathbf{x_1}.\lambda\mathbf{x_2}.\lambda\mathbf{x_3}.\mathbf{d}^*(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}))^*, \mathbf{c_0}\big).$$

As $\mathbb{V}_\alpha \models c_0$ **set** for some α, we may proceed by induction on α. $\mathbb{V}_\alpha \models c_0$ **set** implies that there are $\beta < \alpha$ and a_0, f_0 such that

$$\begin{aligned}
&c_0 = \sup(a_0, f_0) = \mathbf{sup}^*(a_0, f_0), && (5)\\
&\mathbb{U}_\beta \models a_0 \ \ \mathbf{set},\\
&\forall n \in \omega[\,\mathbb{U} \models n \in a_0 \rightarrow \mathbb{V}_\beta \models \{f_0\}(n) \ \ \mathbf{set}],\\
&\forall n, m \in \omega[\,\mathbb{U} \models n = m \in a_0 \rightarrow \mathbb{V}_\beta \models \{f_0\}(n) = \{f_0\}(m)].
\end{aligned}$$

The inductive assumption then yields

$$\forall n \in \omega\big[\,\mathbb{U} \models n \in a_0 \rightarrow \phi(\mathbf{T_V^*}(\{\mathbf{f_0}\}(\mathbf{n}), \lambda\mathbf{x_1}.\lambda\mathbf{x_2}.\lambda\mathbf{x_3}.\mathbf{d}^*(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})), \{\mathbf{f_0}\}(\mathbf{n}))\big];$$

hence,

$$\forall n \in \omega\big[\,\mathbb{U} \models n \in a_0 \rightarrow \phi((\boldsymbol{\lambda}^* \lambda v.\mathbf{T_V^*}(\mathbf{v}, \lambda\mathbf{x_1}.\lambda\mathbf{x_2}.\lambda\mathbf{x_3}.\mathbf{d}^*(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})))(\mathbf{n}), \{\mathbf{f_0}\}(\mathbf{n}))\big].$$

Employing this and (5), the general assumption (4) yields

$$\phi(d^*(a_0, f_0, \boldsymbol{\lambda}^*(\lambda v.\mathbf{T_V^*}(\mathbf{v}, \lambda\mathbf{x_1}.\lambda\mathbf{x_2}.\lambda\mathbf{x_3}.\mathbf{d}^*(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}))), \mathbf{sup}^*(\mathbf{a_0}, \mathbf{f_0}))),$$

where g corresponds to $\boldsymbol{\lambda}^*(\lambda v.\mathbf{T_V^*}(\mathbf{v}, \lambda\mathbf{x_1}.\lambda\mathbf{x_2}.\lambda\mathbf{x_3}.\mathbf{d}^*(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})))$. In view of the translation of $\mathbf{T_V}$ under *, the latter is equivalent to

$$\phi(\mathbf{T_V^*}(\mathbf{sup}^*(\mathbf{a_0}, \mathbf{f_0}), \lambda\mathbf{x_1}.\lambda\mathbf{x_2}.\lambda\mathbf{x_3}.\mathbf{d}^*(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})), \mathbf{sup}^*(\mathbf{a_0}, \mathbf{f_0})),$$

and thus

$$\phi(\mathbf{T_V^*}(\mathbf{c_0}, \lambda\mathbf{x_1}.\lambda\mathbf{x_2}.\lambda\mathbf{x_3}.\mathbf{d}^*(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})), \mathbf{c_0})$$

[10] To be precise, $d^*(a, f, g)$ is short for $(d(x_1, x_2, x_3))^*[a/x_1, f/x_2, g/x_3]$.

since $c_0 = \mathbf{sup}^*(a_0, f_0)$. $\Box$

The preceding proof also yields the soundness of **V**–equality.

Type theories with a universe **U** lend themselves to a natural strengthening by the principle of **U**–induction. **U**–induction was employed by Aczel ([A 82],[A 86]) for the interpretation of certain presentation axioms in type theory.

Definition 4.12 *The rule of* **U**–*induction formalizes that the universe* **U** *is inductively defined. To put this rule on paper we resort to linear notation; the judgements to the right within brackets are discharged assumptions.*

$$\frac{\begin{array}{l} \text{Set } C(z)\ [z \in \mathbf{U}] \\ a_0 \in \mathbf{U} \\ d_{N_0} \in C(\mathbf{N_0}) \\ d_{N_1} \in C(\mathbf{N_1}) \\ d_N \in C(\mathbf{N}) \\ d_\Pi(A,F,c,e) \in C(\mathbf{\Pi}(A,F))[A \in \mathbf{U}, F \in A \to \mathbf{U}, c \in C(A), e \in \mathbf{\Pi}(A,F)] \\ d_\Sigma(A,F,c,e) \in C(\mathbf{\Sigma}(A,F))[A \in \mathbf{U}, F \in A \to \mathbf{U}, c \in C(A), e \in \mathbf{\Sigma}(A,F)] \\ d_+(A,B,c,d) \in C(A+B)[A \in \mathbf{U}, B \in \mathbf{U}, c \in C(A), d \in C(B)] \\ d_I(A,c,a,b) \in C(\mathbf{I}(A,a,b))[A \in \mathbf{U}, c \in C(A), a \in A, b \in A] \end{array}}{\mathbf{T_U}(a_0, d_{N_0}, d_{N_1}, d_N, (A,F,c,e)d_\Pi, (A,F,c,e)d_\Sigma, (A,B,c,d)d_+, (A,B,a,b)d_I) \in C(a_0)}$$

If $h_{\mathbf{U}}(a_0)$ denotes the term to the left of $C(a_0)$, the equality rules are given by (where $(\boldsymbol{\lambda}x)t := \boldsymbol{\lambda}((x)t)$)

$$\begin{aligned} h_{\mathbf{U}}(\mathbf{N_i}) &= d_{N_i} \\ h_{\mathbf{U}}(\mathbf{N}) &= d_N \\ h_{\mathbf{U}}(\mathbf{\Pi}(A,F)) &= d_\Pi(A,F,h_{\mathbf{U}}(A),(\boldsymbol{\lambda}x)h_{\mathbf{U}}(F(x))) \\ h_{\mathbf{U}}(\mathbf{\Sigma}(A,F)) &= d_\Sigma(A,F,h_{\mathbf{U}}(A),(\boldsymbol{\lambda}x)h_{\mathbf{U}}(F(x))) \\ h_{\mathbf{U}}(A+B) &= d_+(A,B,h_{\mathbf{U}}(A),h_{\mathbf{U}}(B)) \\ h_{\mathbf{U}}(\mathbf{I}(A,a,b)) &= d_I(A,h_{\mathbf{U}}(A),a,b). \end{aligned}$$

The language of type theories with **U**–elimination needs a little adjustment in that we would rather consider $\mathbf{N_2}, \mathbf{N_3}, \ldots$ as defined by $\mathbf{N_2} := \mathbf{N_1} + \mathbf{N_1}$, $\mathbf{N_3} := \mathbf{N_1} + \mathbf{N_2}$, etc. The assignment * of 4.7 and the interpretation $^\wedge$ of 4.10 can be extended so as to accommodate the constant $\mathbf{T_U}$. More precisely, there is a term $\mathbf{T^*_U}$ of **EON** such that, using the abbreviations $\lambda\vec{x} := \lambda x_1.\lambda x_2.\lambda x_3$ and $H(a) := \mathbf{T^*_U}(\mathbf{a}, \mathbf{d_0}, \mathbf{d_1}, \mathbf{d_2}, \lambda\vec{\mathbf{x}}.\mathbf{f}, \lambda\vec{\mathbf{x}}.\mathbf{g}, \lambda\vec{\mathbf{x}}.\mathbf{h}, \lambda\vec{\mathbf{x}}.\mathbf{l})$ we can prove in **EON** plus $\forall x\,(x \,\varepsilon\, \mathbf{N})$ that

$$H(a) \simeq \begin{cases} d_0 & \text{if } \ a = (4,1) \\ d_1 & \text{if } \ a = (4,2) \\ d_2 & \text{if } \ a = (4,0) \\ f(a_1,a_2,H(a_1),(\lambda x.x)(\lambda u.H(a_2(u)))) & \text{if } \ a = (0,(a_1,a_2)) \\ g(a_1,a_2,H(a_1),(\lambda x.x)(\lambda u.H(a_2(u)))) & \text{if } \ a = (1,(a_1,a_2)) \\ h(a_1,a_2,H(a_1),H(a_2)) & \text{if } \ a = (2,(a_1,a_2)) \\ l(a_1,H(a_1),a_2,a_3) & \text{if } \ a = (3,(a_1,a_2,a_3)) \\ \Omega & \text{if } \ \text{else.} \end{cases}$$

The definition of $\mathbf{T^*_U}$ uses the Recursion Theorem 1.6 and definition by cases, which is justified under the assumption $\forall x(x \,\varepsilon\, \mathbf{N})$ (cf. [Be 85],XI.2). Note that $(\forall x(x \,\varepsilon\, \mathbf{N}))^\diamond$ is derivable in **KP** and thus $\mathbf{KP} \vdash \psi^\diamond$ holds for any theorem ψ of **EON** + $\forall x(x \,\varepsilon\, \mathbf{N})$.

Theorem 4.13 $\mathbf{ML_1V}+\mathbf{U}$*–induction is interpretable in* **KP** *via* $^\wedge$.

Proof: The soundness of **U**–introduction under $^\wedge$ follows readily by induction on the stages of the relation $\mathbb{U} \models n$ **set**. $\Box$

Theorem 4.14 *The following theories have the same proof–theoretic strength:*

$$\mathbf{KP}, \mathbf{ID}, \mathbf{ML_1V}, \mathbf{ML_1V}^i, \mathbf{ML_1W} + \mathbf{U}\text{–}\textit{induction}, \mathbf{ML_1V} + \mathbf{U}\text{–}\textit{induction}, \mathbf{CZF},$$
$$\mathbf{CZF}\ \textit{minus Subset Collection}, \mathbf{CZF} + \mathbf{DC} + \Pi\Sigma I - \mathbf{AC} + \Pi\Sigma I - \mathbf{PA}.$$

Here $\mathbf{ML_1V}^i$ *denotes an intensional version of* $\mathbf{ML_1V}$ *as described in [TD 88], Ch.11, Sect. 8. The choice principles* **DC** *(Dependent choices),* $\Pi\Sigma I - \mathbf{AC}$ *(*$\Pi\Sigma I-$ *axiom of choice), and* $\Pi\Sigma I - \mathbf{PA}$ *(*$\Pi\Sigma I-$*presentation axiom) have been isolated in [A 82].*

Proof: We use $T \hookrightarrow T'$ to mean that T is interpretable in T'. $T \equiv T'$ is used to signify proof–theoretic equivalence. Let $\mathbf{ID}^i$ denote the intuitionistic theory of noninterated inductive definitions. It is an old result that $\mathbf{ID}^i \equiv \mathbf{ID}$ (cf. [BFPS 81]). Moreover, from [J 82] it follows that $\mathbf{ID} \equiv \mathbf{KP}$. Drawing on Lemma 3.1, which shows how to formalize inductive definitions in **CZF**, we can consider $\mathbf{ID}^i$ as a subtheory of **CZF**, thus $\mathbf{ID}^i \hookrightarrow \mathbf{CZF}$. On closer scrutiny we observe that Subset Collection was not used in the proof of Lemma 3.1. Whence $\mathbf{ID}^i$ is already interpretable in **CZF** minus Subset Collection.

By [TD 88], Ch. 11, Sect. 8 we get $\mathbf{CZF} \hookrightarrow \mathbf{ML_1V}^i$. Furthermore, $\mathbf{ML_1V}^i$ is a subtheory of $\mathbf{ML_1V}$, and [P 93] provides an interpretation of $\mathbf{ML_1V}$ in $\mathbf{ML}_1 + \mathbf{U}$–induction. From [A 82] we obtain an interpretation of $\mathbf{CZF}+\mathbf{DC}+\Pi\Sigma I - \mathbf{AC}+\Pi\Sigma I - \mathbf{PA}$ in $\mathbf{ML_1V}+\mathbf{U}$–induction. Finally, as $\mathbf{ML_1V}+\mathbf{U}$–induction is interpretable in **KP**, by Theorem 4.13, the proof–theoretic equivalence of all the theories follows. $\Box$

5 Interpretation of $\mathbf{ML_{1W}V}$ in KPi

Before we determine an upper bound for the strength of $\mathbf{ML_1W}$, we shall investigate a weaker theory $\mathbf{ML_{1W}V}$ that allows **W**–formation only for small types. We take interest in this theory since it has the same strength (as we will show) as some well–known classical theories like $\boldsymbol{\Delta^1_2}$– $\mathbf{CA} + \mathbf{BI}$ and permits us to determine the exact strength of Aczel's constructive set theory $\mathbf{CZF} + \mathbf{REA}$.

Definition 5.1 $\mathbf{ML_{1W}V}$ *results from* $\mathbf{ML_1W}$ *by omitting the rule of* **W***–formation and adding the rules pertaining to* **V***, i.e.* **V***–formation,* **V***–introduction,* **V***–elimination, and* **V***–equality.*

Remark 5.2 *First, notice that* $\mathbf{ML_{1W}V}$ *still allows one to derive restricted* **W***–formation, i.e.*

$$\textit{(res–}\mathbf{W}\textit{–formation)}\quad \frac{\begin{array}{cc} & [x \in A] \\ A \in \mathbf{U} & F(x) \in \mathbf{U}\end{array}}{\mathbf{W}(A,F)\ \textit{set}},$$

as **U***–introduction provides*

$$\frac{\begin{array}{cc} & [x \in A] \\ A \in \mathbf{U} & F(x) \in \mathbf{U}\end{array}}{\mathbf{W}(A,F) \in \mathbf{U}}$$

and $\mathbf{U}$*–formation gives*

$$\frac{\mathbf{W}(A,F)\in\mathbf{U}}{\mathbf{W}(A,F)\ set.}$$

Secondly, observe that $\mathbf{ML_1{}_WV}$ *can be viewed as a subtheory of* $\mathbf{ML_1W}$ *by letting* $\mathbf{V} := (\mathbf{W}X\in\mathbf{U}).X$.

Close inspection of the proofs in [A 86] reveals the following results.

Proposition 5.3

1. $\mathbf{CZF}$+$\mathbf{DC}$+$\mathbf{REA}$+$\Pi\Sigma I-\mathbf{AC}$ *is interpretable (via Aczel's interpretation) in* $\mathbf{ML_1{}_WV}$.
2. *If* $\mathbf{U}$*–induction*[11] *is added to* $\mathbf{ML_1{}_WV}$*, then also the principle* $\Pi\Sigma I-\mathbf{PA}$ *is true under this interpretation.* $\Box$

Corollary 5.4 *The proof-theoretic strength of* $\mathbf{ML_1{}_WV}$ *is at least that of* $\Delta^1_2\!-\mathbf{CA}+\mathbf{BI}$.

Proof: This follows from Theorem 3.9 and Proposition 5.3 as $\mathbf{T_0}$ has the same strength as $\Delta^1_2\!-\mathbf{CA}+\mathbf{BI}$, employing [JP 82] and [J 83]. $\Box$

The remainder of this section is devoted to interpreting $\mathbf{ML_1{}_WV}$ in an extension of $\mathbf{KP}$.

Definition 5.5 *By* $\mathbf{KPi}$ *we denote the theory that entails* $\mathbf{KP}$ *and has an axiom* $INACC_r$ *signifying that for each set there is an admissible set that contains it.*

Remark 5.6 *According to [JP 82],* $\mathbf{KPi}$ *has the same proof–theoretic strength as* $\Delta^1_2\!-\mathbf{CA}+\mathbf{BI}$. *In actuality, both theories prove the same statements of second order arithmetic.*

Lemma 5.7 ($\mathbf{KPi}$) *(Δ_0 positive inductive definitions) Let* $\phi(u,x,a_1,\dots,a_n)$ *be a* Δ_0 *formula with free variables among those shown such that each occurrence of* x *in* ϕ *is positive. Set* $\Gamma_{\phi,\vec{a},b}(x) = \{u\in b:\phi(u,x,a_1,\dots,a_n)\}$. *Let* F *be the function defined by* Δ *recursion along the ordinals satisfying*

$$F(\alpha,\vec{a},b) = \Gamma_{\phi,\vec{a},b}(\bigcup\{F(\beta,\vec{a},b):\beta<\alpha\}).$$

Finally, put $I_{\phi,\vec{a},b} = \bigcup\{F(\alpha,\vec{a},b):\alpha\in\mathbf{ON}\}$. *Then* $I_{\phi,\vec{a},b}$ *is a set. We say that* $I_{\phi,\vec{a},b}$ *is the set inductively defined by* ϕ *in the parameters* $\vec{a},b$. $I_{\phi,\vec{a},b}$ *is closed under* $I_{\phi,\vec{a},b}$*, i.e.* $\Gamma_{\phi,\vec{a},b}(I_{\phi,\vec{a},b})\subseteq I_{\phi,\vec{a},b}$. *Moreover, the assignment of* $I_{\phi,\vec{a},b}$ *to each set* $\vec{a},b$ *is* Δ *definable.*

Proof: Select an admissible set $\mathbb{A}$ such that $\vec{a},b\in\mathbb{A}$. Let $I_{\mathbb{A}} = \bigcup\{F(\xi,\vec{a},b):\xi\in\mathbb{A}\}$. Note that $I_{\mathbb{A}}$ is Σ–definable over $\mathbb{A}$ as $\mathbb{A}$ is admissible. We show $F(\alpha,\vec{a},b)\subseteq I_{\mathbb{A}}$ by transfinite induction on α. Suppose $\bigcup\{F(\beta,\vec{a},b):\beta<\alpha\}\subseteq I_{\mathbb{A}}$. Now if $u\in F(\alpha,\vec{a},b)$, then $u\in b$ and $\phi(u,\bigcup\{F(\beta,\vec{a},b):\beta<\alpha\},\vec{a})$. Thus $\phi(u,I_{\mathbb{A}},\vec{a})$ as positivity entails monotonicity. This yields $\mathbb{A}\models\phi(u,\exists\beta[\cdot\in F(\beta)],\vec{a})$,

[11] Here $\mathbf{U}$—induction has to be adapted to $\mathbf{ML_1{}_WV}$ in the obvious way: add

$$d_{\mathbf{W}}(A,F,c,e)\in C(\mathbf{W}(A,F))[A\in\mathbf{U},F\in A\to\mathbf{U},c\in C(A),e\in\mathbf{W}(A,F)]$$

to the hypotheses of the rule in Definition 4.12 and augment the function $\mathbf{T_U}$ by the argument $(A,F,c,e)d_{\mathbf{W}}$. As to equality, we have to add $h_{\mathbf{U}}(\mathbf{W}(A,F)) = d_{\mathbf{W}}(A,h_{\mathbf{U}}(A),(\boldsymbol{\lambda}x)h_{\mathbf{U}}(F(x)))$.

where the latter formula arises from $\phi(u,x,\vec{a})$ by replacing each occurrence of $v\in x$ in ϕ by $\exists\beta[v\in F(\beta)]$. Since the occurrences of x are all positive, $\phi(u,\exists\beta[\cdot\in F(\beta)],\vec{a})$ becomes a Σ formula. Hence, applying Σ reflection in $\mathbb{A}$ (cf. [Ba 75],I.4.3), there is a δ in $\mathbb{A}$ such that

$$\mathbb{A}\models\phi(u,\exists\beta<\delta[\cdot\in F(\beta)],\vec{a}),$$

yielding $u\in F(\delta,\vec{a},b)\subseteq I_{\mathbb{A}}$. Thus we have shown $I_{\mathbb{A}}=I_{\phi,\vec{a},b}$, confirming that $I_{\phi,\vec{a},b}$ is a set. The above proof also shows that $\Gamma_{\phi,\vec{a},b}(I_{\mathbb{A}})\subseteq I_{\mathbb{A}}$, so $I_{\phi,\vec{a},b}$ is closed under $\Gamma_{\phi,\vec{a},b}$. Moreover we have

$$y=I_{\phi,\vec{a},b}\quad\text{iff}\quad\exists\mathbb{A}\,[\text{“}\mathbb{A}\text{ is admissible”}\wedge\vec{a},b\in\mathbb{A}\ \wedge\ y=\bigcup\{F(\xi,\vec{a},b):\xi\in\mathbb{A}\}],$$

showing that the class function $(\vec{a},b\mapsto I_{\phi,\vec{a},b})$ is Σ and therefore also Δ definable. $\Box$

Definition 5.8 (**KPi**) *Next we want to extend the inductive definition of $\mathbb{U}$ and $\mathbb{V}$ of Definition 4.1 so as to allow for an interpretation of res–**W**–formation. To distinguish the new relations, say $R_1^{W,\alpha},\dots,R_7^{W,\alpha}$, from those in 4.1, we shall write $\mathbb{U}_\alpha^W\models$ and $\mathbb{V}_\alpha^W\models$ in place of $\mathbb{U}_\alpha\models$ and $\mathbb{V}_\alpha\models$, respectively. Let $\mathbf{w}(\mathbf{m},\mathbf{n})=\langle\mathbf{8},\langle\mathbf{n},\mathbf{m}\rangle_{\mathbb{N}}\rangle_{\mathbb{N}}$. We now define the relations $\mathbb{U}_\alpha^W\models n=m\in k$, $\mathbb{U}_\alpha^W\models n=m$, $\mathbb{U}_\alpha^W\models$ “f is a family of types over k”, $\mathbb{V}_\alpha^W\models k$ set, and $\mathbb{V}_\alpha^W\models n=m$ by the old clauses of the corresponding relations in Definitions 4.1, together with the new clauses:*

*If $\mathbb{U}_\alpha^W\models a$ **set**, and $\mathbb{U}_\alpha^W\models$ “e is a family of types over a”, then*

1. $\mathbb{U}_{\alpha+1}^W\models\mathbf{w}(\mathbf{a},\mathbf{e})$ **set**,
2. $\mathbb{U}_{\alpha+1}^W\models x\in\mathbf{w}(\mathbf{a},\mathbf{e})$ *iff* xSx,
3. $\mathbb{U}_{\alpha+1}^W\models x=y\in\mathbf{w}(\mathbf{a},\mathbf{e})$ *iff* xSy,

where S is inductively defined (on $\mathbb{N}$) by the rule:

$$\mathbb{U}_\alpha^W\models x=y\in a\ \wedge\ \forall i,j\big[\mathbb{U}_\alpha^W\models i=j\in\{e\}(x)\Rightarrow\ \{u\}(i)S\{v\}(j)\big]\ \ \Rightarrow\ \ \sup(x,u)S\sup(y,v).\tag{6}$$

*Finally, define $\mathbb{U}^W\models k$ **set**, $\mathbb{V}^W\models k$ **set**, $\mathbb{U}^W\models k=n$, and $\mathbb{V}^W\models k=n$ analogously to Definition 4.3.*

*We are to justify that the preceding definition falls under the scope of Σ recursion (on the basis of **KPi**). However, this follows from Lemma 5.7 as S is inductively defined by a Δ_0 formula in the parameters $\mathbb{N},R_1^{W,\alpha},\dots,R_7^{W,\alpha}$.* $\Box$

The analogues of Lemma 4.2 and Proposition 4.4 hold for the new relations. We shall verify that the new relation $\mathbb{U}^W\models\mathbf{w}(\mathbf{a},\mathbf{e})$ **set** provides for an interpretation of the W–type of $\mathbf{ML_1WV}$.

Lemma 5.9 (**KPi**) *Suppose $\mathbb{U}^W\models$ “e is a family of types over a”.*

- *(i) If $\mathbb{U}^W\models n\in a$ and $\mathbb{U}^W\models b\in\big(\{e\}(n)\to\mathbf{w}(\mathbf{a},\mathbf{e})\big)$,* [12] *then $\mathbb{U}^W\models\sup(n,b)\in\mathbf{w}(\mathbf{a},\mathbf{e})$.*
- *(ii) (**w**–induction) If $\phi(u)$ is a set–theoretic formula such that, for all x,b, it holds that $\mathbb{U}^W\models x\in a$, $\mathbb{U}^W\models b\in\big(\{e\}(x)\to\mathbf{w}(\mathbf{a},\mathbf{e})\big)$ and $\forall m[\mathbb{U}^W\models m\in\{e\}(x)\Rightarrow\ \phi(m)]$ imply $\phi(\sup(x,b))$, then $\forall n[\mathbb{U}^W\models n\in\mathbf{w}(\mathbf{a},\mathbf{e})\Rightarrow\ \phi(\mathbf{n})]$.*

[12] $\{e\}(n)\ \to\ \mathbf{w}(\mathbf{a},\mathbf{e})$ denotes $\pi\big(\{e\}(n),\lambda x.\mathbf{w}(\mathbf{a},\mathbf{e})\big)$, where $\lambda x.\mathbf{w}(\mathbf{a},\mathbf{e})$ stands for $\{\bar{\mathbf{k}}\}(\mathbf{w}(\mathbf{a},\mathbf{e}))$. Note that $\{\{\bar{\mathbf{k}}\}(\mathbf{w}(\mathbf{a},\mathbf{e}))\}(\mathbf{x})\simeq\mathbf{w}(\mathbf{a},\mathbf{e})$.

Proof: (i): Choose α such that $\mathbb{U}^W_\alpha \models$ "e is a family of types over a", $\mathbb{U}^W_\alpha \models n \in a$ and $\mathbb{U}^W_{\alpha+1} \models b \in (\{e\}(n) \to \mathbf{w}(\mathbf{a},\mathbf{e}))$. Then $\mathbb{U}^W_{\alpha+1} \models \mathbf{w}(\mathbf{a},\mathbf{e})$ **set**and $\forall i,j\big[\,\mathbb{U}^W_\alpha \models i = j \in \{e\}(n) \Rightarrow\ \mathbb{U}^W_{\alpha+1} \models \{b\}(i) = \{b\}(j) \in \mathbf{w}(\mathbf{a},\mathbf{e})\big]$. Letting S be defined as in (6), the latter reads $\forall i,j[\,\mathbb{U}^W_\alpha \models i = j \in \{e\}(n) \Rightarrow\ \{b\}(i)S\{b\}(j)]$, thus, since also $\mathbb{U}^W_\alpha \models n = n \in a$, the inductive definition of S entails $\sup(n,b)S\sup(n,b)$, which means $\mathbb{U}^W_{\alpha+1} \models \sup(n,b) \in \mathbf{w}(\mathbf{a},\mathbf{e})$, and therefore $\mathbb{U}^W \models \sup(n,b) \in \mathbf{w}(\mathbf{a},\mathbf{e})$.

(ii): Pick α such that $\mathbb{U}^W_\alpha \models$ *"e is a family of types over a"* . Then $\mathbb{U}^W_{\alpha+1} \models \mathbf{w}(\mathbf{a},\mathbf{e})$ **set**. Since all elements of $\mathbf{w}(\mathbf{a},\mathbf{e})$ are added at the same time as $\mathbf{w}(\mathbf{a},\mathbf{e})$ is declared a set, it suffices to show

$$\forall n[\,\mathbb{U}^W_{\alpha+1} \models n \in \mathbf{w}(\mathbf{a},\mathbf{e}) \Rightarrow\ \phi(\mathbf{n})]. \tag{7}$$

Again, let S be the relation from 5.8. (7) then reads:

$$\forall n[nSn \to \phi(n)]. \tag{8}$$

We show (8) by induction on the way S was built up, which comes down to transfinite induction on the ordinals. Assume nSn. Then $n = \sup(x,u)$ for some $x,u \in \mathbb{N}$, $\mathbb{U}^W_\alpha \models x = x \in a$, and

$$\forall i,j[i = j \in \{e\}(x) \Rightarrow\ \{u\}(i)S\{u\}(j)]. \tag{9}$$

Thus, inductively we may assume

$$\forall i\big[\,\mathbb{U}^W_\alpha \models i \in \{e\}(x) \Rightarrow\ \phi(\{u\}(i))\big]. \tag{10}$$

Moreover, by (9), we also have

$$\mathbb{U}^W \models u \in (\{e\}(x) \to \mathbf{w}(\mathbf{a},\mathbf{e})). \tag{11}$$

Employing (10) and (11), and the assumption on ϕ, we get $\phi(\sup(x,u))$, thus $\phi(n)$. $\square$

Definition 5.10 *(Interpretation of* $\mathbf{ML_1WV}$ *in* **KPi***) By now it should be obvious how to extend* $\wedge$ *to an interpretation of* $\mathbf{ML_1WV}$ *in* **KPi**. *First, note that* $\mathbf{ML_1WV}$ *has the new constants* $\mathbf{W}$ *and* $\mathbf{T_W}$ *that need to be translated into terms of* **EON**. *Set* $\mathbf{W}^* := \lambda x.\lambda y.(8,(x,y))$ *and let* $\mathbf{T}^*_\mathbf{W}$ *be determined by the recursion theorem of* **EON** *as to satisfy*

$$\begin{aligned}&\mathbf{T}^*_\mathbf{W}(\mathbf{sup}^*(\mathbf{a},\mathbf{b}),\boldsymbol{\lambda}^*(\lambda\mathbf{x_1}.\lambda\mathbf{x_2}.\lambda\mathbf{x_3}.\mathbf{d}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}))) \simeq\\ &\qquad d(a,b,\boldsymbol{\lambda}^*(\lambda v.\mathbf{T}^*_\mathbf{W}(\mathbf{ap}^*(\mathbf{b},\mathbf{v}),\boldsymbol{\lambda}^*(\lambda\mathbf{x_1}.\lambda\mathbf{x_2}.\lambda\mathbf{x_3}.\mathbf{d}(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3}))))).\end{aligned}$$

To the definition of set terms in 4.9, the following clause has to be added:

> *If* $B(x)$ *and* A *are set terms, and* x *is not free in* A *or in* B*, then* $\mathbf{W}(A,B)$*, is a set term.*

The interpretation $\wedge$ *of 4.10 has to be complemented/changed as follows:*

$$\begin{array}{rcl}(x \in \mathbf{W}(A,B))^\wedge & \textit{is} & \exists u[(\mathbf{W}(A,B)^* \simeq u)^\wedge \wedge \mathbb{U}^W \models x \in u]\\ (x = y \in \mathbf{W}(A,B))^\wedge & \textit{is} & \exists u[(\mathbf{W}(A,B)^* \simeq u)^\wedge \wedge \mathbb{U}^W \models x = y \in u]\\ (u \in \mathbf{U})^\wedge & \textit{is} & \mathbb{U}^W \models u\ \mathbf{set}\\ (u = v \in \mathbf{U})^\wedge & \textit{is} & \mathbb{U}^W \models u = v\\ (u \in \mathbf{V})^\wedge & \textit{is} & \mathbb{V}^W \models u\ \mathbf{set}\\ (u = v \in \mathbf{V})^\wedge & \textit{is} & \mathbb{V}^W \models u = v\end{array}$$

Theorem 5.11 *If ϕ is a judgement of* $\mathbf{ML_{1W}V}$ *not of the form "A set," then* $\mathbf{KPi} \vdash (\Phi)^\wedge$.

Proof: The soundness of the rules pertaining to $\mathbf{W}$ follows from Lemma 5.9 and the choice of $\mathbf{T}^*_{\mathbf{W}}$, more precisely, $\mathbf{U}$–introduction and $\mathbf{U}$–elimination are taken care of by 5.9(i) and 5.9(ii), respectively. $\square$

In view of the inductive definition of $\mathbb{U}^W \models n$ **set**, it is clear that the interpretation $^\wedge$ also validates $\mathbf{U}$–induction in the form adapted to $\mathbf{ML_{1W}V}$.

Theorem 5.12 *If Φ is a judgement of* $\mathbf{ML_{1W}V}$+$\mathbf{U}$*–induction not of the form "A set", then* $\mathbf{KPi} \vdash (\Phi)^\wedge$.

Theorem 5.13 *The following theories* $\mathbf{ML_{1W}V}$, $\mathbf{ML_{1W}V}$+$\mathbf{U}$*–induction,* $\mathbf{T_0}$, Δ^1_2– $\mathbf{CA}+\mathbf{BI}$, $\mathbf{KPi}$ $\mathbf{CZF}+\mathbf{REA}$ $\mathbf{CZF}+\mathbf{REA}$ *minus Subset Collection and* $\mathbf{CZF}+\mathbf{REA}+\mathbf{DC}+\Pi\Sigma I-\mathbf{AC}+\Pi\Sigma I-\mathbf{PA}$ *are of the same proof–theoretic strength.*

Proof: This follows from 4.3, 5.4, 5.6, 5.11, 3.3 and 5.12. $\square$

6 The system $\mathbf{ML_{1W}}$

The system $\mathbf{ML_{1W}V}$ has an inductive type that does not belong to $\mathbf{U}$, i.e. the type $\mathbf{V}$. Here the question arises whether $\mathbf{V}$ really contributes to the proof–theoretic strength of $\mathbf{ML_{1W}V}$.

Definition 6.1 We denote by $\mathbf{ML_{1W}}$ the theory resulting from $\mathbf{ML_{1W}V}$ by depriving it of the type $\mathbf{V}$.

Since $\mathbf{ML_1V}$ is considerably stronger than $\mathbf{ML_1}$ by [A 77] and Theorem 4.14, one might gather that this also holds true of $\mathbf{ML_{1W}V}$ and $\mathbf{ML_{1W}}$. However, this is not the case.

In this paper we have so far avoided advanced techniques from ordinal–theoretic proof theory and rather combined "soft" proof–theoretic methods (like interpreting a theory in another theory) with results from the literature, however, some of them heavily rely on ordinal analyses of theories. Notwithstanding that it would be possible to carry out well–ordering proofs for ordinal notation systems in $\mathbf{ML_{1W}}$ that would show $\mathbf{ML_{1W}}$ to be of the strength of Δ^1_2– $\mathbf{CA}+\mathbf{BI}$, we shall continue in using interpretations for determining the strength of $\mathbf{ML_{1W}}$. Unfortunately, the type $\mathbf{V}$ is of central importance in Aczel's interpretation of $\mathbf{CZF}+\mathbf{REA}$ in $\mathbf{ML_{1W}V}$. Our approach will therefore be to design an intuitionistic theory of second order arithmetic that is sufficient for carrying out the well–ordering proofs in [J 83] and can be interpreted in $\mathbf{ML_{1W}}$.

Definition 6.2 *(The theory* **IARI***)* **IARI** *is a theory in the language of second order arithmetic with set variables. The constants and function symbols are 0 (zero), S (successor), + (plus), $\cdot$ (times), and $\langle\, , \rangle$, $(\,)_0$, $(\,)_1$ for an injective pairing function and its inverses. The only predicate symbol is $=$ for equality on the natural numbers. The logical rules of* **IARI** *are those of intuitionistic second order arithmetic. The arithmetic axioms are the usual defining axioms for $0, S, +, \cdot$ and $\forall n \forall m[(\langle n,m\rangle)_0 = n \wedge (\langle n,m\rangle)_1 = m]$. Equality for sets will be considered a defined notion, that is to say $X = Y \;:=\; \forall n[n \in X \leftrightarrow n \in Y]$.*

In addition to the usual axioms for intuitionistic second order logic, axioms are (the universal closures of):

1. **Induction***:*
$$\phi(0) \wedge \forall n[\phi(n) \to \phi(n+1)] \to \forall n\phi(n)$$
for all formulae ϕ.

2. **Arithmetic Comprehension Schema***:*
$$\exists X \forall n[n \in X \leftrightarrow \psi(x)]$$
for ψ arithmetical (parameters allowed).

3. **Replacement***:*
$$\forall X[\forall n \in X \exists\,!Y \phi(n,Y) \to \exists Z \forall n \in X\, \phi(n,(Z)_n)]$$
for all formulas ϕ. Here $\phi(n,(Z)_n)$ arises from $\phi(n,Z)$ by replacing each occurrence $t \in Z$ in the formula by $\langle n,t\rangle \in Z$.

4. **Inductive Generation***:*
$$\forall U \forall X \exists Y \big[\, WP_U(X,Y) \wedge (\forall n[\forall k(k <_X n \to \phi(k)) \to \phi(n)] \to \forall m \in Y\, \phi(m))\big],$$
for all formulas ϕ, where $k <_X n$ abbreviates $\langle k,n\rangle \in X$ and $WP_U(X,Y)$ stands for
$$Prog_U(X,Y) \wedge \forall Z[Prog_U(X,Z) \to Y \subseteq Z]$$
with $Prog_U(X,Y)$ being $\forall n \in U[\forall k(k <_X n \to k \in Y) \to n \in Y]$.

Remark 6.3 *(***IARI***) Note that $WP_U(X,Y)$ and $WP_U(X,Y')$ imply $Y = Y'$, i.e. $\forall n(n \in Y \leftrightarrow n \in Y')$. Therefore, if $WP_U(X,Y)$, then*
$$\forall n \in U[\forall k <_X n\, \phi(k) \to \phi(n)] \to \forall m \in Y\, \phi(m)$$
holds for all formulae ϕ.

The latter principle will be referred to as *"induction over the well–founded part of $<_X$"*. In the rest of this section we shall write $\mathrm{WF}(U,X)$ for the (extensionally) uniquely determined Y which satisfies $\mathrm{WP}_U(X,Y)$.

The main tool for performing the wellordering proof of [J 82] in **IARI** is the following principle of transfinite recursion.

Proposition 6.4 (**IARI**) *If $\mathrm{WP}_U(X,Y)$ and $\forall n \in Y \forall W \exists! V\, \psi(n,W,V)$, then there exists Z such that*
$$\forall n \in Y\, \psi(n, \bigcup\{(Z)_k : k <_X n\}, (Z)_n).$$

Proof: The proof proceeds by induction over Y, i.e. by induction over the well–founded part of $<_X$ with respect to U, and is similar to the proof of definition by recursion in **CZF** (cf. Proposition 2.1); here Replacement is used instead of Strong Collection, and Arithmetic Comprehension replaces Bounded Separation. □

Next we go about interpreting **IARI** in $\mathbf{ML_{1W}}$.

Definition 6.5 *(Interpretation of* **IARI** *in* $\mathbf{ML_{1W}}$*)*

Terms. Each function symbol for an n-place primitive recursive function is interpreted as a function $f^\diamond \in N \to^n N$ in type theory, where $N \to^0 X := N$ and $N \to^{n+1} X := N \to (N \to^n X)$. We define

$$\begin{aligned} s^\diamond &:= (\lambda x)\mathbf{s_N}(x), \\ (P^n_k)^\diamond &:= (\lambda x_1)\cdots(\lambda x_n)x_k, \\ (C^n_k)^\diamond &:= (\lambda x_1)\cdots(\lambda x_n)\mathbf{s}^k_{\mathbf{N}}(0), (\text{ where } \mathbf{s}^0_{\mathbf{N}} := \mathbf{0}; \mathbf{s}^{\mathbf{k+1}}_{\mathbf{N}} := \mathbf{ap}(\mathbf{s_N}, \mathbf{s}^{\mathbf{k}}_{\mathbf{N}})) \end{aligned}$$

$$((\mathrm{Sub}^n_m))^\diamond[g, h_1, \ldots, h_n] \ := \ (\lambda x_1)\cdots(\lambda x_n)\mathbf{ap}(g^\diamond, \mathbf{ap}(h^\diamond_1, x_1, \ldots, x_n), \ldots, \mathbf{ap}(h^\diamond_m, x_1, \ldots, x_n))$$

$$((\mathrm{Rec}_n[g, h]))^\diamond \ := \ (\lambda x_1)\cdots(\lambda x_{n+1})\mathbf{R}(x_{n+1}, \mathbf{ap}(g^\diamond, x_1, \ldots, x_n), (x, y)\mathbf{ap}(h^\diamond, x_1, \ldots, x_n, x, y)).$$

For terms we let $0^\diamond := \mathbf{0}$, $x^\diamond := x$ if x is a numerical variable, and if t has the form $f(t_1, \ldots, t_n)$ let

$$t^\diamond := f^\diamond(t^\diamond_1, \ldots, t^\diamond_n) := \mathbf{ap}(\ldots \mathbf{ap}(f^\diamond, t^\diamond_1), \ldots, t^\diamond_n).$$

It is obvious that we should interpret the numerical variables of **IARI** in type theory as ranging over $\mathbf{N}$. It might be less obvious how to model the set variables of **IARI**. This is done by defining the *weak power set* of $\mathbf{N}$, $\mathfrak{P}(\mathbf{N})$, as the set of of predicates on $\mathbf{N}$ with truth conditions in the universe $\mathbf{U}$, i.e. $\mathfrak{P}(\mathbf{N}) := \mathbf{N} \to \mathbf{U}$. For $n \in \mathbf{N}$ and $X \in \mathfrak{P}(\mathbf{N})$, membership of n in X is defined by

$$n \mathrel{\dot\in} X \ := \ \mathbf{ap}(X, n).$$

With this notion of power set, *comprehension* with respect to a property $\phi(n) \in \mathbf{U}$ $(n \in \mathbf{N})$ is immediate, by letting

$$\{n \in \mathbf{N} : \phi(\mathbf{n})\} \ := \ (\boldsymbol{\lambda}\mathbf{x})\phi(\mathbf{x}) \in \mathfrak{P}(\mathbf{N}). \tag{12}$$

The subset relation ($\dot\subset$) and extensional equality ($\doteq$) on $\mathfrak{P}(\mathbf{N})$ are defined in the obvious way as follows.

$$X \mathrel{\dot\subset} Y \ := \ \forall n \in \mathbf{N}(\mathbf{n} \mathrel{\dot\in} \mathbf{X} \to \mathbf{n} \mathrel{\dot\in} \mathbf{Y})$$

and

$$X \doteq Y \ := \ X \mathrel{\dot\subset} Y \wedge Y \mathrel{\dot\subset} X.$$

If $X_i \in \mathfrak{P}(\mathbf{N})$ $(\mathbf{i} \in \mathbf{I})$ is a family of sets with $I \in \mathbf{U}$, let

$$\dot{\bigcup}\{X_i : i \in I\} \ := \ \{n \in \mathbf{N} : \exists \mathbf{i} \in \mathbf{I}(\mathbf{n} \mathrel{\dot\in} \mathbf{X_i})\}.$$

We shall write $s =_{\mathbf{N}} t$ for $\mathbf{I}(\mathbf{N}, s, t)$. Variables n, k, m are supposed to range over $\mathbf{N}$, thus, e.g., $\forall k$ abbreviates $\forall k \in \mathbf{N}$. $\forall n \mathrel{\dot\in} X(\ldots)$ is shorthand for $\forall n \in \mathbf{N}(n \mathrel{\dot\in} X \to \ldots)$. Formulas of second order arithmetic are now translated as type expressions as follows:

$$\begin{aligned} [s = t]^\diamond &:= s^\diamond =_{\mathbf{N}} t^\diamond, \\ [s \in X]^\diamond &:= s^\diamond \mathrel{\dot\in} X, \\ [\phi \circ \psi]^\diamond &:= \phi^\diamond \circ \psi^\diamond \text{ for } \circ \in \{\wedge, \vee, \to\}, \\ \perp^\diamond &:= \mathbf{I}(\mathbf{N}, \mathbf{s_N}(0), 0), \\ [Qy\, \phi]^\diamond &:= (Qy \in \mathbf{N})\phi^\diamond \quad \text{for } Q \in \{\forall, \exists\}, \\ [QX\, \phi]^\diamond &:= (QX \in \mathfrak{P}(\mathbf{N}))\phi^\diamond \quad \text{for } \mathbf{Q} \in \{\forall, \exists\}. \end{aligned}$$

Definition 6.6 *If F is a family of propositions (i.e. types) over the type A we call F a* species *over A.*

In the following deductions in type theory are presented in an informal style. The following lemma deals with the interpretation of Replacement.

Lemma 6.7 *Suppose $A \in \mathfrak{P}(\mathbf{N})$. Let F be a species over $\mathbf{N} \times \mathfrak{P}(\mathbf{N})$ such that*

$$\forall n \dot{\in} A\, \exists X \in \mathfrak{P}(\mathbf{N})\, \mathbf{F}(\mathbf{n}, \mathbf{X}) \text{ true}, \tag{13}$$

$$\forall n \dot{\in} A\, \forall X, Y \in \mathfrak{P}(\mathbf{N})\, [\mathbf{X} \doteq \mathbf{Y} \wedge \mathbf{F}(\mathbf{n}, \mathbf{X}) \rightarrow \mathbf{F}(\mathbf{n}, \mathbf{Y})] \text{ true}, \tag{14}$$

$$\forall n \dot{\in} A\, \forall X, Y \in \mathfrak{P}(\mathbf{N})\, [\mathbf{F}(\mathbf{n}, \mathbf{X}) \wedge \mathbf{F}(\mathbf{n}, \mathbf{Y}) \rightarrow \mathbf{X} \doteq \mathbf{Y}] \text{ true}. \tag{15}$$

Then there exists $C \in \mathfrak{P}(\mathbf{N})$ such that, with $g(n) := \{m \in \mathbf{N} : \exists k \dot{\in} C(\langle n, m\rangle^{\diamond} =_{\mathbf{N}} k)\}$,

$$\forall n \dot{\in} A\, F(n, g(n)) \text{ true}.$$

Proof: From (13) it follows $\forall n[\exists v \in A(n) \rightarrow \exists X \in \mathfrak{P}(\mathbf{N})\, \mathbf{F}(\mathbf{n}, \mathbf{X})]$ true, thus $\forall n \forall v \in A(n)\, \exists X \in \mathfrak{P}(\mathbf{N})\, \mathbf{F}(\mathbf{n}, \mathbf{X})$ true, and hence[13]

$$\forall z \in (\boldsymbol{\Sigma} v \in \mathbf{N}\, A(v))\, \exists X \in \mathfrak{P}(\mathbf{N})\, \mathbf{F}(\mathbf{p_0}(\mathbf{z}), \mathbf{X}) \text{ true}.$$

As a result, the axiom of choice in type theory provides us with a function $h \in (\boldsymbol{\Sigma} v \in \mathbf{N}\, A(v)) \rightarrow \mathfrak{P}(\mathbf{N})$ so that

$$\forall z \in (\boldsymbol{\Sigma} v \in \mathbf{N}\, A(v))\, F(\mathbf{p_0}(z), h(z)) \text{ true}.$$

Now define $g \in \mathbf{N} \rightarrow \mathfrak{P}(\mathbf{N})$ by

$$g(n) \;:=\; \dot{\bigcup}\{h(z) : z \in (\boldsymbol{\Sigma} v \in \mathbf{N}\, A(v)) \wedge \mathbf{p_0}(z) =_{\mathbf{N}} n\}.$$

As a consequence of (15) we have

$$\forall z, z' \in (\boldsymbol{\Sigma} v \in \mathbf{N}\, A(v))\, [\mathbf{p_0}(z) =_{\mathbf{N}} \mathbf{p_0}(z') \rightarrow h(z) \doteq h(z')] \text{ true}. \tag{16}$$

Now if $n \dot{\in} A$ true we may pick $z \in (\boldsymbol{\Sigma} v \in \mathbf{N}\, A(v))$ such that $\mathbf{p_0}(z) =_{\mathbf{N}} n$ true. Then $g(n) \doteq h(z)$ true by (16) and, as $F(n, h(z))$ true, we obtain $F(n, g(n))$ true using (14). Finally, set

$$C \;:=\; \{z \in \mathbf{N} : \exists v \in \mathbf{N}\, \exists u \dot{\in} g(v)(\langle v, u\rangle^{\diamond} =_{\mathbf{N}} z)\} \qquad \Box$$

The next lemma addresses Inductive Generation.

Lemma 6.8 *Let $U, X \in \mathfrak{P}(\mathbf{N})$ and define $n \dot{<}_X m := \langle n, m\rangle^{\diamond} \dot{\in} X$. Then there exists $Y \in \mathfrak{P}(\mathbf{N})$ such that*

$$\forall n \dot{\in} U[\forall k(k \dot{<}_X n \rightarrow k \dot{\in} Y) \rightarrow n \dot{\in} Y] \text{ true} \tag{17}$$

and for each species G on $\mathbf{N}$,

$$\forall n \dot{\in} U[\forall k(k \dot{<}_X n \rightarrow G(k)) \rightarrow G(n)] \rightarrow \forall n \dot{\in} Y\, G(n) \text{ true}. \tag{18}$$

[13] $\mathbf{p_0}, \mathbf{p_1}$ are defined by $\mathbf{p_0}(c) := \mathbf{E}(c, (x, y).x)$ and $\mathbf{p_1}(c) := \mathbf{E}(c, (x, y).y)$, where $\mathbf{E}$ is the eliminatory constant related to $\boldsymbol{\Sigma}$.

Proof: Let $A = (\mathbf{\Sigma}n \in \mathbf{N})U(n)$. Let B be the species on A defined by $B(v) := (\mathbf{\Sigma}k \in \mathbf{N})(k \dot{<}_X \mathbf{p_0}(v))$. Then $B \in A \to \mathfrak{P}(\mathbf{N})$. Set $W := \mathbf{W}(A, B)$. Define $\Gamma : \mathfrak{P}(\mathbf{N}) \to \mathfrak{P}(\mathbf{N})$ by

$$\Gamma(Z) := \{n \in \mathbf{N} : n \,\dot{\in}\, U \wedge \forall k(k \dot{<}_X n \to k \,\dot{\in}\, Z)\}.$$

Define $F : W \to \mathfrak{P}(\mathbf{N})$ by transfinite recursion on W so that

$$F(\mathbf{sup}(a, f)) := \Gamma(\dot{\bigcup}\{F(f(s)) : s \in B(a)\})$$

for $a \in A$, $f \in B(a) \to W$. Finally set

$$Y := \dot{\bigcup}\{F(s) : s \in W\}.$$

Then $Y \in \mathfrak{P}(\mathbf{N})$. To verify (17), assume $n \,\dot{\in}\, U$ true and $\forall k(k \dot{<}_X n \to n \,\dot{\in}\, Y)$ true. Pick $x_0 \in U(n)$ and let $v_0 := (n, x_0)$. Then $v_0 \in A$ and $\forall z \in B(v_0)\, \exists s \in W\, [\mathbf{p_0}(z) \,\dot{\in}\, F(s)]$ true. Employing the axiom of choice in type theory, there is an $f \in B(v_0) \to W$ such that

$$\forall z \in B(v_0)[\mathbf{p_0}(z) \,\dot{\in}\, F(f(z))] \text{ true.}$$

Therefore we get

$$\forall k(k \dot{<}_X n \to k \,\dot{\in} \dot{\bigcup}\{F(f(z)) : z \in B(v_0)\}) \text{ true.}$$

Thus $n \,\dot{\in}\, F(\mathbf{sup}(v_0, f))$ true. As $F(\mathbf{sup}(v_0, f)) \,\dot{\subset}\, Y$ true, this implies $n \,\dot{\in}\, Y$ true.

To verify (18) let G be a species over $\mathbf{N}$ such that

$$\forall n \,\dot{\in}\, U[\forall k(k \dot{<}_X n \to G(k)) \to G(n)] \text{ true.} \tag{19}$$

We use transfinite induction over W to verify $\forall s \in W[\forall k \,\dot{\in}\, F(s)\, G(k))]$ true which comprehends (18). So assume $a \in A$, $f \in B(a) \to W$ and $\forall x \in B(a) \forall k \,\dot{\in}\, F(f(x))\, G(k)$ true. We wish to show $\forall n \,\dot{\in}\, F(\mathbf{sup}(f, a))\, G(n)$ true. So assume $n \,\dot{\in}\, F(\mathbf{sup}(f, a))$ true. This means

$$n \,\dot{\in}\, \Gamma(\dot{\bigcup}\{F(f(s)) : s \in B(a)\}) \text{ true,}$$

thus $n \,\dot{\in}\, U$ true and $\forall k(k \dot{<}_X n \to \exists x \in B(a)\, [k \,\dot{\in}\, F(f(x))])$ true. As $\forall x \in B(a)\, \forall k \,\dot{\in}\, F(f(x))\, G(k)$ true, we obtain $\forall k[k \dot{<}_X n \to G(k)]$ true. Therefore $G(n)$ true using (19) and $n \,\dot{\in}\, U$ true. □

In actuality, the intensional version[14] $\mathbf{ML^i_{1W}}$ of $\mathbf{ML_{1W}}$ suffices for the interpretation of **IARI**.

Theorem 6.9 *Any theorem of* **IARI** *is true on interpretation in* $\mathbf{ML^i_{1W}}$*, that is to say, for each theorem* ϕ *of* **IARI**, $\phi^\diamond$ *is a proposition of* $\mathbf{ML^i_{1W}}$ *and* $\mathbf{ML^i_{1W}} \vdash t \in \phi^\diamond$*, for a suitable closed term* t*.*

Proof: By induction on the length of deductions in **IARI**. A natural deduction formulation of **IARI** is most convenient. That $^\diamond$ validates intuitionistic arithmetic is, for example, proved in [Be 85],XI.17 and [TD 88],Ch. 11, Sect. 4. It is also routine to check that $^\diamond$ validates the (intuitionistic) laws for the second order quantifiers. As to equality, note that

$$\forall u, v \in \mathbf{N}\, \forall X, Y \in \mathfrak{P}(\mathbf{N})\, [\mathbf{u} =_{\mathbf{N}} \mathbf{v} \to (\mathbf{u} \,\dot{\in}\, \mathbf{X} \leftrightarrow \mathbf{v} \,\dot{\in}\, \mathbf{Y})]$$

[14]For a definition of the intensional versions of type theories see [TD 88],Ch.11,Sect.5.

holds true in $\mathbf{ML^i_{1W}}$ by the very definitions of $=_{\mathbf{N}}$ and $\dot{\in}$. Therefore, if **IA** is obtained from **IARI** by retaining only the part without Arithmetic Comprehension, Replacement, and Inductive Generation, the theorem holds with **IA** in place of **IARI**. We thus only have to check those axioms. As to Arithmetic Comprehension, note that if $\phi(x, \vec{y}, \vec{Z})$ is an arithmetic formula then, for $n, \vec{m} \in \mathbf{N}$ and $\vec{X} \in \mathfrak{P}(\mathbf{N})$, $\phi(n, \vec{m}, \vec{X})^\diamond$ is a small type, that is to say $\phi(n, \vec{m}, \vec{X})^\diamond \in \mathbf{U}$; and therefore the validity of Arithmetic Comprehension is confirmed by (12). Replacement is taken care of by Lemma 6.7 and Inductive Generation follows from Lemma 6.8. □

To determine a lower bound for the strength of **IARI** we show that the well–ordering proof of [J 83] can be carried out in **IARI**, thereby confirming that **IARI** has the same proof–theoretic strength as $\mathbf{\Delta^1_2}$– $\mathbf{CA} + \mathbf{BI}$.

In what follows we assume familiarity with [J 83]. We shall only dwell on those parts of the latter paper that require an alternative development in **IARI**. Just by glancing through [J 83] it becomes clear that everything up to Lemma 2.14 is provable in **IARI**, where of course classifications are conceived as sets of natural numbers and the role of $\mathbf{T_0}$'s Inductive Generation is now taken over by the Inductive Generation of **IARI**, that is to say $i(X, R)$ must be replaced with $\mathbf{WF}(X, R)$.

The first place that needs to undergo a major change is [J 83] Lemma 2.15 since it draws on the operation w which is defined there by invoking the recursion theorem of $\mathbf{T_0}$. Instead, we shall use transfinite recursion over distinguished sets in the form of Proposition 6.4.

Definition 6.10 *(***IARI***) In keeping with [J 83], let* $(\mathbf{OT}, <_{\mathbf{OT}})$ *denote the notation system. We identify* $<_{\mathbf{OT}}$ *with the set* $\{\langle n, m\rangle : n <_{\mathbf{OT}} m\}$*. Let Q be a distinguished set. Then* $WF(Q, <_{\mathbf{OT}}) = Q$*. So we can define a set Z^Q by transfinite recursion over Q by*

$$(Z^Q)_a \;=\; \begin{cases} (Z^Q)_{Sa} & \text{if } a \notin CT_0 \\ WF(\{u \in \mathbf{OT} : Sa = 0\}, <_{\mathbf{OT}}) & \text{if } a = 0 \\ WF(\mathfrak{M}_a, <) & \text{if } a \in CT, \end{cases}$$

where for $u \in CT$,

$$\begin{aligned} \mathfrak{M}_u &= \{x \in \mathbf{OT} : Sx \leq_{\mathbf{OT}} u;\ \forall v \in W_u\, [K_v \subset W_u]\}, \\ W_u &= \begin{cases} (Z^Q)_{u^*} & \text{if } u \in L0 \\ \bigcup\{(Z^Q)_{u[x]} : x \in Q;\ x <_{\mathbf{OT}} t(u)\} & \text{if } u \notin L0. \end{cases} \end{aligned}$$

In order for the latter to be a legitimate transfinite recursion over Q we must guarantee that, for $u \in Q$, if $x \in Q$ and $x <_{\mathbf{OT}} t(u)$, then $u[x] \in Q$ and $u[x] <_{\mathbf{OT}} u$. But this follows from [J 83] Lemma 2.6.

Remark 6.11 *(***IARI***) For distinguished sets Q, P and $x \in Q \cap P$, it follows from [J 83], Lemma 2.13 that $(Q)_x = (P)_x$. Thus we may define an operation $\mathfrak{C}$ on the whole of*

$$\mathfrak{W} \;:=\; \bigcup\{Q : Q \text{ is distinguished}\}$$

by letting

$$\mathfrak{C}(x) \;=\; (Z^Q)_x \text{ for some } Q \text{ with } x \in Q.$$

Lemma 6.12 (**IARI**) *For all $a \in \mathfrak{W}$, $\mathfrak{C}(a)$ is a distinguished set.*

Proof: Let Q be a distinguished set such that $a \in Q$. Use transfinite induction on Q to verify that $\mathfrak{C}(a)$ is distinguished. Note that, for all $a \in Q$, $(Z^Q)_a = \mathfrak{C}(a)$. The proof of the lemma is then the same as for the statement "$a \in Q \to \exists X \mathcal{C}(X, a)$" in [J 83] Theorem 2.4. $\square$

If one now replaces the assertion "$\exists X \mathcal{C}(X, a)$" with "$\mathfrak{C}(a)$ *is distinguished*" and "$\mathcal{W}$" with "$\mathfrak{W}$" in the remainder of [J 83] (from Theorem 2.4 till the end), the same proofs will work. The upshot is that **IARI** has at least the proof–theoretic strength of Δ^1_2– $\mathbf{CA} + \mathbf{BI}$. Hence, in view of Theorem 5.13, we have ascertained the strength of $\mathbf{ML_{1W}}$.

Theorem 6.13 $\mathbf{ML_{1W}}$ *has the same proof–theoretic strength as the theories listed in Theorem 5.13, in particular the same strength as* Δ^1_2– $\mathbf{CA} + \mathbf{BI}$. *The same holds for* $\mathbf{ML^i_{1W}}$.

Remark 6.14 *There is also a subsystem of* $\mathbf{ML_{1W}}$ *that has exactly the strength of* $\boldsymbol{\Delta^1_2}$–$\mathbf{CA}$. *This theory arises from* $\mathbf{ML_{1W}}$ *by further demanding that* $\mathbf{W}$*–elimination has to be restricted to families in* $\mathbf{U}$.

7 The Strength of $\mathbf{ML_1W}$

For an interpretation of $\mathbf{ML_1W}$ in set theory one also has to deal with the $\mathbf{W}$–types "at large," i.e. those which are not ranging over $\mathbf{U}$.

Definition 7.1 *Let* $\mathbf{KP_1W}$ *be defined as follows. The language of* $\mathbf{KP_1W}$ *is the language of set theory augmented by infinitely many constants* $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \ldots$ *. The axioms of* $\mathbf{KP_1W}$ *consist of the axioms of* $\mathbf{KP}$ *and the formulae:*

1. *"$\mathbb{A}_i$ is a transitive non-empty set" ($i \in \mathbb{N}$)*
2. *$\mathbb{A}_i \in \mathbb{A}_j$ if $i < j$*
3. *$\phi^{\mathbb{A}_0}$ if ϕ is an axiom of* $\mathbf{KPi}$
4. *$\psi^{\mathbb{A}_{i+1}}$ if ψ is an axiom of* $\mathbf{KP}$ *($i \in \mathbb{N}$)*

Definition 7.2 *(Interpretation of* $\mathbf{ML_1W}$ *in* $\mathbf{KP_1W}$*) This interpretation (we call it* $^\wedge$ *again) is defined in the same way as in 5.10 except for the* $\mathbf{W}$*-type whose interpretation we take from 5.8.*

$$\begin{array}{rcl}(u \in \mathbf{W}(A, B))^\wedge & := & uSu, \\ (u = v \in \mathbf{W}(A, B))^\wedge & := & uSv,\end{array}$$

where S is inductively defined (on $\mathbb{N}$) by the rule:

> *(*) If $(x = y \in A)^\wedge$ and $\forall x', y' \in \mathbb{N}\big[(x' = y' \in B(x))^\wedge \to \{u\}(x')S\{v\}(y')\big]$, then* $\sup(x, u) S \sup(y, v)$.

Note that in order for S to be definable in $\mathbf{KP_1W}$*, $\{\langle x, y\rangle \in \mathbb{N} \times \mathbb{N} : (x = y \in A)^\wedge\}$ and $\{\langle x, y', x\rangle \in \mathbb{N}^3 : (x' = y' \in B(x))^\wedge\}$ have to be sets. Thus in order for this interpretation to make sense, we have to verify that each set term A of* $\mathbf{ML_1W}$ *gets interpreted by a set, i.e. $\{x : (x \in A)^\wedge\}$ is a set using only means available in* $\mathbf{KP_1W}$.

Lemma 7.3 *For C a set term we denote by $|C|$ the number of steps it takes to generate C (by the clauses of 4.9). If C is a set term of* $\mathbf{ML_1W}$ *with free variables among $\vec{z}\,(= z_1, \ldots, z_n)$, then* $\mathbf{KP_1W}$ *proves*

$$\{\langle u, \vec{z}\rangle \in \mathbb{N}^{n+1} : (u \in C)^\wedge\} \in \mathbb{A}_{|C|}$$

and

$$\{\langle u, v, \vec{z}\rangle \in \mathbb{N}^{n+2} : (u = v \in C)^\wedge\} \in \mathbb{A}_{|C|}.$$

Proof: We proceed by (meta) induction on the generation of C. The assertions are obvious when C is $\mathbf{N}$ or $\mathbf{N_k}$.

1. If C is $\mathbf{U}$, then $(u \in C)^\wedge$ is $\mathbb{U}^W \models u$ **set**. Now, $\{u : \mathbb{U}^W \models u \textbf{ set}\}$ is Σ definable in **KPi** and therefore this class has a definition wherein all quantifiers are restricted to $\mathbb{A}_0$. Thus $\{u : \mathbb{U}^W \models u \textbf{ set}\}$ is an element of $\mathbb{A}_1$ by Δ_0 separation in $\mathbb{A}_1$. Similarly, $\{\langle u, v\rangle \in \mathbb{N}^2 : \mathbb{U}^W \models u = v\} \in \mathbb{A}_1$.

2. If C is $\Pi(A, B)$, then $\{\langle u, \vec{z}\rangle \in \mathbb{N}^{n+1} : (u \in C)^\wedge\}$ is

$$\begin{aligned}\{\langle u, \vec{z}\rangle \in \mathbb{N}^{n+1} :\ & \forall x \in \mathbb{N}[(x \in A)^\wedge \to (ux \in B(x))^\wedge] \wedge \\ & \forall x, y \in \mathbb{N}[(x = y \in A)^\wedge \to (ux = uy \in B(x))^\wedge]\}.\end{aligned}$$

   By induction hypothesis, we obtain

$$\begin{aligned}\{\langle u, \vec{z}, x\rangle \in \mathbb{N}^{n+1} : (x \in A)^\wedge\} &\in \mathbb{A}_n, \\ \{\langle v, x, \vec{z}\rangle \in \mathbb{N}^{n+2} : (v \in B(x))^\wedge\} &\in \mathbb{A}_n, \\ \{\langle x, y, \vec{z}\rangle \in \mathbb{N}^{n+2} : (x = y =\in A)^\wedge\} &\in \mathbb{A}_n, \\ \{\langle v, w, x, \vec{z}\rangle \in \mathbb{N}^{n+3} : (v = w \in B(x))^\wedge\} &\in \mathbb{A}_n,\end{aligned}$$

   where $n = \max(|A|, |B(x)|)$.[15] As $\{\langle u, \vec{z}\rangle \in \mathbb{N}^{n+1} : (u \in C)^\wedge\}$ is Δ_0 definable, using the above sets as parameters, this is a set in $\mathbb{A}_n$ and thus also in $\mathbb{A}_{n+1}$, which is $\mathbb{A}_{|C|}$.

   The argument for $\{\langle u, v, \vec{z}\rangle \in \mathbb{N}^{n+2} : (u = v \in C)^\wedge\} \in \mathbb{A}_{|C|}$ proceeds along the same lines. The same arguments apply when C is a set term via one of the clauses (2)–(6) of 4.9

3. Suppose C is $\mathbf{W}(A, B)$. Inductively we get that

$$\{\langle x, y, \vec{z}\rangle \in \mathbb{N}^{n+2} : (x = y \in A)^\wedge\} \quad \text{and} \quad \{\langle x, y', \vec{z}\rangle \in \mathbb{N}^{n+2} : (x' = y' \in B)^\wedge\}$$

   are elements of $\mathbb{A}_n$ with $n = \max(|A|, |B(x)|)$. Now let $S_{\vec{z}}$ be the relation inductively defined in 7.2(*). The proof of Lemma 5.7 reveals that $\{\langle x', y', \vec{z}\rangle : x' S_{\vec{z}} y'\}$ is Σ_1 definable over $\mathbb{A}_n$, whence $\{\langle x', y', \vec{z}\rangle : x' S_{\vec{z}} y'\} \in \mathbb{A}_{n+1}$. Consequently,

$$\{\langle u, \vec{z}\rangle \in \mathbb{N}^{n+1} : (u \in \mathbf{W}(A, B))^\wedge\}$$

   and

$$\{\langle u, v, \vec{z}\rangle \in \mathbb{N}^{n+2} : (u = v \in \mathbf{W}(A, B))^\wedge\}$$

   are also elements of $\mathbb{A}_{n+1}$. □

[15] This case shows the need for the extra parameters $\vec{z}$ in the formulation of the lemma

Theorem 7.4 *If Φ is a judgement of* $\mathbf{ML_1W}$ *not of the form "A set", then* $\mathbf{KP_1W} \vdash \Phi^\wedge$.

Proof: By Lemma 7.3, $^\wedge$ really provides a syntactic translation of Φ into a formula of set theory. On the strength of its definition, it is clear that the rules for the **W**–types are sound under this interpretation. $\square$

By now we know that $\mathbf{KP_1W}$ is an upper bound for the proof–theoretic strength of $\mathbf{ML_1W}$. Actually, this bound is sharp according to Setzer's thesis [S 93]. As regards a proof, we would have to employ rather advanced techniques from Gentzen–style proof theory, including an ordinal analysis of $\mathbf{KP_1W}$. However, once these techniques are digested it is fairly easy to provide an ordinal analysis of $\mathbf{KP_1W}$ that yields the right bound. Indeed, the paper [R 91] permits one to almost read off the ordinal of $\mathbf{KP_1W}$ which is $\psi\Omega_1(I + \omega)$, where I is a notation for the first inaccessible ordinal. The next step would then consist in showing that, for all (meta) n, $\mathbf{ML_1W}$ proves the well–foundedness of the notation system up to α_n, where $\alpha_n = \psi\Omega_1\Omega_{I+n}$. This would show that $\mathbf{ML_1W}$ has the proof–theoretic ordinal $sup\{\alpha_n : n < \omega\} = \psi\Omega_1\Omega_{I+\omega}$.

8 Conclusions

Similar results can be established for the theories $\mathbf{ML}_n\mathbf{W}$. As a rule of thumb, the tower of n universes in $\mathbf{ML}_n\mathbf{W}$ corresponds to n recursively inaccessible universes in classical Kripke–Platek set theory.

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