The Disjunction and Related Properties for Constructive Zermelo-Fraenkel Set Theory

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Abstract

This paper proves that the *disjunction property*, the *numerical existence property*, *Church's rule*, and several other metamathematical properties hold true for Constructive Zermelo-Fraenkel Set Theory, **CZF**, and also for the theory **CZF** augmented by the Regular Extension Axiom.

As regards the proof technique, it features a self-validating semantics for **CZF** that combines realizability for extensional set theory and truth. The technique applies to full intuitionistic Zermelo-Fraenkel set theory, **IZF**, as well.

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1 Introduction

While Constructive Zermelo-Fraenkel Set Theory, **CZF**, has gained the status of a standard reference theory for developing constructive predicative mathematics (cf. [1, 2, 3, 4]) surprisingly little is known about certain pleasing metamathematical properties such as the disjunction and the numerical existence property that are often considered to be hallmarks of intuitionistic theories.

Definition 1.1 Let T be a theory whose language, $L(T)$, encompasses the language of set theory. Moreover, for simplicity, we shall assume that $L(T)$ has a constant ω denoting the set of von Neumann natural numbers and for each n a constant $\bar{n}$ denoting the n-th element of ω.

1. T has the *disjunction property*, **DP**, if whenever $T \vdash \psi \vee \theta$ holds for sentences ψ and θ of T, then $T \vdash \psi$ or $T \vdash \theta$.

2. T has the *numerical existence property*, **NEP**, if whenever $T \vdash (\exists x{\in}\omega)\phi(x)$ holds for a formula $\phi(x)$ with at most the free variable x, then $T \vdash \phi(\bar{n})$ for some n.

3. T has the *existence property*, **EP**, if whenever $T \vdash \exists x\phi(x)$ holds for a formula $\phi(x)$ having at most the free variable x, then there is a formula $\vartheta(x)$ with exactly x free, so that

$$T \vdash \exists! x\,[\vartheta(x) \wedge \phi(x)].$$

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4. T has the *weak existence property*, **wEP**, if whenever

$$T \vdash \exists x \phi(x)$$

holds for a formula $\phi(x)$ having at most the free variable x, then there is a formula $\vartheta(x)$ with exactly x free, so that

$$\begin{array}{lll} T & \vdash & \exists! x\, \vartheta(x), \\ T & \vdash & \forall x\, [\vartheta(x) \to \exists u\, u \in x], \\ T & \vdash & \forall x\, [\vartheta(x) \to \forall u \in x\, \phi(u)]. \end{array}$$

5. T is closed under *Church's rule*, **CR**, if whenever $T \vdash (\forall x \in \omega)(\exists y \in \omega)\phi(x, y)$ holds for some formula of T with at most the free variables shown, then, for some number e,

$$T \vdash (\forall x \in \omega)\phi(x, \{\bar{e}\}(x)),$$

where $\{e\}(x)$ stands for the result of applying the e-th partial recursive function to x.

6. T is closed under the *Extended Church's rule*, **ECR**, if whenever

$$T \vdash (\forall x \in \omega)[\neg\psi(x) \;\to\; (\exists y \in \omega)\phi(x, y)]$$

holds for formulae of T with at most the free variables shown, then, for some number e,

$$T \vdash (\forall x \in \omega)[\neg\psi(x) \;\to\; \{\bar{e}\}(x) \in \omega \;\wedge\; \phi(x, \{\bar{e}\}(x))].$$

Note that $\neg\psi(x)$ could be replaced by any formula which is provably equivalent in T to its double negation. This comprises arithmetic formulae that are both $\vee$-free and $\exists$-free.

7. Let $f : \omega \to \omega$ convey that f is a function from ω to ω. T is closed under the variant of *Church's rule*, $\mathbf{CR_1}$, if whenever $T \vdash \exists f\, [f : \omega \to \omega \;\wedge\; \psi(f)]$ (with $\psi(f)$ having no variables but f), then, for some number e, $T \vdash (\forall x \in \omega)(\exists y \in \omega)(\{\bar{e}\}(x) = y) \;\wedge\; \psi(\{\bar{e}\})$.

8. T is closed under the *Unzerlegbarkeits rule*, **UZR**, if whenever $T \vdash \forall x[\psi(x) \;\vee\; \neg\psi(x)]$, then

$$T \vdash \forall x\, \psi(x) \;\vee\; \forall x\, \neg\psi(x).$$

9. T is closed under the *Uniformity rule*, **UR**, if whenever $T \vdash \forall x\, (\exists y \in \omega)\psi(x, y)$, then

$$T \vdash (\exists y \in \omega)\, \forall x\, \psi(x, y).$$

Slightly abusing terminology, we shall also say that T enjoys any of these properties if this, strictly speaking, holds only for a definitional extension of T.

Actually, **DP** follows easily from **NEP**, and conversely, **DP** implies **NEP** for systems containing a modicum of arithmetic (see [12]).

Also note that **ECR** entails **CR**, taking $\psi(x)$ to be $x \neq x$.

Of course, classical (recursive and consistent) set theories do not have the **DP** nor the **NEP**. **ZF** and **ZFC** are known not to have the existence property either. But classical set theories can have the **EP** as, for instance, the theories $\mathbf{ZF} + V = L$ and $\mathbf{ZF} + V = OD$, where $V = OD$ expresses that all sets are ordinal definable.

Realizability semantics are of paramount importance in the study of intuitionistic theories. They were first proposed by Kleene [16] in 1945. It appears that the first realizability definition for set theory was given by Tharp [30] who used (indices of) Σ_1 definable partial (class) functions as realizers. This form of realizability is a straightforward extension of Kleene's 1945 realizability for numbers in that a realizer for a universally quantified statement $\forall x\phi(x)$ is an index e of a Σ_1 partial function such that $\{e\}(x)$ is a realizer for $\phi(x)$ for all sets x. In the same vein, e realizes $\exists x\phi(x)$ if e is a pair $\langle a, e'\rangle$ with e' being a realizer for $\phi(a)$. A markedly different strand of realizability originates with Kreisel's and Troelstra's [20] definition of realizability for second order Heyting arithmetic and the theory of species. Here, the clauses for the realizability relation $\Vdash$ relating to second order quantifiers are: $e \Vdash \forall X\phi(X) \Leftrightarrow \forall X\, e \Vdash \phi(X)$, $e \Vdash \exists X\phi(X) \Leftrightarrow \exists X\, e \Vdash \phi(X)$. This type of realizability does not seem to give any constructive interpretation to set quantifiers; realizing numbers "pass through" quantifiers. However, one could also say that thereby the collection of sets of natural numbers is generically conceived. On the intuitionistic view, the only way to arrive at the truth of a statement $\forall X\phi(X)$ is a proof. A collection of objects may be called generic if no member of it has an intensional aspect that can make any difference to a proof.

Kreisel-Troelstra realizability was applied to systems of higher order arithmetic and set theory by Friedman [11]. A realizability-notion akin to Kleene's slash [17, 18] was extended to various intuitionistic set theories by Myhill [25, 26]. [25] showed that intuitionistic **ZF** with Replacement instead of Collection (dubbed $\mathbf{IZF}_R$ henceforth) has the **DP**, **NEP**, and **EP**. [26] proved that the constructive set theory **CST** enjoys the **DP** and the **NEP**, and that the theory without the axioms of countable and dependent choice, $\mathbf{CST}^-$, also has the **EP**. It was left open in [26] whether the full existence property holds in the presence of relativized dependent choice, **RDC**. Friedman and Ščedrov [14] then established that $\mathbf{IZF}_R + \mathbf{RDC}$ satisfies the **EP** also. The Myhill-Friedman approach [25, 26] proceeds in two steps. The first, which appears to make the whole procedure non effective, consists in finding a conservative extension T' of the given theory T which contains names for all the objects asserted to exist in T. T' is obtained by inductively adding names and defining an increasing sequence of theories T_α through all the countable ordinals $\alpha < \omega_1$ and letting $T' = \bigcup_{\alpha<\omega_1} T_\alpha$.[1] The second step consists in defining a notion of realizablity for T' which is a variant of Kleene's "slash".

Several systems of set theory for the constructive mathematical practice were propounded by Friedman in [13]. The metamathematical properties of these theories and several others as well were subsequently investigated by Beeson [6, 7]. In particular, Beeson showed that **IZF** has the **DP** and **NEP**. He used a combination of Kreisel-Troelstra realizability and Kleene's [16, 17, 18, 19] q-realizability. However, while Myhill and Friedman developed realizablity directly for extensional set theories, Beeson engineered his realizability for non-extensional set theories and obtained results for the extensional set theories of [13] only via an interpretation in their non-extensional counterparts. This detour had the disadvantage that in many cases (where the theory does not have full Separation or Powerset) the **DP** and **NEP** for the corresponding extensional set theory T-*ext* could only be established for a restricted class of formulas; [6] Theorem 5.2 proves that **NEP** holds for T-*ext* when $T\text{-}ext \vdash (\exists x \in \omega)(x \in Q)$, where Q is a definable set of T. It appears unlikely that the Myhill-Friedman techniques or Beeson's detour through q-realizability for non-extensional set theories can be employed to yield the **DP** and **NEP** for **CZF**. The theories considered by Myhill and Friedman have Replacement instead of Collection and, in all probability, their approach is limited to such theories, whereas Beeson's techniques yield numerical explicit definability, not for all formulae $\varphi(u)$, but only for $\varphi(u)$ of the form $u \in Q$, where Q is a specific definable set. But there was another approach available. McCarty [22, 23] adapted Kreisel-Troelstra realizability directly to extensional

[1] This type of construction is due to J.R. Moschovakis [24] §8&9.

set theories. [22, 23], though, were concerned with realizability for intuitionistic Zermelo-Fraenkel set theory (having Collection instead of Replacement), **IZF**, and employed transfinite iterations of the powerset operation through all the ordinals in defining a realizability (class) structure. Moreover, in addition to the powerset axiom this approach also availed itself of unfettered separation axioms. At first blush, this seemed to render the approach unworkable for **CZF** as this theory lacks the powerset axiom and has only bounded separation. Notwithstanding that, it was shown in [28] that these obstacles can be overcome. Indeed, this notion of realizability provides a self-validating semantics for **CZF**, viz. it can be formalized in **CZF** and demonstrably in **CZF** it can be verified that every theorem of **CZF** is realized.

The current paper introduces a new realizability structure $\mathtt{V}^*$, which arises by amalgamating the realizability structure with the universe of sets in a coherent, albeit rather complicated way. The main semantical notion presented and utilized in this paper combines realizability for extensional set theory over $\mathtt{V}^*$ with truth in the background universe V. A combination of realizability with truth has previously been considered in the context of realizability notions for first and higher order arithmetic. It was called rnt-realizability in [31]. The main metamathematical results obtained via this tool are the following.

Theorem 1.2 *The* **DP** *and the* **NEP** *hold true for* **CZF** *and* **CZF + REA***. Both theories are closed under* **CR***,* **ECR***,* $\mathbf{CR_1}$*,* **UZR***, and* **UR***, too.*

One also obtains another proof of Beeson's result that **IZF** has the **DP** and the **NEP** and a proof that **IZF** is closed under **CR**, **ECR**, $\mathbf{CR_1}$, **UZR**, and **UR**. There are a number of further metamathematical results that can be obtained via this technology. They will be presented in section 9. For example, it will be shown that Markov's principle can be added to any of the foregoing theories.

The question of whether **CZF** has the existence property is currently unanswered. The proof of the failure of **EP** for **IZF** due to Friedman and Ščedrov [15] seems to single out Collection as the culprit. Inspection of that proof also reveals that **IZF** doesn't have the **wEP** either. However, that proof does not seem to carry over to **CZF** since the refutation of **EP** for **IZF** uses existential statements of the form

$$\exists b\,[\forall u{\in}a\,\exists y\,\varphi(u,y)\;\rightarrow\;\forall u{\in}a\,\exists y{\in}b\,\varphi(u,y)],$$

that are always deducible in **IZF** by employing Collection and full Separation, but, in general, are not deducible in **CZF**. We conjecture that **EP** fails for **CZF** on account of Subset Collection (and maybe Collection).

2 The system CZF

In this section we will summarize the language and axioms for **CZF**. The language of **CZF** is based on the same first order language as that of classical Zermelo-Fraenkel Set Theory, **ZF** whose only non-logical symbol is $\in$. The logic of **CZF** is intuitionistic first order logic with equality. Among its non-logical axioms are *Extensionality*, *Pairing* and *Union* in their usual forms. **CZF** has additionally axiom schemata which we will now proceed to summarize.

Infinity: $\exists x\forall u[u{\in}x \leftrightarrow (\emptyset = u \,\vee\, \exists v{\in}x\; u = v+1)]$ where $v+1 = v\cup\{v\}$.

Set Induction: $\forall x[\forall y \in x\phi(y) \rightarrow \phi(x)] \rightarrow \forall x\phi(x)$

Bounded Separation: $\forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \phi(x)]$

for all *bounded* formulae ϕ. A set-theoretic formula is *bounded* or *restricted* if it is constructed from prime formulae using $\neg, \wedge, \vee, \rightarrow, \forall x{\in}y$ and $\exists x{\in}y$ only.

Strong Collection: For all formulae ϕ,

$$\forall a[\forall x \in a \exists y \phi(x,y) \;\rightarrow\; \exists b\,[\forall x \in a\,\exists y \in b\,\phi(x,y) \wedge \forall y \in b\,\exists x \in a\,\phi(x,y)]].$$

Subset Collection: For all formulae ψ,

$$\forall a \forall b \exists c \forall u\,[\forall x \in a\,\exists y \in b\;\psi(x,y,u) \;\rightarrow$$
$$\exists d \in c\,[\forall x \in a\,\exists y \in d\,\psi(x,y,u) \wedge \forall y \in d\,\exists x \in a\,\psi(x,y,u)]].$$

The first large set axiom proposed in the context of constructive set theory was the *Regular Extension Axiom*, **REA**, which was introduced to accommodate inductive definitions in **CZF** (cf. [3]).

Definition 2.1 A is inhabited if $\exists x\, x \in A$. An inhabited set A is *regular* if A is transitive, and for every $a \in A$ and set $R \subseteq a \times A$ if $\forall x \in a\,\exists y\,(\langle x,y\rangle \in R)$, then there is a set $b \in A$ such that

$$\forall x \in a\,\exists y \in b\,(\langle x,y\rangle \in R) \;\wedge\; \forall y \in b\,\exists x \in a\,(\langle x,y\rangle \in R).$$

In particular, if $R:\, a \rightarrow A$ is a function, then the image of R is an element of A.

The *Regular Extension Axiom*, **REA**, is as follows: *Every set is a subset of a regular set.*

In what follows, we shall assume that the language of **CZF** has a constant ω denoting the set of von Neumann natural numbers and for each n a constant $\bar{n}$ denoting the n-th element of ω. Of course, one also has to add axioms pertaining to ω and the constants $\bar{n}$. For ω one can take the axiom $\forall u[u{\in}\omega \leftrightarrow (\emptyset = u \vee \exists v{\in}\omega\; u = v+1)]$ where $v+1 = v \cup \{v\}$.

This definitional (hence conservative) extension of the original system makes it easier to state the numerical existence property for **CZF**.

3 Some background on applicative structures

In order to define a realizability interpretation we must have a notion of realizing functions on hand. A particularly general and elegant approach to realizability builds on structures which have been variably called *partial combinatory algebras*, *applicative structures*, or *Schönfinkel algebras*. These structures are best described as the models of a theory **APP**. The following presents the main features of **APP**; for full details cf. [9, 10, 7, 32]. The language of **APP** is a first-order language with a ternary relation symbol App, a unary relation symbol N (for a copy of the natural numbers) and equality, $=$, as primitives. The language has an infinite collection of variables, denoted $x, y, z, \ldots$, and nine distinguished constants: $\mathbf{0}, \mathbf{s}_N, \mathbf{p}_N, \mathbf{k}, \mathbf{s}, \mathbf{d}, \mathbf{p}, \mathbf{p_0}, \mathbf{p_1}$ for, respectively, zero, successor on N, predecessor on N, the two basic combinators, definition by cases, pairing and the corresponding two projections. There is no arity associated with the various constants. The *terms* of **APP** are just the variables and constants. We write $t_1 t_2 \simeq t_3$ for $\mathrm{App}(t_1, t_2, t_3)$.

Formulae are then generated from atomic formulae using the propositional connectives and the quantifiers.

In order to facilitate the formulation of the axioms, the language of **APP** is expanded definitionally with the symbol $\simeq$ and the auxiliary notion of an *application term* is introduced. The set of application terms is given by two clauses:

1. all terms of **APP** are application terms; and
2. if s and t are application terms, then (st) is an application term.

For s and t application terms, we have auxiliary, defined formulae of the form:

$$s \simeq t \quad := \quad \forall y(s \simeq y \leftrightarrow t \simeq y),$$

if t is not a variable. Here $s \simeq a$ (for a a free variable) is inductively defined by:

$$s \simeq a \quad \text{is} \quad \begin{cases} s = a, & \text{if } s \text{ is a term of } \mathbf{APP}, \\ \exists x, y[s_1 \simeq x \,\wedge\, s_2 \simeq y \,\wedge\, \text{App}(x, y, a)] & \text{if } s \text{ is of the form } (s_1 s_2). \end{cases}$$

Some abbreviations are $t_1 t_2 \ldots t_n$ for $((\ldots(t_1 t_2)\ldots)t_n)$; $t\downarrow$ for $\exists y(t \simeq y)$ and $\phi(t)$ for $\exists y(t \simeq y \wedge \phi(y))$.

Some further conventions are useful. Systematic notation for *n-tuples* is introduced as follows: (t) is t, (s,t) is $\mathbf{p}st$, and $(t_1, \ldots, t_n)$ is defined by $((t_1, \ldots, t_{n-1}), t_n)$. In this paper, the **logic** of **APP** is assumed to be that of intuitionistic predicate logic with identity. **APP**'s **non-logical axioms** are the following:

Applicative Axioms

1. $\text{App}(a, b, c_1) \,\wedge\, \text{App}(a, b, c_2) \,\rightarrow\, c_1 = c_2$.
2. $(\mathbf{k}ab)\downarrow \,\wedge\, \mathbf{k}ab \simeq a$.
3. $(\mathbf{s}ab)\downarrow \,\wedge\, \mathbf{s}abc \simeq ac(bc)$.
4. $(\mathbf{p}a_0 a_1)\downarrow \,\wedge\, (\mathbf{p_0}a)\downarrow \,\wedge\, (\mathbf{p_1}a)\downarrow \,\wedge\, \mathbf{p_i}(\mathbf{p}a_0 a_1) \simeq a_i$ for $i = 0, 1$.
5. $N(c_1) \,\wedge\, N(c_2) \,\wedge\, c_1 = c_2 \rightarrow \mathbf{d}abc_1c_2\downarrow \,\wedge\, \mathbf{d}abc_1c_2 \simeq a$.
6. $N(c_1) \,\wedge\, N(c_2) \,\wedge\, c_1 \neq c_2 \rightarrow \mathbf{d}abc_1c_2\downarrow \,\wedge\, \mathbf{d}abc_1c_2 \simeq b$.
7. $\forall x\,(N(x) \rightarrow [\mathbf{s}_N x\downarrow \,\wedge\, \mathbf{s}_N x \neq \mathbf{0} \,\wedge\, N(\mathbf{s}_N x)])$.
8. $N(\mathbf{0}) \,\wedge\, \forall x\,(N(x) \,\wedge\, x \neq \mathbf{0} \rightarrow [\mathbf{p}_N x\downarrow \,\wedge\, \mathbf{s}_N(\mathbf{p}_N x) = x])$.
9. $\forall x\,[N(x) \rightarrow \mathbf{p}_N(\mathbf{s}_N x) = x]$
10. $\varphi(\mathbf{0}) \,\wedge\, \forall x[N(x) \,\wedge\, \varphi(x) \rightarrow \varphi(\mathbf{s}_N x)] \,\rightarrow\, \forall x[N(x) \rightarrow \varphi(x)]$.

Let $\mathbf{1} := \mathbf{s}_N\,\mathbf{0}$. The applicative axioms entail that $\mathbf{1}$ is an application term that evaluates to an object falling under N but distinct from $\mathbf{0}$, i.e., $\mathbf{1}\downarrow$, $N(\mathbf{1})$ and $\mathbf{0} \neq \mathbf{1}$.

Employing the axioms for the combinators $\mathbf{k}$ and $\mathbf{s}$ one can deduce an abstraction lemma yielding λ-terms of one argument. This can be generalized using n–tuples and projections.

Lemma 3.1 (cf. [9]) *(***Abstraction Lemma***) For each application term t there is a new application term t^* such that the parameters of t^* are among the parameters of t minus $x_1, \ldots, x_n$ and such that*

$$\mathbf{APP} \vdash t^*\downarrow \,\wedge\, t^* x_1 \ldots x_n \simeq t.$$

$\lambda(x_1, \ldots, x_n).t$ is written for t^*.

The most important consequence of the Abstraction Lemma is the Recursion Theorem. It can be derived in the same way as for the λ–calculus (cf. [9], [10], [7], VI.2.7). Actually, one can prove a uniform version of the following in **APP**.

Corollary 3.2 (Recursion Theorem)

$$\forall f \exists g \forall x_1 \ldots \forall x_n \, g(x_1, \ldots, x_n) \simeq f(g, x_1, \ldots, x_n).$$

The "standard" applicative structure is **Kl** in which the universe $|\mathbf{Kl}|$ is ω and $\mathrm{App}^{\mathbf{Kl}}(x, y, z)$ is Turing machine application:

$$\mathrm{App}^{\mathbf{Kl}}(x, y, z) \quad \text{iff} \quad \{x\}(y) \simeq z.$$

The primitive constants of **APP** are interpreted over $|\mathbf{Kl}|$ in the obvious way. For details see [22],chap.3,sec.2 or [7], VI.2.7. In the following we will be solely concerned with the standard applicative structure **Kl**. We will also be assuming that the notion of an applicative structure and in particular the structure **Kl** have been formalized in **CZF**, and that **CZF** proves that **Kl** is a model of **APP**.

4 The general realizability structure

If a is an ordered pair, i.e., $a = \langle x, y \rangle$ for some sets x, y, then we use $1^{st}(a)$ and $2^{nd}(a)$ to denote the first and second projection of a, respectively; that is, $1^{st}(a) = x$ and $2^{nd}(a) = y$. For a class X we denote by $\mathcal{P}(X)$ the class of all sets y such that $y \subseteq X$.

Definition 4.1 Ordinals are transitive sets whose elements are transitive also. As per usual, we use lower case Greek letters to range over ordinals.

$$\begin{aligned}
\mathtt{V}^*_\alpha &= \bigcup_{\beta \in \alpha} \{\langle a, b \rangle : \ a \in \mathtt{V}_\beta; \ b \subseteq \omega \times \mathtt{V}^*_\beta; \ (\forall x \in b) \, 1^{st}(2^{nd}(x)) \in a\} \qquad (1) \\
\mathtt{V}_\alpha &= \bigcup_{\beta \in \alpha} \mathcal{P}(\mathtt{V}_\beta) \\
\mathtt{V}^* &= \bigcup_{\alpha} \mathtt{V}^*_\alpha \\
\mathtt{V} &= \bigcup_{\alpha} \mathtt{V}_\alpha .
\end{aligned}$$

As the power set operation is not available in **CZF** it is not clear whether the classes $\mathtt{V}$ and $\mathtt{V}^*$ can be formalized in **CZF**. However, employing the fact that **CZF** accommodates inductively defined classes this can be demonstrated in the same vein as in [28], Lemma 3.4.

The definition of $\mathtt{V}^*_\alpha$ in (1) is perhaps a bit involved. Note first that all the elements of $\mathtt{V}^*$ are ordered pairs $\langle a, b \rangle$ such that $b \subseteq \omega \times \mathtt{V}^*$. For an ordered pair $\langle a, b \rangle$ to enter $\mathtt{V}^*_\alpha$ the first conditions to be met are that $a \in \mathtt{V}_\beta$ and $b \subseteq \omega \times \mathtt{V}^*_\beta$ for some $\beta \in \alpha$. Furthermore, it is required that a contains enough elements from the transitive closure of b in that whenever $\langle e, c \rangle \in b$ then $1^{st}(c) \in a$.

Lemma 4.2 (**CZF**).

(i) $\mathtt{V}$ and $\mathtt{V}^*$ *are cumulative: for* $\beta \in \alpha$, $\mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$ and $\mathtt{V}^*_\beta \subseteq \mathtt{V}^*_\alpha$.

(ii) For all sets a, $a \in \mathtt{V}$.

(iii) If a, b are sets, $b \subseteq \omega \times \mathtt{V}^*$ and $(\forall x \in b)\, 1^{st}(2^{nd}(x)) \in a$, then $\langle a, b\rangle \in \mathtt{V}^*$.

Proof: (i) is immediate by Definition 4.1. (ii) is proved by $\in$-induction on a. Suppose $a \subseteq \mathtt{V}$. Then $\forall x \in a\, \exists \alpha\, x \in \mathtt{V}_\alpha$, thus, using Strong Collection, there exists a set of ordinals D such that $\forall x \in a\, \exists \alpha \in D\, x \in \mathtt{V}_\alpha$. Now let $D' = \{\alpha + 1 :\ \alpha \in D\}$ and $\delta = \bigcup D'$ (where $\alpha + 1 := \alpha \cup \{\alpha\}$). Then δ is an ordinal as well, and $\forall \alpha \in D\, \alpha \in \delta$. Thus it follows that $a \subseteq \bigcup_{\alpha\in\delta} \mathtt{V}_\alpha$, so that by (i) we get that $a \subseteq \mathtt{V}_\delta$, and hence $a \in \mathtt{V}_{\delta+1}$.

For (iii), suppose a, b are sets such that $b \subseteq \omega \times \mathtt{V}^*$. By (ii) we find an ordinal γ such that $a \in \mathtt{V}_\gamma$. Moreover,

$$\forall x \in b\, \exists \alpha\, \exists e \in \omega\, \exists z \in \mathtt{V}^*_\alpha\; x = \langle e, z\rangle,$$

and therefore, proceeding as in (ii), we put to use Strong Collection to obtain an ordinal δ such that

$$\forall x \in b\, \exists \alpha \in \delta\, \exists e \in \omega\, \exists z \in \mathtt{V}^*_\alpha\; x = \langle e, z\rangle.$$

And hence, $b \subseteq \omega \times \bigcup_{\alpha\in\delta} \mathtt{V}^*_\alpha$, so that by (i) we have $b \subseteq \omega \times \mathtt{V}^*_\delta$. If we now pick an ordinal η such that $\gamma, \delta \in \eta$ then $\langle a, b\rangle \in \mathtt{V}_\eta \times \mathcal{P}(\omega \times \mathtt{V}^*_\eta)$ and therefore $\langle a, b\rangle \in \mathtt{V}^*_{\eta+1}$. □

5 Defining realizability

We now proceed to define a notion of realizability over $\mathtt{V}^*$. We use lower case gothic letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}, \mathfrak{n}, \mathfrak{m}, \mathfrak{p}, \mathfrak{q} \ldots$ as variables to range over elements of $\mathtt{V}^*$ while variables $e, c, d, f, g, \ldots$ will be reserved for elements of ω. Each element $\mathfrak{a}$ of $\mathtt{V}^*$ is an ordered pair $\langle x, y\rangle$, where $x \in \mathtt{V}$ and $y \subseteq \omega \times \mathtt{V}^*$; and we define the components of $\mathfrak{a}$ by

$$\begin{aligned} \mathfrak{a}^\circ &:= 1^{st}(\mathfrak{a}) = x \\ \mathfrak{a}^* &:= 2^{nd}(\mathfrak{a}) = y. \end{aligned}$$

Lemma 5.1 *For every $\mathfrak{a} \in \mathtt{V}^*$, if $\langle e, \mathfrak{c}\rangle \in \mathfrak{a}^*$ then $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$.*

Proof: This is immediate by the definition of $\mathtt{V}^*$. □

If φ is a sentence with parameters in $\mathtt{V}^*$, then φ° denotes the formula obtained from φ by replacing each parameter $\mathfrak{a}$ in φ with $\mathfrak{a}^\circ$.

Definition 5.2 Bounded quantifiers will be treated as quantifiers in their own right, i.e., bounded and unbounded quantifiers are treated as syntactically different kinds of quantifiers.

We define $e \Vdash_{rt} \phi$ for sentences ϕ with parameters in $\mathtt{V}^*$. (The subscript $_{rt}$ is supposed to serve as a reminder of "realizability with truth".)

$$\begin{aligned} e \Vdash_{rt} \mathfrak{a} \in \mathfrak{b} \quad &\text{iff} \quad \mathfrak{a}^\circ \in \mathfrak{b}^\circ \;\wedge\; \exists \mathfrak{c}\, [\langle (e)_0, \mathfrak{c}\rangle \in \mathfrak{b}^* \;\wedge\; (e)_1 \Vdash_{rt} \mathfrak{a} = \mathfrak{c}] \\ e \Vdash_{rt} \mathfrak{a} = \mathfrak{b} \quad &\text{iff} \quad \mathfrak{a}^\circ = \mathfrak{b}^\circ \;\wedge\; \forall f \forall \mathfrak{c}\, [\langle f, \mathfrak{c}\rangle \in \mathfrak{a}^* \;\rightarrow\; (e)_0 f \Vdash_{rt} \mathfrak{c} \in \mathfrak{b}] \\ &\qquad \wedge\; \forall f \forall \mathfrak{c}\, [\langle f, \mathfrak{c}\rangle \in \mathfrak{b}^* \;\rightarrow\; (e)_1 f \Vdash_{rt} \mathfrak{c} \in \mathfrak{a}] \\ e \Vdash_{rt} \phi \wedge \psi \quad &\text{iff} \quad (e)_0 \Vdash_{rt} \phi \;\wedge\; (e)_1 \Vdash_{rt} \psi \end{aligned}$$

$$
\begin{array}{lll}
e \Vdash_{rt} \phi \vee \psi & \text{iff} & [(e)_0 = 0 \wedge (e)_1 \Vdash_{rt} \phi] \vee [(e)_0 \neq 0 \wedge (e)_1 \Vdash_{rt} \psi] \\
e \Vdash_{rt} \neg\phi & \text{iff} & \neg\phi^\circ \wedge \forall f\, \neg f \Vdash_{rt} \phi \\
e \Vdash_{rt} \phi \to \psi & \text{iff} & (\phi^\circ \to \psi^\circ) \wedge \forall f\, [f \Vdash_{rt} \phi \to ef \Vdash_{rt} \psi] \\
e \Vdash_{rt} (\forall x \in \mathfrak{a})\, \phi & \text{iff} & (\forall x \in \mathfrak{a}^\circ)\phi^\circ \wedge \\
 & & \forall f\, \forall \mathfrak{b}(\langle f, \mathfrak{b}\rangle \in \mathfrak{a}^* \to ef \Vdash_{rt} \phi[x/\mathfrak{b}]) \\
e \Vdash_{rt} (\exists x \in \mathfrak{a})\phi & \text{iff} & \exists \mathfrak{b}\, (\langle (e)_0, \mathfrak{b}\rangle \in \mathfrak{a}^* \wedge (e)_1 \Vdash_{rt} \phi[x/\mathfrak{b}]) \\
e \Vdash_{rt} \forall x\phi & \text{iff} & \forall \mathfrak{a}\, e \Vdash_{rt} \phi[x/\mathfrak{a}] \\
e \Vdash_{rt} \exists x\phi & \text{iff} & \exists \mathfrak{a}\, e \Vdash_{rt} \phi[x/\mathfrak{a}]
\end{array}
$$

Notice that $e \Vdash_{rt} u \in v$ and $e \Vdash_{rt} u = v$ can be defined for arbitrary sets u, v, viz., not just for $u, v \in \mathtt{V}^*$. The definitions of $e \Vdash_{rt} u \in v$ and $e \Vdash_{rt} u = v$ fall under the scope of definitions by transfinite recursion. More precisely, the (class) functions

$$
\begin{aligned}
F_\in(u, v) &= \{e \in \omega : e \Vdash_{rt} u \in v\} \\
G_=(u, v) &= \{e \in \omega : e \Vdash_{rt} u = v\}
\end{aligned}
$$

can be defined (simultaneously) on $\mathrm{V} \times \mathrm{V}$ by recursion on the relation

$$
\langle c, d\rangle \lhd \langle a, b\rangle \quad \text{iff} \quad (c = a \wedge d \in \mathbf{TC}(b)) \vee (d = b \wedge c \in \mathbf{TC}(a)), \tag{2}
$$

where $\mathbf{TC}(x)$ stands for the transitive closure of a set x. This principle is a consequence of Set Induction and Strong Collection (or Replacement).

Definition 5.3 By $\in$-recursion we define for every set x a set x^{st} as follows:

$$
x^{st} = \langle x, \{\langle 0, u^{st}\rangle : u \in x\}\rangle. \tag{3}
$$

Lemma 5.4 *For all sets x, $x^{st} \in \mathtt{V}^*$ and $(x^{st})^\circ = x$.*

Proof: We prove this by $\in$-induction on x. So assume that for all $u \in x$, $u^{st} \in \mathtt{V}^*$ and $(u^{st})^\circ = u$. Then $\{\langle 0, u^{st}\rangle : u \in x\} \subseteq \omega \times \mathtt{V}^*$ and, for all $u \in x$, $(u^{st})^\circ \in x$. Thus, by Lemma 4.2(iii), $x^{st} \in \mathtt{V}^*$. Also, $(x^{st})^\circ = x$. □

Lemma 5.5 *If $\psi(\mathfrak{b}^\circ)$ holds for all $\mathfrak{b} \in \mathtt{V}^*$ then $\forall x\, \psi(x)$.*

Proof: Let x be an an arbitrary set. By the previous result we have $x^{st} \in \mathtt{V}^*$ and $(x^{st})^\circ = x$. Whence $\psi((x^{st})^\circ)$ and therefore $\psi(x)$ holds. □

Lemma 5.6 *If $\mathfrak{a} \in \mathtt{V}^*$ and $(\forall \mathfrak{b} \in \mathtt{V}^*)[\mathfrak{b}^\circ \in \mathfrak{a}^\circ \to \psi(\mathfrak{b}^\circ)]$ then $(\forall x \in \mathfrak{a}^\circ)\psi(x)$.*

Proof: Let $x \in \mathfrak{a}^\circ$ and put $\mathfrak{b}_x := \langle x, \emptyset\rangle$. Lemma 4.2(iii) ensures that $\mathfrak{b}_x \in \mathtt{V}^*$. Moreover, $x = \mathfrak{b}_x^\circ$ and $\mathfrak{b}_x^\circ \in \mathfrak{a}^\circ$; whence $\psi(x)$. □

Lemma 5.7 *If $e \Vdash_{rt} \phi$ then ϕ°.*

Proof: This follows by induction on the generation of ϕ, with the aid of Lemma 5.1 in the case of a bounded $\exists$-quantifier and Lemma 5.5 in the case of an unbounded $\forall$-quantifier. $\square$

Our hopes for showing **DP** and **NEP** for **CZF** and related systems rest on the following results.

Lemma 5.8 *If $e \Vdash_{rt} (\exists x \in \mathfrak{a})\phi$ then*

$$\exists \mathfrak{b}\,(\langle (e)_0, \mathfrak{b}\rangle \in \mathfrak{a}^* \ \wedge\ \phi^\circ[x/\mathfrak{b}^\circ]).$$

Proof: Obvious by 5.7. $\square$

Lemma 5.9 *If $e \Vdash_{rt} \phi \vee \psi$ then*

$$[(e)_0 = 0 \ \wedge\ \phi^\circ] \ \vee\ [(e)_0 \neq 0 \ \wedge\ \psi^\circ].$$

Proof: Obvious by 5.7. $\square$

Lemma 5.10 *Negated formulae are self-realizing, that is to say, if ψ is a statement with parameters in $\mathtt{V}^*$, then*

$$\neg\psi^\circ \ \rightarrow\ 0 \Vdash_{rt} \neg\psi.$$

Proof: Assume $\neg\psi^\circ$. From $f \Vdash_{rt} \psi$ we would get ψ° by Lemma 5.8. But this is absurd. Hence $\forall f\, \neg f \Vdash_{rt} \psi$, and therefore $0 \Vdash_{rt} \neg\psi$. $\square$

Definition 5.11 Let t be an application term and ψ be a formula of set theory. Then $t \Vdash_{rt} \psi$ is short for $(\exists e \in \omega)[t \simeq e \ \wedge\ e \Vdash_{rt} \psi]$.

5.1 The soundness theorem for intuitionistic predicate logic with equality

Lemma 5.12 *There are closed application terms $\mathbf{i_r}, \mathbf{i_s}, \mathbf{i_t}, \mathbf{i_0}, \mathbf{i_1}$ such that for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathtt{V}^*$,*

1. $\mathbf{i}_r \Vdash_{rt} \mathfrak{a} = \mathfrak{a}$.
2. $\mathbf{i}_s \Vdash_{rt} \mathfrak{a} = \mathfrak{b} \rightarrow \mathfrak{b} = \mathfrak{a}$.
3. $\mathbf{i}_t \Vdash_{rt} (\mathfrak{a} = \mathfrak{b} \wedge \mathfrak{b} = \mathfrak{c}) \ \rightarrow\ \mathfrak{a} = \mathfrak{c}$.
4. $\mathbf{i}_0 \Vdash_{rt} (\mathfrak{a} = \mathfrak{b} \wedge \mathfrak{b} \in \mathfrak{c}) \ \rightarrow\ \mathfrak{a} \in \mathfrak{c}$.
5. $\mathbf{i}_1 \Vdash_{rt} (\mathfrak{a} = \mathfrak{b} \wedge \mathfrak{c} \in \mathfrak{a}) \ \rightarrow\ \mathfrak{c} \in \mathfrak{b}$.
6. *Moreover, for each formula $\varphi(v, u_1, \ldots, u_r)$ of **CZF** all of whose free variables are among $v, u_1, \ldots, u_r$ there exists a closed application term $\mathbf{i}_\varphi$ such that for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}_1, \ldots, \mathfrak{c}_r \in \mathtt{V}^*$,*

$$\mathbf{i}_\varphi \Vdash_{rt} \varphi(\mathfrak{a}, \vec{\mathfrak{c}}) \wedge \mathfrak{a} = \mathfrak{b} \ \rightarrow\ \varphi(\mathfrak{b}, \vec{\mathfrak{c}}),$$

where $\vec{\mathfrak{c}} = \mathfrak{c}_1, \ldots, \mathfrak{c}_r$.

Proof: Realizers for the above formulas can be taken from [22], chapter 2, sections 5 and 6 or from [21] Theorem 14. One has to apply the recursion theorem to construct the desired realizers. (1) is proved by induction on the transitive closure of $\mathfrak{a}$. No induction is needed for (2). (3) and (4) are proved simultaneously by induction on a ternary version of the relation $\lhd$ in (2). (5) follows directly from (3) and (4). (6) is proved inductively on the build-up of φ, utilizing (3)-(5) in the atomic cases.

Note that the above-mentioned realizers also realize the corresponding universal closures owing to the "genericity" of realizers of universal statements, i.e.,

$$e \Vdash_{rt} \forall v \psi(v) \text{ iff } \forall \mathfrak{a}\, e \Vdash_{rt} \psi(\mathfrak{a}).$$

□

Theorem 5.13 *Let $\mathcal{D}$ be a proof in intuitionistic predicate logic with equality of a formula $\varphi(u_1, \ldots, u_r)$ of* **CZF** *all of whose free variables are among $u_1, \ldots, u_r$. Then there exists a closed application term $e_{\mathcal{D}}$ such that* **CZF** *proves*

$$e_{\mathcal{D}} \Vdash_{rt} \forall u_1 \ldots \forall u_r\, \varphi(u_1, \ldots, u_r).$$

Proof: Having addressed equality in Lemma 5.12, we only need to concern ourselves with pure logic. With the exception of the logical principles

$$\forall u \in a\, \varphi(u) \;\leftrightarrow\; \forall u\, [u \in a \rightarrow \varphi(u)], \tag{4}$$

$$\exists u \in a\, \varphi(u) \;\leftrightarrow\; \exists u\, [u \in a \wedge \varphi(u)], \tag{5}$$

which relate bounded to unbounded quantifiers, the proof is similar to the one for second order Heyting arithmetic (cf. [31]). Details can be found in [22], chapter 2, sections 5 and 6, and [21], Theorem 12. Let $\mathfrak{a} \in \mathtt{V}^*$ and φ be a formula with parameters in $\mathtt{V}^*$. We find a realizer for the formula of (4) as follows:

$$\begin{array}{rl}
 & e \Vdash_{rt} \forall u\, [u \in \mathfrak{a} \rightarrow \varphi(u)] \\
\Leftrightarrow & \forall \mathfrak{b} \in \mathtt{V}^*\; e \Vdash_{rt} [\mathfrak{b} \in \mathfrak{a} \rightarrow \varphi(\mathfrak{b})] \\
\Leftrightarrow & \forall \mathfrak{b} \in \mathtt{V}^*\; ([\mathfrak{b}^\circ \in \mathfrak{a}^\circ \rightarrow \varphi^\circ(\mathfrak{b}^\circ)] \;\wedge\; \forall f \in \omega[(f \Vdash_{rt} \mathfrak{b} \in \mathfrak{a}) \rightarrow ef \Vdash_{rt} \varphi(\mathfrak{b})]) \\
\overset{L.5.6}{\Leftrightarrow} & (\forall u \in \mathfrak{a}^\circ)\varphi^\circ(u) \;\wedge \\
 & \forall \mathfrak{b} \in \mathtt{V}^*\, \forall f \in \omega\, ([\exists \mathfrak{c}\, (\langle (f)_0, \mathfrak{c}\rangle \in \mathfrak{a}^* \;\wedge\; (f)_1 \Vdash_{rt} \mathfrak{b} = \mathfrak{c}] \;\rightarrow\; ef \Vdash_{rt} \varphi(\mathfrak{b})) \\
\Rightarrow & (\forall u \in \mathfrak{a}^\circ)\varphi^\circ(u) \;\wedge \\
 & \forall \mathfrak{c} \in \mathtt{V}^*\, \forall f \in \omega\, [(\langle (f)_0, \mathfrak{c}\rangle \in \mathfrak{a}^* \;\wedge\; (f)_1 \Vdash_{rt} \mathfrak{c} = \mathfrak{c}) \;\rightarrow\; ef \Vdash_{rt} \varphi(\mathfrak{c})] \\
\Rightarrow & (\forall u \in \mathfrak{a}^\circ)\varphi^\circ(u) \;\wedge\; \forall \langle g, \mathfrak{c}\rangle \in \mathfrak{a}^*\; e\, (\mathbf{p} g \mathbf{i}_r) \Vdash_{rt} \varphi(\mathfrak{c}) \\
\Rightarrow & \lambda g.e\, (\mathbf{p} g \mathbf{i}_r) \Vdash_{rt} \forall u \in \mathfrak{a}\, \varphi(u).
\end{array}$$

Conversely, we have:

$$\begin{array}{rl}
 & e \Vdash_{rt} \forall u \in \mathfrak{a}\, \varphi(u) \\
\Leftrightarrow & (\forall u \in \mathfrak{a}^\circ)\varphi^\circ(u) \;\wedge\; \forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}\;\; ef \Vdash_{rt} \varphi(\mathfrak{c}) \\
\Rightarrow & \forall \mathfrak{b} \in \mathtt{V}^*\, ([\mathfrak{b}^\circ \in \mathfrak{a}^\circ \rightarrow \varphi^\circ(\mathfrak{b}^\circ)] \;\wedge \\
 & \forall g \in \omega\, [\exists \mathfrak{c}\, (\langle (g)_0, \mathfrak{c}\rangle \in \mathfrak{a}^* \;\wedge\; (g)_1 \Vdash_{rt} \mathfrak{b} = \mathfrak{c}) \;\rightarrow\; \mathbf{i}_\varphi(\mathbf{p}(g)_1(e\,(g)_0)) \Vdash_{rt} \varphi(\mathfrak{b})]) \\
\Rightarrow & \lambda g.\mathbf{i}_\varphi(\mathbf{p}(g)_1(e\,(g)_0)) \Vdash_{rt} \forall u[u \in \mathfrak{a} \rightarrow \varphi(u)].
\end{array}$$

The constants $\mathbf{i}_r, \mathbf{i}_\varphi$ are from Lemma 5.12. Letting $\mathbf{m}$ be

$$\mathbf{p}(\lambda e.\lambda g.e\,(\mathbf{p}g\mathbf{i}_r))(\lambda e.\lambda g.\mathbf{i}_\varphi(\mathbf{p}(g)_1(e\,(g)_0))),$$

we get

$$\mathbf{m} \Vdash_{rt} \forall\vec{w}\forall v(\forall u \in v\,\varphi(u) \;\leftrightarrow\; \forall u\,[u \in v \to \varphi(u)]),$$

where $\forall\vec{w}$ quantifies over the remaining free variables of φ.

Similarly one finds $\bar{\mathbf{m}}$ such that

$$\bar{\mathbf{m}} \Vdash_{rt} \forall\vec{w}\forall v(\exists u \in v\,\varphi(u) \;\leftrightarrow\; \exists u\,[u \in v \;\wedge\; \varphi(u)]).$$

□

5.2 Realizability for bounded formulae

In the following we shall often have occasion to employ the fact that for a bounded formula $\varphi(v)$ with parameters from $\mathtt{V}^*$ and $x \subseteq \mathtt{V}^*$,

$$\{\langle e, \mathfrak{c}\rangle :\; e \in \omega \;\wedge\; \mathfrak{c} \in x \;\wedge\; e \Vdash_{rt} \varphi(\mathfrak{c})\}$$

is a set. To prove this we shall consider an extended class of formulae.

Definition 5.14 The *extended bounded formulae* are the smallest class of formulas containing the formulae of the form $x \in y$, $x = y$, $e \Vdash_{rt} x \in y$, $e \Vdash_{rt} x = y$ (where x, y are variables or elements of $\mathtt{V}^*$) which is closed under $\wedge, \vee, \neg, \to$ and bounded quantification.

Lemma 5.15 (**CZF**) *Separation holds for extended bounded formulae, i.e., for every extended bounded formula $\varphi(v)$ and set x, $\{v \in x \,:\, \varphi(v)\}$ is a set.*

Proof: Since $F_\in$ and $G_=$ are provably total functions of **CZF**, formulas of the form $e \Vdash_{rt} x \in y$ and $e \Vdash_{rt} x = y$ can be treated in the context of **CZF** as though they were atomic symbols of the language. This follows from [27], Proposition 2.4 or [4], Proposition 11.12. □

Lemma 5.16 (**CZF**) *Let $\varphi(v, u_1, \ldots, u_r)$ be a bounded formula of* **CZF** *all of whose free variables are among $u_1, \ldots, u_r$. Then there there is an extended bounded formula $\widetilde{\varphi}(v, u_1, \ldots, u_r)$ and $f_\varphi \in \omega$ such that for all $a_1, \ldots, a_r \in \mathtt{V}^*$ and $e \in \omega$,*

$$e \Vdash_{rt} \varphi(\vec{\mathfrak{a}}) \quad \textit{iff} \quad \widetilde{\varphi}(f_\varphi e, \vec{\mathfrak{a}}).$$

Proof: We proceed by induction on the generation of φ. For an atomic formula φ, the assertion follows with $\widetilde{\varphi} \equiv \varphi$ and f_φ being an index for the identity function. The assertion easily follows from the respective inductive assumptions if φ is of the form $\varphi_0 \wedge \varphi_1$ or $\varphi_0 \vee \varphi_1$.

Now suppose φ is of the form $\forall x \in w\,\psi(x, \vec{u}, w)$. Inductively we then have for all $\mathfrak{b}, \mathfrak{c}, \vec{\mathfrak{a}} \in \mathtt{V}^*$ and $e' \in \omega$,

$$e' \Vdash_{rt} \psi(\mathfrak{b}, \vec{\mathfrak{a}}, \mathfrak{c}) \quad \text{iff} \quad \widetilde{\psi}(f_\psi e', \mathfrak{b}, \vec{\mathfrak{a}}, \mathfrak{c})$$

for some extended bounded formula $\widetilde{\psi}$. Hence, by the definition of realizability for bounded formulae, we can readily construct the desired extended formula $\widetilde{\varphi}$ from $\widetilde{\psi}$.

The case of a bounded existential quantifier is similar to the preceding case. □

Corollary 5.17 (**CZF**) *Let $\varphi(v)$ be a bounded formula with parameters from $\mathtt{V}^*$ and $x \subseteq \mathtt{V}^*$. Then*

$$\{\langle e, \mathfrak{c}\rangle : \ e \in \omega \ \wedge \ \mathfrak{c} \in x \ \wedge \ e \Vdash_{rt} \varphi(\mathfrak{c})\}$$

is a set.

Proof: The above class is a set by the previous two lemmas. □

6 The soundness theorem for CZF

Theorem 6.1 *For every theorem θ of* **CZF**, *there exists an application term t such that*

$$\mathbf{CZF} \vdash \ (t \Vdash_{rt} \theta).$$

Moreover, the proof of this soundness theorem is effective in that the application term t can be effectively constructed from the **CZF** *proof of θ.*

Proof: In view of Theorem 5.13 it suffices to address the axioms of **CZF**. We treat them one after the other.

(**Extensionality**): Let $\theta(i, e, x, y)$ be the formula

$$\forall c \in \omega \forall \mathfrak{c}\, [\langle c, \mathfrak{c}\rangle \in x^* \ \rightarrow \ (e)_i c \Vdash_{rt} \mathfrak{c} \in y].$$

We then have the following equivalences:

$$\begin{aligned} & e \Vdash_{rt} [(\forall x \in \mathfrak{a})(x \in \mathfrak{b}) \ \wedge \ (\forall x \in \mathfrak{b})(x \in \mathfrak{a})] \\ \Leftrightarrow \ & (\forall x \in \mathfrak{a}^\circ)(x \in \mathfrak{b}^\circ) \ \wedge \ (\forall x \in \mathfrak{b}^\circ)(x \in \mathfrak{a}^\circ) \ \wedge \ \theta(0, \mathfrak{a}, \mathfrak{b}) \ \wedge \ \theta(1, \mathfrak{b}, \mathfrak{a}) \\ \Leftrightarrow \ & \mathfrak{a}^\circ = \mathfrak{b}^\circ \ \wedge \ \theta(0, \mathfrak{a}, \mathfrak{b}) \ \wedge \ \theta(1, \mathfrak{b}, \mathfrak{a}) \\ \Leftrightarrow \ & e \Vdash_{rt} \mathfrak{a} = \mathfrak{b}. \end{aligned}$$

As a result, $\mathbf{p}(\lambda x.x)(\lambda x.x) \Vdash_{rt}$ *Extensionality.*

(**Pair**): We need to guarantee the existence of an $e \in \omega$ such that

$$\forall \mathfrak{a}, \mathfrak{b} \in \mathtt{V}^* \, \exists \mathfrak{c} \in \mathtt{V}^* \; e \Vdash_{rt} \mathfrak{a} \in \mathfrak{c} \wedge \mathfrak{b} \in \mathfrak{c}. \tag{6}$$

Set $e = \mathbf{p}(\mathbf{p}0\mathbf{i}_r)(\mathbf{p}0\mathbf{i}_r)$ and let $\mathfrak{c} = \langle x, y\rangle$, where $x = \{\mathfrak{a}^\circ, \mathfrak{b}^\circ\}$ and $y = \{\langle 0, \mathfrak{a}\rangle, \langle 0, \mathfrak{b}\rangle\}$. By Lemma 4.2, $\mathfrak{c} \in \mathtt{V}^*$. We then have $\mathfrak{a}^\circ, \mathfrak{b}^\circ \in \mathfrak{c}^\circ$ and $\langle 0, \mathfrak{a}\rangle, \langle 0, \mathfrak{b}\rangle \in \mathfrak{c}^*$ as well as $\mathbf{i}_r \Vdash_{rt} \mathfrak{a} = \mathfrak{a}$ and $\mathbf{i}_r \Vdash_{rt} \mathfrak{b} = \mathfrak{b}$, showing that $e \Vdash_{rt} \mathfrak{a} \in \mathfrak{c} \wedge \mathfrak{b} \in \mathfrak{c}$ holds.

(**Union**): For each $\mathfrak{a} \in \mathtt{V}^*$, put

$$\begin{aligned} Un(\mathfrak{a}) &= \langle \bigcup \mathfrak{a}^\circ, A\rangle, \text{ where} \\ A &= \{\langle h, \mathfrak{b}\rangle : \ \exists \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^* \ \langle h, \mathfrak{b}\rangle \in \mathfrak{c}^*\}. \end{aligned}$$

Note that $\langle h, \mathfrak{b}\rangle \in A$ implies $\langle h, \mathfrak{b}\rangle \in \mathfrak{c}^*$ for some $\langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*$, which yields $\mathfrak{b}^\circ \in \mathfrak{c}^\circ$ and $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$, and thus $\mathfrak{b}^\circ \in \bigcup \mathfrak{a}^\circ$. Hence, by Lemma 4.2, we have $Un(\mathfrak{a}) \in \mathtt{V}^*$.

Let $e = \lambda u.\lambda v.\mathbf{p}v\mathbf{i}_r$. Suppose $\langle f, \mathfrak{c}\rangle \in \mathfrak{a}^* \wedge \mathfrak{c}^\circ \in \mathfrak{a}^\circ$. We want to show that

$$ef \quad \Vdash_{rt} \quad (\forall u \in \mathfrak{c})(u \in Un(\mathfrak{a})). \tag{7}$$

To this end assume that $\langle h, \mathfrak{b}\rangle \in \mathfrak{c}^* \wedge \mathfrak{b}^\circ \in \mathfrak{c}^\circ$. Put $\mathfrak{q} := Un(\mathfrak{a})$. From $\mathfrak{c}^\circ \in \mathfrak{a}^\circ \wedge \mathfrak{b}^\circ \in \mathfrak{c}^\circ$ we get that $\mathfrak{b}^\circ \in \bigcup \mathfrak{a}^\circ$, and hence $\mathfrak{c}^\circ \in \mathfrak{q}^\circ$. As $(efh)_0 = h$, we have $\langle (efh)_0, \mathfrak{b}\rangle \in \mathfrak{c}^*$, so that $\langle (efh)_0, \mathfrak{b}\rangle \in y$ and hence $\langle (efh)_0, \mathfrak{b}\rangle \in \mathfrak{q}^*$. Since also $\mathbf{i}_r \Vdash_{rt} \mathfrak{b} = \mathfrak{b}$, it follows that $efh \Vdash_{rt} \mathfrak{b} \in Un(\mathfrak{a})$. This shows (7). From (7) we get $e \Vdash_{rt} \forall a\, \exists q\, (\forall w \in a)(\forall u \in w)(u \in q)$, as desired.

(**Bounded Separation**): Let $\varphi(x)$ be a bounded formula with parameters in $\mathtt{V}^*$. This time we need to find $e, e' \in \omega$ such that for all $\mathfrak{a} \in \mathtt{V}^*$ there exists a $\mathfrak{b} \in \mathtt{V}^*$ such that

$$(e \Vdash_{rt} \forall x \in \mathfrak{b}\,[x \in \mathfrak{a} \wedge \varphi(x)]) \wedge (e' \Vdash_{rt} \forall x \in \mathfrak{a}[\varphi(x) \to x \in \mathfrak{b}]). \tag{8}$$

For $\mathfrak{a} \in \mathtt{V}^*$, define

$$\begin{aligned} Sep(\mathfrak{a}, \varphi) &= \{\langle \mathbf{p}fg, \mathfrak{c}\rangle : f, g \in \omega \wedge \langle g, \mathfrak{c}\rangle \in \mathfrak{a}^* \wedge f \Vdash_{rt} \varphi[x/\mathfrak{c}]\}, \\ \mathfrak{b} &= \langle \{x \in \mathfrak{a}^\circ : \varphi^\circ(x)\}, Sep(\mathfrak{a}, \varphi)\rangle. \end{aligned}$$

By Corollary 5.17, $Sep(\mathfrak{a}, \varphi)$ is a set, and hence $\mathfrak{b}$ is a set. To ensure that $\mathfrak{b} \in \mathtt{V}^*$ let $\langle h, \mathfrak{c}\rangle \in Sep(\mathfrak{a}, \varphi)$. Then $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}^*$ and $f \Vdash_{rt} \varphi[x/\mathfrak{c}]$ for some $f, g \in \omega$. Thus $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$ and, by Lemma 5.7, $\varphi^\circ[x/\mathfrak{c}^\circ]$, yielding $\mathfrak{c}^\circ \in \{x \in \mathfrak{a}^\circ : \varphi^\circ(x)\}$. Therefore, by Lemma 4.2, we have $\mathfrak{b} \in \mathtt{V}^*$.

To verify (8), first assume $\langle h, \mathfrak{c}\rangle \in \mathfrak{b}^*$ and $\mathfrak{c}^\circ \in \mathfrak{b}^\circ$. Then $h = \mathbf{p}fg$ for some $f, g \in \omega$ and $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}^*$ and $f \Vdash_{rt} \varphi[x/\mathfrak{c}]$. Since $\mathfrak{c}^\circ \in \mathfrak{b}^\circ$ holds, it follows that $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$. As a result, $\mathfrak{c}^\circ \in \mathfrak{a}^\circ \wedge \langle g, \mathfrak{c}\rangle \in \mathfrak{a}^* \wedge \mathbf{i}_r \Vdash_{rt} \mathfrak{c} = \mathfrak{c}$, and consequently we have $\mathbf{p}(h)_1\mathbf{i}_r \Vdash_{rt} \mathfrak{b} \in \mathfrak{a}$ and $(h)_0 \Vdash_{rt} \varphi[x/\mathfrak{c}]$. Moreover, we have $(\forall x \in \mathfrak{b}^\circ)(x \in \mathfrak{a}^\circ \wedge \varphi^\circ(x))$. Therefore with $e = \mathbf{p}(\mathbf{p}(\lambda u.(u)_1)\mathbf{i}_r)(\lambda u.(u)_0)$, we get $e \Vdash_{rt} \forall x \in \mathfrak{b}\,[x \in \mathfrak{a} \wedge \varphi(x)]$.

Now assume $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}$, $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$ and $f \Vdash_{rt} \varphi[x/\mathfrak{c}]$. Then $\langle \mathbf{p}fg, \mathfrak{c}\rangle \in \mathfrak{b}^*$ and also $\mathfrak{c}^\circ \in \mathfrak{b}^\circ$ as $\varphi^\circ[x/\mathfrak{c}^\circ]$ is a consequence of $f \Vdash_{rt} \varphi[x/\mathfrak{c}]$ by Lemma 5.7. Therefore $\mathbf{p}(\mathbf{p}fg)\mathbf{i}_r \Vdash_{rt} \mathfrak{c} \in \mathfrak{b}$. Finally, by the very definition of $\mathfrak{b}$ we have $(\forall x \in \mathfrak{a}^\circ)[\varphi^\circ(x) \to x \in \mathfrak{b}^\circ]$, and hence with $e' = \lambda u.\lambda v.\mathbf{p}(\mathbf{p}vu)\mathbf{i}_r$ we get $e' \Vdash_{rt} (\forall x \in \mathfrak{a})[\varphi(x) \to x \in \mathfrak{b}]$.

(**Set Induction**): Let $\varphi(y)$ be a formula with parameters in $\mathtt{V}^*$ and at most y free. We are to construct an application term t so that

$$t \Vdash_{rt} \theta \to \psi,$$

where θ is the formula $\forall a\,[(\forall y \in a\,\phi(y)) \to \varphi(a)]$ and ψ is $\forall a\, \varphi(a)$. We clearly have $\theta^\circ \to \psi^\circ$ since this is an instance of Set Induction. It therefore suffices to to find a term t such that whenever $g \Vdash_{rt} \theta$ then $tg \Vdash_{rt} \varphi(\mathfrak{a})$ holds for all $\mathfrak{a} \in \mathtt{V}^*$. So assume that for all $\mathfrak{a} \in \mathtt{V}^*$,

$$g \Vdash_{rt} (\forall y \in \mathfrak{a}\, \varphi(y)) \to \varphi(\mathfrak{a}). \tag{9}$$

Note that (9) entails that $\forall a\,[(\forall y \in a\, \varphi^\circ(y)) \to \varphi^\circ(a)]$, utilizing Lemma 5.7, 5.5. Hence, by Set Induction, we have that for all $\mathfrak{b} \in \mathtt{V}^*$,

$$\varphi^\circ(\mathfrak{b}^\circ). \tag{10}$$

Now, suppose $\mathfrak{a} \in \mathtt{V}^*_\alpha$ and that we have found an e such that for all $\mathfrak{b} \in \bigcup_{\beta\in\alpha} \mathtt{V}^*_\beta$, $e \Vdash_{rt} \varphi(\mathfrak{b})$. Thus, if $\langle f, \mathfrak{b}\rangle \in \mathfrak{a}^*$, then $\mathfrak{b} \in \bigcup_{\beta\in\alpha} \mathtt{V}^*_\beta$, and hence $e \Vdash_{rt} \varphi(\mathfrak{b})$, so that in view of (10) and (9),

$$\lambda u.\mathbf{k}eu \Vdash_{rt} \forall y \in \mathfrak{a}\varphi(y) \quad \text{and} \quad g(\lambda u.\mathbf{k}eu) \Vdash_{rt} \varphi(\mathfrak{a}). \tag{11}$$

With the aid of the recursion theorem for applicative structures we can effectively cook up an application term t such that $tf \simeq f(\lambda u.\mathbf{k}(tf)u)$ holds for all f. If we now put $e := tg$ in the above, we see by induction on α that $tg \Vdash_{rt} \varphi(\mathfrak{a})$ and hence

$$t \Vdash_{rt} \forall a\,[(\forall y \in a\varphi(y)) \rightarrow \varphi(a)] \rightarrow \forall a\,\varphi(a).$$

(**Infinity**): The most obvious candidate to represent ω in $\mathtt{V}^*$ is $\underline{\omega}$, which is given via an injection of ω into $\mathtt{V}^*$. Set

$$\underline{n} = \langle n, \{\langle k, \underline{k}\rangle : k < n\}\rangle \tag{12}$$
$$\underline{\omega} = \langle \omega, \{\langle n, \underline{n}\rangle : n \in \omega\}\rangle. \tag{13}$$

Note that $\underline{n}^\circ = n$ and $\underline{\omega}^\circ = \omega$. Clearly, by Lemma 4.2, $\underline{n}, \underline{\omega} \in \mathtt{V}^*$.

In order to show realizability of the Infinity axiom, we first have to write it out in full detail. Let $\bot_v$ be the formula $\forall u \in v\ \neg\, u = u$ and let $SC(u,v)$ be the formula $\forall y \in v\,[y = u \ \vee\ y \in u] \ \wedge\ [u \in v \ \wedge\ \forall y \in u\, y \in v]$. Then Infinity amounts to the sentence

$$\exists x\,(\forall v \in x\,[\bot_v \ \vee \exists u \in x\, SC(u,v)] \ \wedge\ \forall v[(\bot_v \ \vee \exists u \in x\, SC(u,v)) \rightarrow v \in x]). \tag{14}$$

Abbreviating the formula of (14) by $\exists x\, \vartheta_{inf}(x)$ it is obvious that $\vartheta^\circ_{inf}(\underline{\omega}^\circ)$ holds.

Now, suppose $\langle f, \mathfrak{c}\rangle \in \underline{\omega}^*$. Then $f = n$ and $\mathfrak{c} = \underline{n}$ for some $n \in \omega$. If $n = 0$ then $\underline{n} = \langle 0, 0\rangle$ and therefore $0 \Vdash_{rt} \bot_{\mathfrak{c}}$. Otherwise we have $n = k+1$ for some $k \in \omega$. If $\langle m, \underline{m}\rangle \in \underline{n}^*$ then $m = k$ or $m \in k$, so that $\mathbf{i}_r \Vdash_{rt} \underline{m} = \underline{k}$ or $\mathbf{p}m\mathbf{i}_r \Vdash_{rt} \underline{m} \in \underline{k}$, and whence $\mathbf{d}(\mathbf{p}0\mathbf{i}_r)(\mathbf{p}1(\mathbf{p}m\mathbf{i}_r))m\,k \Vdash_{rt} (\underline{m} = \underline{k} \vee \underline{m} \in \underline{k})$. As a result of the foregoing we have $\ell(k) \Vdash_{rt} \forall y \in \underline{n}\,(y = \underline{k} \vee y \in \underline{k})$, where $\ell(k) := \lambda z.\mathbf{d}(\mathbf{p}0\mathbf{i}_r)(\mathbf{p}1(\mathbf{p}z\mathbf{i}_r))z\,k$. Note both that $\mathbf{p}k\mathbf{i}_r \Vdash_{rt} \underline{k} \in \underline{n}$ and $\lambda z.\mathbf{p}z\mathbf{i}_r \Vdash_{rt} (\forall y \in \underline{k})\, y \in \underline{n}$, and hence $\wp(k) \Vdash_{rt} \underline{k} \in \underline{n} \ \wedge\ (\forall y \in \underline{k})\, y \in \underline{n}$, where $\wp(k) := \mathbf{p}(\mathbf{p}k\mathbf{i}_r)(\lambda z.\mathbf{p}z\mathbf{i}_r)$. Also note that $k = \mathbf{p}_N\, n$. With

$$t(n) \ := \ \mathbf{p}(\mathbf{p}_N n)(\mathbf{p}(\ell(\mathbf{p}_N\, n))(\wp(\mathbf{p}_N\, n)))$$

we thus obtain $t(n) \Vdash_{rt} \exists u \in \underline{\omega}\, SC(u, \underline{n})$. In conclusion, as $n = 0$ or $n = k+1$ for some $k \in \omega$ and $n = f$ and $\underline{n} = \mathfrak{c}$ we arrive at $\mathbf{d}(\mathbf{p}00)(\mathbf{p}1t(f))f0 \Vdash_{rt} [\bot_{\mathfrak{c}} \ \vee \exists u \in \underline{\omega}\, SC(u, \mathfrak{c})]$. Hence we have

$$\mathbf{q}^+ \Vdash_{rt} \forall v \in \underline{\omega}\,[\bot_v \ \vee \exists u \in \underline{\omega}\, SC(u,v)] \tag{15}$$

where $\mathbf{q}^+ := \lambda f.\mathbf{d}(\mathbf{p}00)(\mathbf{p}1t(f))f0$.

Conversely assume $\mathfrak{a} \in \mathtt{V}^*$ and

$$e \ \Vdash_{rt} \ \bot_{\mathfrak{a}} \vee \exists u \in \underline{\omega}\, SC(u, \mathfrak{a}). \tag{16}$$

Then either $(e)_0 = 0$ and $(e)_1 \Vdash_{rt} \bot_{\mathfrak{a}}$ or $(e)_0 = 1$ and $(e)_1 \Vdash_{rt} \exists u \in \underline{\omega}\, SC(u, \mathfrak{a})$.

The first case scenario yields $\bot_{\mathfrak{a}^\circ}$ by Lemma 5.7 and thus $\mathfrak{a}^\circ = 0$. Moreover, it yields $\mathfrak{a} = \langle 0, 0\rangle$. To see this assume $\langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*$. Then $(e)_1 f \Vdash_{rt} \neg\, \mathfrak{c} = \mathfrak{c}$, which means that $\forall g \in \omega\ \neg\, g \Vdash_{rt} \mathfrak{c} = \mathfrak{c}$. However, as $\mathbf{i}_r \Vdash_{rt} \mathfrak{c} = \mathfrak{c}$ this is absurd, showing $\mathfrak{a}^* = 0$. The latter yields $\mathbf{i}_r \Vdash_{rt} \underline{0} = \mathfrak{a}$ and thus

$$\mathbf{p}(e)_0\mathbf{i}_r \Vdash_{rt} \mathfrak{a} \in \underline{\omega}. \tag{17}$$

The second scenario entails that $((e)_1)_0 = n$ for some $n \in \omega$ as well as $((e)_1)_1 \Vdash_{rt} SC(\underline{n}, \mathfrak{a})$. Therefore we can conclude that $t_1 \Vdash_{rt} \forall y \in \mathfrak{a}\,(y = \underline{n} \vee y \in \underline{n})$, $t_2 \Vdash_{rt} \underline{n} \in \mathfrak{a}$, and $t_3 \Vdash_{rt} \forall y \in \underline{n}\, y \in \mathfrak{a}$ with $s := ((e)_1)_1$, $t_1 := (s)_0$, $t_2 := ((s)_1)_0$ and $t_3 := ((s)_1)_1$. Our first aim is to construct a closed application term $\mathbf{q}^\#$ such that $\mathbf{q}^\# \Vdash_{rt} \mathfrak{a} = \underline{n+1}$. To this end assume first that $\langle f, \mathfrak{c}\rangle \in \mathfrak{a}$. Then

$t_1 f \Vdash_{rt} \mathfrak{c} = \underline{n} \vee \mathfrak{c} \in \underline{n}$ and $(t_1 f)_0 = 0$ or $(t_1 f)_0 = 1$. From $(t_1 f)_0 = 0$ we obtain $(t_1 f)_1 \Vdash_{rt} \mathfrak{c} = \underline{n}$, and hence $\mathbf{p}n(t_1 f)_1 \Vdash_{rt} \mathfrak{c} \in \underline{n+1}$. If, on the other hand, $(t_1 f)_0 = 1$, we conclude that $(t_1 f)_1 \Vdash_{rt} \mathfrak{c} \in \underline{n}$, which entails that $((t_1 f)_1)_0 = k$ and $((t_1 f)_1)_1 \Vdash_{rt} \mathfrak{c} = \underline{k}$ for some $k \in n$, and hence $\mathbf{p}r_0 r_1 \Vdash_{rt} \mathfrak{c} \in \underline{n+1}$ where $r_i := ((t_1 f)_1)_i$. To summarize, we have

$$\langle f, \mathfrak{c} \rangle \in \mathfrak{a} \quad \rightarrow \quad q_1(f) \Vdash_{rt} \mathfrak{c} \in \underline{n+1}, \tag{18}$$

where $q_1(f) := \mathbf{d}(\mathbf{p}n(t_1 f)_1)(\mathbf{p}r_0 r_1)(t_1 f)_0 0$.

Next assume that $\langle f, \mathfrak{c} \rangle \in \underline{n+1}$. Then $f = k$ and $\mathfrak{c} = \underline{k}$ for some $k \in n+1$. We thus have $k = n \vee k \in n$. $k = n$ yields $t_2 \Vdash_{rt} \mathfrak{c} \in \mathfrak{a}$, while $k \in n$ yields $t_3 k \Vdash_{rt} \underline{k} \in \mathfrak{a}$, so that $t_3 k \Vdash_{rt} \mathfrak{c} \in \mathfrak{a}$. Thus, since $f = k$ we get $q_2(f) \Vdash_{rt} \mathfrak{c} \in \mathfrak{a}$ with $q_2(f) := \mathbf{d} t_2 (t_3 f) f n$. In conclusion,

$$\langle f, \mathfrak{c} \rangle \in \underline{n+1} \quad \rightarrow \quad q_2(f) \Vdash_{rt} \mathfrak{c} \in \mathfrak{a}. \tag{19}$$

With $\mathbf{q}^{\#} := \mathbf{p}(\lambda f. q_1(f))(\lambda f. q_2(f))$, (18) and (19) entail that $\mathbf{q}^{\#} \Vdash_{rt} \mathfrak{a} = \underline{n+1}$, and thus $\mathbf{p}(n+1)\mathbf{q}^{\#} \Vdash_{rt} \mathfrak{a} \in \underline{\omega}$.

The upshot of the foregoing is that from (16) we have concluded that (17) holds if $(e)_0 = 0$ and that (19) holds if $(e)_0 = 1$. Also note that $(e)_0 = 1$ entails $n + 1 = \mathbf{s}_N n = \mathbf{s}_N ((e)_1)_0$. Thus we arrive at $\ell^{\diamond}(e) \Vdash_{rt} \mathfrak{a} \in \underline{\omega}$ with $\ell^{\diamond}(e) := \mathbf{d}(\mathbf{p}(e)_0 \mathbf{i}_r)(\mathbf{p}(\mathbf{s}_N ((e)_1)_0)\mathbf{q}^{\#})(e)_0 0$. Using lambda-abstraction on e, it follows that

$$\lambda e.\ell^{\diamond}(e) \quad \Vdash_{rt} \quad \forall v[(\bot_v \ \vee \exists u \in \underline{\omega}\, SC(u,v)) \ \rightarrow \ v \in \underline{\omega}]. \tag{20}$$

Finally, (15) and (20) show that $\mathbf{p}\mathbf{q}^{+}(\lambda e.\ell^{\diamond}(e))$ provides a realizer for the Infinity axiom.

(**Strong Collection**): Let $a \in \mathtt{V}^*$ and assume that $g \Vdash_{rt} \forall x \in \mathfrak{a}\, \exists y\, \varphi(x,y)$. Then we have

$$\forall x \in \mathfrak{a}^{\circ}\, \exists y\, \varphi^{\circ}(x,y) \tag{21}$$

and whenever $\langle f, \mathfrak{b} \rangle \in \mathfrak{a}^*$ then $gf \Vdash_{rt} \exists y\, \varphi(\mathfrak{b}^{\circ}, y)$, i.e., $\exists \mathfrak{c} \in \mathtt{V}^*\ gf \Vdash_{rt} \varphi(\mathfrak{b}, \mathfrak{c})$. By invoking Strong Collection in the background universe, there exists a set D such that

$$\forall \langle f, \mathfrak{b} \rangle \in \mathfrak{a}^*\, \exists \mathfrak{c} \in \mathtt{V}^*\, [\langle \mathbf{p}(gf)f, \mathfrak{c} \rangle \in D \ \wedge \ gf \Vdash_{rt} \varphi(\mathfrak{b}, \mathfrak{c})], \qquad \text{and} \tag{22}$$

$$\forall z \in D\, \exists \langle f, \mathfrak{b} \rangle \in \mathfrak{a}^*\ \exists \mathfrak{c} \in \mathtt{V}^*\, [z = \langle \mathbf{p}(gf)f, \mathfrak{c} \rangle \ \wedge \ gf \Vdash_{rt} \varphi(\mathfrak{b}, \mathfrak{c})]. \tag{23}$$

In particular, $D \subseteq \omega \times \mathtt{V}^*$. (22) also implies that

$$\forall \langle h, \mathfrak{c} \rangle \in D\, \exists \mathfrak{b}^{\circ} \in \mathfrak{a}^{\circ}\ \varphi^{\circ}(\mathfrak{b}^{\circ}, \mathfrak{c}^{\circ}). \tag{24}$$

Moreover, applying Strong Collection to (21) there exists a set E such that

$$\forall x \in \mathfrak{a}^{\circ}\, \exists y \in E\, \varphi^{\circ}(x,y) \ \wedge \ \forall y \in E\, \exists x \in \mathfrak{a}^{\circ}\, \varphi^{\circ}(x,y).$$

Now let

$$\begin{aligned} Y &= E \cup \{\mathfrak{c}^{\circ} : \ \exists k\ \langle k, \mathfrak{c} \rangle \in D\}, \\ \mathfrak{d} &= \langle Y, D \rangle. \end{aligned}$$

Note that if $\langle k, \mathfrak{c} \rangle \in D$ then $\mathfrak{c}^{\circ} \in Y$. By 4.2 we have $\mathfrak{d} \in \mathtt{V}^*$ and owing to (24) we get

$$\forall x \in \mathfrak{a}^{\circ}\, \exists y \in Y\, \varphi^{\circ}(x,y) \ \wedge \ \forall y \in Y\, \exists x \in \mathfrak{a}^{\circ}\, \varphi^{\circ}(x,y). \tag{25}$$

We need to construct application terms e, e' from g such that

$$e \Vdash_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{d}\, \varphi(x,y), \tag{26}$$
$$e' \Vdash_{rt} \forall y \in \mathfrak{d}\, \exists x \in \mathfrak{a}\, \varphi(x,y). \tag{27}$$

For (26), let $\langle f, \mathfrak{b}\rangle \in \mathfrak{a}^*$. Then, by (22), there exists $\mathfrak{c}$ such that $\langle \mathbf{p}(gf)f, \mathfrak{c}\rangle \in D$ and $gf \Vdash_{rt} \varphi(\mathfrak{b}, \mathfrak{c})$, and hence $\mathbf{p}(\mathbf{p}(gf)f)(gf) \Vdash_{rt} \exists y \in \mathfrak{d}\, \varphi(x,y)$; so that with $e = \lambda u.\mathbf{p}(\mathbf{p}(gu)u)(gu)$ and taking (25) into account, we obtain (26).

To show (27), let $\langle h, \mathfrak{c}\rangle \in \mathfrak{d}^*$, i.e., $\langle h, \mathfrak{c}\rangle \in D$. Owing to (23) there exists $\langle f, \mathfrak{b}\rangle \in \mathfrak{a}^*$ and there exists $g \in \omega$ such that $h = \mathbf{p}(gf)f$ and $gf \Vdash_{rt} \varphi(\mathfrak{b}, \mathfrak{c})$. Thus, letting $e' = \lambda v.\mathbf{p}(v)_1(v)_0$, we have $e'h \Vdash_{rt} \varphi(\mathfrak{b}, \mathfrak{c})$. Since by (25) we also know that $\forall y \in \mathfrak{d}^\circ\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x,y)$ holds, (27) follows.

Letting $\vartheta(u,z)$ be the conjunction of the formulas $\forall x \in u\, \exists y \in z\, \varphi(x,y)$ and $\forall y \in z\, \exists x \in u\, \varphi(x,y)$, we also have

$$\forall x \in \mathfrak{a}^\circ\, \exists y\, \varphi^\circ(x,y) \;\rightarrow\; \exists z\, \vartheta^\circ(\mathfrak{a}^\circ, z) \tag{28}$$

by Strong Collection. Thus, on account of (26), (27) and (28) we arrive at

$$\mathbf{p}(\lambda g.e)(\lambda g.e') \Vdash_{rt} \forall x \in \mathfrak{a}\, \exists y\, \varphi(x,y) \;\rightarrow\; \exists z\, \vartheta(\mathfrak{a}, z),$$

as desired.

(**Subset Collection**): Let $\mathfrak{a}, \mathfrak{b} \in \mathtt{V}^*$ and $\varphi(x,u,y)$ be a formula with at most the free variables exhibited and parameters in $\mathtt{V}^*$. We would like to find a realizer $\mathbf{r}$ such that

$$\mathbf{r} \Vdash_{rt} \exists q \forall u [\forall x \in \mathfrak{a}\, \exists y \in \mathfrak{b}\, \varphi(x,y,u) \rightarrow \exists v \in q\, \varphi'(\mathfrak{a}, v, u)], \tag{29}$$

where $\varphi'(\mathfrak{a}, v, u)$ abbreviates the formula

$$\forall x \in \mathfrak{a}\, \exists y \in v\, \varphi(x,y,u) \;\wedge\; \forall y \in v\, \exists x \in \mathfrak{a}\, \varphi(x,y,u).$$

Set

$$B \;=\; \{\langle \mathbf{p}ef, \mathfrak{d}\rangle \;:\; e, f \in \omega \;\wedge\; ef\downarrow \;\wedge\; \langle (ef)_0, \mathfrak{d}\rangle \in \mathfrak{b}^*\}.$$

Note that B is a set. Now, let $\psi(e, f, \mathfrak{c}, u, z)$ be the formula

$$u \in \mathtt{V}^* \;\wedge\; e, f \in \omega \;\wedge\; ef\downarrow \;\wedge \exists \mathfrak{d}\, [\langle \mathbf{p}ef, \mathfrak{d}\rangle = z \;\wedge\; \langle (ef)_0, \mathfrak{d}\rangle \in \mathfrak{b}^* \;\wedge\; (ef)_1 \Vdash_{rt} \varphi(\mathfrak{c}, \mathfrak{d}, u)].$$

By invoking Subset Collection there exists a set D such that

$$\forall u \forall e\, [\forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists z \in B \psi(e, f, \mathfrak{c}, u, z) \;\rightarrow\; \exists w \in D\, \psi'(\mathfrak{a}^*, e, u)], \tag{30}$$

where $\psi'(\mathfrak{a}^*, e, u, w)$ is the conjunction of the formulas $\forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists z \in w\, \psi(e, f, \mathfrak{c}, u, z)$ and $\forall z \in w\, \exists \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \psi(e, f, \mathfrak{c}, u, z)$. Letting $\hat{D} := \{w \cap B :\; w \in D\}$, (30) implies

$$\forall u \forall e\, [\forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists z \in B \psi(e, f, \mathfrak{c}, u, z) \;\rightarrow\; \exists w \in \hat{D}\, \psi'(\mathfrak{a}^*, e, u, w)]. \tag{31}$$

Using Subset Collection again, there exists a set C such that

$$\forall u [\forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{b}^\circ\, \varphi^\circ(x,y,u) \;\rightarrow\; \exists v \in C\, \vartheta(\mathfrak{a}^\circ, v, u)], \tag{32}$$

where $\vartheta(z,v,u)$ stands for the conjunction of the formulas $\forall x \in z\, \exists y \in v\, \varphi^\circ(x,y,u)$ and $\forall y \in v\, \exists x \in z\, \varphi^\circ(x,y,u)$. Next, we define the witness in $\mathtt{V}^*$ for the existential quantifier $\exists q$ in (29). Let

$$\begin{aligned} \mathcal{W} &:= \{\langle v \cup \{\mathfrak{c}^\circ : \exists h\, \langle h, \mathfrak{c}\rangle \in w\}, w\rangle : v \in C \wedge w \in \hat{D}\}, \\ E &:= C \cup \{\mathfrak{z}^\circ : \mathfrak{z} \in \mathcal{W}\}, \\ E^+ &:= \{\langle 0, \mathfrak{z}\rangle : \mathfrak{z} \in \mathcal{W}\} \\ \mathfrak{e} &:= \langle E, E^+\rangle. \end{aligned} \tag{33}$$

As $B \subseteq \omega \times \mathtt{V}^*$ we get $w \subseteq \omega \times \mathtt{V}^*$ whenever $w \in \hat{D}$, and hence, by Lemma 4.2, $\mathfrak{z} \in \mathtt{V}^*$ holds for all $\mathfrak{z} \in \mathcal{W}$. Thus, for $\mathfrak{z} \in \mathcal{W}$, we have $\langle 0, \mathfrak{z}\rangle \in \omega \times \mathtt{V}^*$ and $\mathfrak{z}^\circ \in E$, so that $\mathfrak{e} \in \mathtt{V}^*$ by Lemma 4.2.

Now let $e \in \omega$ and let $\mathfrak{p} \in \mathtt{V}^*$ satisfy

$$e \Vdash_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{b}\, \varphi(x,y,\mathfrak{p}). \tag{34}$$

Thus we get

$$\begin{aligned} &\forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{b}^\circ\, \varphi^\circ(x,y,\mathfrak{p}^\circ) \quad \text{and} \\ &\forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists \mathfrak{d}\, [\langle (ef)_0, \mathfrak{d}\rangle \in \mathfrak{b}^* \wedge (ef)_1 \Vdash_{rt} \varphi(\mathfrak{c},\mathfrak{d},\mathfrak{p})]. \end{aligned} \tag{35}$$

Hence $\forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists z \in B\, \psi(e,f,\mathfrak{c},\mathfrak{p},z)$ and therefore, by (31), there exists $w \in \hat{D}$ such that $\forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists z \in w\, \psi(e,f,\mathfrak{c},\mathfrak{p},z)$ and $\forall z \in w\, \exists \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \psi(e,f,\mathfrak{c},\mathfrak{p},z)$, so that by unravelling the definition of ψ we get

$$\forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists \mathfrak{d}\, [\langle \mathbf{p}ef, \mathfrak{d}\rangle \in w \wedge \langle (ef)_0, \mathfrak{d}\rangle \in \mathfrak{b}^* \wedge (ef)_1 \Vdash_{rt} \varphi(\mathfrak{c},\mathfrak{d},\mathfrak{p})], \tag{36}$$

$$\forall \langle g, \mathfrak{d}\rangle \in w\, \exists \mathfrak{c}\, [\langle (g)_1, \mathfrak{c}\rangle \in \mathfrak{a}^* \wedge \langle (\hat{g})_0, \mathfrak{d}\rangle \in \mathfrak{b}^* \wedge (\hat{g})_1 \Vdash_{rt} \varphi(\mathfrak{c},\mathfrak{d},\mathfrak{p})], \tag{37}$$

where $\hat{g} := (g)_0(g)_1$. (34) also entails that $\forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{b}^\circ\, \varphi^\circ(x,y,\mathfrak{p}^\circ)$ so that, by (32), there exists $v \in C$ satisfying $\forall x \in \mathfrak{a}^\circ\, \exists y \in v\, \varphi^\circ(x,y,\mathfrak{p}^\circ)$ and $\forall y \in v\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x,y,\mathfrak{p}^\circ)$. Let $\mathfrak{z} := \langle v \cup \{\mathfrak{d}^\circ : \exists h\, \langle h, \mathfrak{d}\rangle \in w\}, w\rangle$. Then $\mathfrak{z} \in \mathcal{W}$ and $\langle 0, \mathfrak{z}\rangle \in \mathfrak{e}^*$. By (37) and Lemma 5.7, if $\langle h, \mathfrak{d}\rangle \in w$ then $\varphi^\circ(\mathfrak{c}^\circ, \mathfrak{d}^\circ, \mathfrak{p}^\circ)$ for some $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$, and hence $\forall y \in \mathfrak{z}^\circ\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x,y,\mathfrak{p}^\circ)$. So we can conclude that

$$\forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{z}^\circ\, \varphi^\circ(x,y,\mathfrak{p}^\circ) \wedge \forall y \in \mathfrak{z}^\circ\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x,y,\mathfrak{p}^\circ). \tag{38}$$

(36) and (37) also imply that

$$\forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists \mathfrak{d}\, [\langle \mathbf{p}ef, \mathfrak{d}\rangle \in \mathfrak{z}^* \wedge (ef)_1 \Vdash_{rt} \varphi(\mathfrak{c},\mathfrak{d},\mathfrak{p})], \tag{39}$$

$$\forall \langle g, \mathfrak{d}\rangle \in \mathfrak{z}^*\, \exists \mathfrak{c}\, [\langle (g)_1, \mathfrak{c}\rangle \in \mathfrak{a}^* \wedge ((g)_0(g)_1)_1 \Vdash_{rt} \varphi(\mathfrak{c},\mathfrak{d},\mathfrak{p})] \tag{40}$$

so that with

$$\begin{aligned} \mathbf{m}_0 &:= \lambda f.\mathbf{p}(\mathbf{p}ef)(ef)_1, \\ \mathbf{m}_1 &:= \lambda g.\mathbf{p}((g)_0(g)_1)_0((g)_0(g)_1)_1 \end{aligned}$$

we obtain from (38), (39) and (40) that

$$\begin{aligned} \mathbf{m}_0 &\Vdash_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{z}\, \varphi(x,y,\mathfrak{p}), \\ \mathbf{m}_1 &\Vdash_{rt} \forall y \in \mathfrak{z}\, \exists x \in \mathfrak{a}\, \varphi(x,y,\mathfrak{p}). \end{aligned}$$

As a result of the foregoing we have

$$\mathbf{p}\mathbf{m}_0\mathbf{m}_1 \Vdash_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{z}\, \varphi(x,y,\mathfrak{p}) \wedge \forall y \in \mathfrak{z}\, \exists x \in \mathfrak{a}\, \varphi(x,y,\mathfrak{p}). \tag{41}$$

Thus far we have shown that (34) implies (41). In consequence of this and (32) and the fact that $C \subseteq \mathfrak{e}^\circ$, we arrive at

$$\lambda e.\mathbf{p}0(\mathbf{p}\mathbf{m}_0\mathbf{m}_1) \Vdash_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{b}\, \varphi(x,y,u) \ \rightarrow\ \exists v \in \mathfrak{e}\, \varphi'(\mathfrak{a}, v, \mathfrak{p})$$

as $\langle 0, \mathfrak{z}\rangle \in \mathfrak{e}^*$. As a result, we get (29) with $\mathbf{r} := \lambda e.\mathbf{p}0(\mathbf{p}\mathbf{m}_0\mathbf{m}_1)$. □

7 The soundness theorem for CZF + REA

Next we show that the regular extension axiom holds in $\mathtt{V}^*$ if it holds in the background universe.

Lemma 7.1 (**CZF**)

(i) If B is a regular set with $2 \in B$, then B is closed under unordered and ordered pairs, i.e., whenever $x, y \in B$, then $\{x, y\}, \langle x, y\rangle \in B$.

(ii) If B is a regular set, then $B \cap \mathtt{V}^*$ is a set.

Proof: For (i) see [28], Lemma 6.1 (1). (ii) is proved in the same vein as [28], Lemma 6.1 (2). □

Theorem 7.2 *For every axiom θ of* **CZF + REA***, there exists a closed application term t such that*

$$\mathbf{CZF} + \mathbf{REA} \vdash\ (t \Vdash_{rt} \theta).$$

Proof: In view of theorem 6.1, we need only find a realizer for the axiom **REA**. Let $\mathfrak{a} \in \mathtt{V}^*$. Due to **REA** there exists a regular set B such that $\mathfrak{a}, 2, \omega \in B$. Let

$$\begin{aligned} A &:= B \cap \mathtt{V}^* \\ \mathfrak{c} &:= \langle B, \{\langle 0, \mathfrak{z}\rangle : \mathfrak{z} \in A\}\rangle. \end{aligned}$$

By Lemma 7.1(ii), A is a set and hence $\mathfrak{c}$ is a set. Moreover, as $A \subseteq \mathtt{V}^*$, it follows that $\{\langle 0, \mathfrak{z}\rangle : \mathfrak{z} \in A\} \subseteq \omega \times \mathtt{V}^*$ and we observe that $\mathfrak{z} \in A$ entails $\mathfrak{z}^\circ \in B$ as B is transitive. Therefore, by Lemma 4.2 (iii), $\mathfrak{c} \in \mathtt{V}^*$. As $\mathfrak{a} \in B$ and B is transitive it follows that $\mathfrak{a}^\circ \in B$, thus $\mathfrak{a}^\circ \in \mathfrak{c}^\circ$. Note also that $\langle 0, \mathfrak{a}\rangle \in \mathfrak{c}^*$. Thus we conclude that

$$\mathbf{p}0\mathbf{i}_r \Vdash_{rt} \mathfrak{a} \in \mathfrak{c}. \tag{42}$$

With $\tilde{\mathbf{m}} := \lambda x.\lambda y.\mathbf{p}0\mathbf{i}_r$ and $\tilde{\mathbf{n}} := \mathbf{p}0(\mathbf{p}0\mathbf{i}_r)$ one realizes transitivity and inhabitedness of $\mathfrak{c}$, respectively, i.e.,

$$\mathbf{p}\tilde{\mathbf{m}}\tilde{\mathbf{n}} \ \Vdash_{rt}\ \forall u \in \mathfrak{c}\, \forall v \in u\ v \in \mathfrak{c}\ \wedge\ \exists x \in \mathfrak{c}\, x \in \mathfrak{c}. \tag{43}$$

It is also the case that $\mathfrak{a}^\circ \in \mathfrak{c}^\circ\ \wedge\ \mathbf{Reg}(\mathfrak{c}^\circ)$ holds. Next we would like to find a realizer $\mathbf{q}$ such that

$$\mathbf{q} \Vdash_{rt} \mathbf{Reg}(\mathfrak{c}). \tag{44}$$

To this end, suppose that $\langle 0, \mathfrak{b}\rangle \in \mathfrak{c}^*$, $f \in \omega$, and $\varphi(x, y)$ is a formula with parameters in $\mathtt{V}^*$ such that

$$f \Vdash_{rt} \forall x \in \mathfrak{b}\, \exists y \in \mathfrak{c}\, \varphi(x, y). \tag{45}$$

Note that all elements of $\mathfrak{c}^*$ are of the form $\langle 0, u\rangle$. As B is transitive and B is closed under taking pairs we have $\mathfrak{c}^* \subseteq B$, and thus(45) yields

$$\begin{aligned}&\forall\mathfrak{p}\forall e\,(\langle e,\mathfrak{p}\rangle \in \mathfrak{b}^* \;\rightarrow \\ &\qquad \exists z \in B\,\exists\mathfrak{q}\,[z = \langle e,\mathfrak{q}\rangle \,\wedge\, (fe)_0 = 0 \,\wedge\, \langle 0,\mathfrak{q}\rangle \in \mathfrak{c}^* \,\wedge\, (fe)_1 \Vdash_{rt} \varphi(\mathfrak{p},\mathfrak{q})]).\end{aligned} \tag{46}$$

Utilizing the regularity of B, there exists $\hat{\mathbf{u}} \in B$ such that

$$\forall\mathfrak{p}\forall e\,[\langle e,\mathfrak{p}\rangle \in \mathfrak{b}^* \;\rightarrow\; \exists z \in \hat{\mathbf{u}}\,\exists\mathfrak{q}(z = \langle e,\mathfrak{q}\rangle \,\wedge\, \langle 0,\mathfrak{q}\rangle \in \mathfrak{c}^* \,\wedge\, (fe)_1 \Vdash_{rt} \varphi(\mathfrak{p},\mathfrak{q}))]; \tag{47}$$

$$\forall z \in \hat{\mathbf{u}}\,\exists\mathfrak{p}, e\,[\langle e,\mathfrak{p}\rangle \in \mathfrak{d}^* \,\wedge\, \exists\mathfrak{q}(\langle 0,\mathfrak{q}\rangle \in \mathfrak{c}^* \,\wedge\, z = \langle e,\mathfrak{q}\rangle \,\wedge\, (fe)_1 \Vdash_{rt} \varphi(\mathfrak{p},\mathfrak{q}))]. \tag{48}$$

From (48) it follows that $\hat{\mathbf{u}} \subseteq \omega \times A \subseteq \omega \times \mathtt{V}^*$, and thus with

$$\mathfrak{u} \;:=\; \langle\{\mathfrak{p}^\circ \mid \exists e \in \omega\,\langle e,\mathfrak{p}\rangle \in \hat{\mathbf{u}}\}, \hat{\mathbf{u}}\rangle$$

we have $\mathfrak{u} \in \mathtt{V}^*$ by 4.2. Moreover, the function $\langle e,\mathfrak{p}\rangle \mapsto \mathfrak{p}^\circ$ defined on $\hat{\mathbf{u}}$ maps into B, so that by the regularity of B we have $\{\mathfrak{p}^\circ \mid \exists e \in \omega\,\langle e,\mathfrak{p}\rangle \in \hat{\mathbf{u}}\} \in B$ and hence $\mathfrak{u} \in B \cap \mathtt{V}^* = A$, which yields $\langle 0,\mathfrak{u}\rangle \in \mathfrak{c}^*$. So we get

$$\mathbf{p}0\mathbf{i}_r \Vdash_{rt} \mathfrak{u} \in \mathfrak{c}. \tag{49}$$

Letting $s(f) := \lambda e.\mathbf{p}e(fe)_1$, (47) and (48) yield

$$s(f) \Vdash_{rt} \forall x \in \mathfrak{b}\,\exists y \in \mathfrak{u}\,\varphi(x,y), \tag{50}$$

$$s(f) \Vdash_{rt} \forall y \in \mathfrak{u}\,\exists x \in \mathfrak{b}\,\varphi(x,y). \tag{51}$$

As $\mathfrak{c}^\circ = B$ and B is regular we also have

$$\begin{aligned}&\forall b \in \mathfrak{c}^\circ\,(\forall x \in b\,\exists y \in \mathfrak{c}^\circ\,\varphi^\circ(x,y) \;\rightarrow \\ &\qquad \exists u \in \mathfrak{c}^\circ\,[\forall x \in b\,\exists y \in u\,\varphi^\circ(x,y) \,\wedge\, \forall y \in u\,\exists x \in b\,\varphi^\circ(x,y)]).\end{aligned} \tag{52}$$

Hence, letting $\tilde{\mathbf{q}} := \lambda f.\mathbf{p}(\mathbf{p}0\mathbf{i}_r)(\mathbf{p}s(f)s(f))$, (49), (50), (51) and (52) entail that

$$\begin{aligned}&\tilde{\mathbf{q}} \Vdash_{rt} \forall b \in \mathfrak{c}\,(\forall x \in b\,\exists y \in \mathfrak{c}\,\varphi(x,y) \;\rightarrow \\ &\qquad \exists u \in \mathfrak{c}\,[\forall x \in b\,\exists y \in u\,\varphi(x,y) \,\wedge\, \forall y \in u\,\exists x \in b\,\varphi(x,y)]).\end{aligned} \tag{53}$$

Choosing $\varphi(x,y)$ to be the formula $\mathfrak{r} \subseteq b \times \mathfrak{c} \,\wedge\, \langle x,y\rangle \in \mathfrak{r}$, we deduce from (53) and (43) that

$$\mathbf{p}(\mathbf{p}\tilde{\mathbf{m}}\tilde{\mathbf{n}})\tilde{\mathbf{q}} \Vdash_{rt} \mathbf{Reg}(\mathfrak{c}).$$

Thus, in view of (42), we conclude that

$$\mathbf{p}(\mathbf{p0i}_r)(\mathbf{p}(\mathbf{p}\tilde{\mathbf{m}}\tilde{\mathbf{n}})\tilde{\mathbf{q}}) \Vdash_{rt} \forall a\,\exists c\,[a \in c \,\wedge\, \mathbf{Reg}(c)].$$

□

Remark 7.3 Theorem 7.2 holds also for **CZF** augmented by other large set axioms such as *"Every set is contained in an inaccessible set"* or *"Every set is contained in a Mahlo set"*. For definitions of "inaccessible set" and "Mahlo set" see [4, 8].

8 Proof of Theorem 1.2

(**DP**): Suppose that **CZF** $\vdash \psi \vee \theta$ with sentences ψ and θ. By Theorem 6.1 there exists a closed application term t such that **CZF** proves $t \Vdash_{rt} (\psi \vee \theta)$. Then $t \simeq n$ for a natural number n.[2] Moreover, since **CZF** contains arithmetic, **CZF** $\vdash t \simeq \bar{n}$, and thus, by Lemma 5.9, **CZF** proves

$$[(\bar{n})_0 = 0 \ \wedge \ \phi] \ \ \vee \ \ [(\bar{n})_0 \neq 0 \ \wedge \ \psi].$$

Now, either $(n)_0 = 0$ or $(n)_0 \neq 0$. In the first case we have **CZF** $\vdash (\bar{n})_0 = 0$ and therefore **CZF** $\vdash \phi$, while in the second case we have **CZF** $\vdash (\bar{n})_0 \neq 0$ and therefore **CZF** $\vdash \psi$.

(**NEP**): Suppose that **CZF** $\vdash (\exists x{\in}\omega)\varphi(x)$ holds for a formula $\varphi(x)$ with at most the free variable x. By Theorem 6.1 there exists a closed application term t such that **CZF** proves $t \Vdash_{rt} (\exists x{\in}\underline{\omega})\phi(x)$. Then there exists a natural number e such that $t \simeq e$ and **CZF** $\vdash t \simeq \bar{e}$. By Lemma 5.8, **CZF** then proves $\exists \mathfrak{b}\, [\langle (\bar{e})_0, \mathfrak{b}\rangle \in \underline{\omega} \ \wedge \ \varphi(\mathfrak{b}^\circ)]$. Owing to the definition of $\underline{\omega}$, we get **CZF** $\vdash \phi(\bar{n})$, where $n = (e)_0$.

(**ECR**): Suppose

$$\textbf{CZF} \vdash (\forall x \in \omega)[\neg\psi(x) \ \rightarrow \ (\exists y{\in}\omega)\vartheta(x,y)]$$

holds for formulae $\psi(x)$ and $\vartheta(x,y)$ with at most the free variables shown. By Theorem 6.1 we find a closed application t such that **CZF** proves $t \Vdash_{rt} (\forall x \in \underline{\omega})[\neg\psi(x) \ \rightarrow \ (\exists y{\in}\underline{\omega})\vartheta(x,y)]$. Note that $\langle m, \mathfrak{b}\rangle \in \underline{\omega}^*$ implies $\mathfrak{b} = \underline{m}$ and hence $\mathfrak{b}^\circ = m$. Unravelling the definition of $\underline{\omega}$, we get that $\langle m, \mathfrak{b}\rangle \in \underline{\omega}^*$ implies $\mathfrak{b} = \underline{m}$ and hence $\mathfrak{b}^\circ = m$. Thus, by definition of $\Vdash_{rt}$ for bounded quantifiers,

$$\textbf{CZF} \ \ \vdash \ \ (\forall n, f \in \omega)[f \Vdash_{rt} \neg\psi(\underline{n}) \ \rightarrow \ tnf \Vdash_{rt} (\exists y \in \underline{\omega})\vartheta(\underline{n}, y)],$$

so that in view of Lemma 5.10 we arrive at

$$\textbf{CZF} \ \ \vdash \ \ (\forall n \in \omega)[\neg\psi(n) \ \rightarrow \ tn0 \Vdash_{rt} (\exists y \in \underline{\omega})\vartheta(\underline{n}, y)]. \tag{54}$$

Letting $\hat{tn} := tn0$, further unravelling yields

$$\textbf{CZF} \ \ \vdash \ \ (\forall n \in \omega)(\neg\psi(n) \ \rightarrow \ \exists \mathfrak{b}\, [\langle (\hat{t}n)_0, \mathfrak{b}\rangle \in \underline{\omega}^* \ \wedge \ (\hat{t}n)_1 \Vdash_{rt} \vartheta(\underline{n}, \mathfrak{b}^\circ)]). \tag{55}$$

Owing to Lemma 5.8, (55) entails that

$$\textbf{CZF} \ \ \vdash \ \ (\forall u \in \omega)[\neg\psi(u) \ \rightarrow \ (\hat{t}u)_0 \in \omega \ \wedge \ \vartheta(u, (\hat{t}u)_0)]. \tag{56}$$

Set $s := \lambda u.(\hat{t}u)_0$. Then **CZF** $\vdash s \simeq \bar{e}$ for some number e. From (56) we can thus infer that

$$\textbf{CZF} \ \ \vdash \ \ (\forall u \in \omega)[\neg\psi(u) \ \rightarrow \ \{\bar{e}\}(u) \in \omega \ \wedge \ \vartheta(u, \{\bar{e}\}(u))]. \tag{57}$$

(**CR**): We already observed that (**CR**) is a consequence of (**ECR**).

(**UZR**): Suppose **CZF** $\vdash \forall x[\psi(x) \vee \neg\psi(x)]$. By Theorem 6.1 there exists a closed application term t such that **CZF** $\vdash t \Vdash_{rt} \forall x[\psi(x) \ \vee \ \neg\psi(x)]$, whence

$$\textbf{CZF} \vdash \forall \mathfrak{a} \ t \Vdash_{rt} [\psi(\mathfrak{a}) \ \vee \ \neg\psi(\mathfrak{a})].$$

[2]Note that here our reasoning goes beyond what is provable in **CZF** since we tacitly assume that **CZF** is sound for statements of the form $t\downarrow$, i.e., if **CZF** $\vdash t\downarrow$ then $t\downarrow$ is true, and hence there exists an integer n such that t evaluates to n.

Moreover, $t \simeq n$ for a natural number n and thus $\mathbf{CZF} \vdash t \simeq \bar{n}$, so that, by Lemma 5.9, **CZF** proves

$$\forall \mathfrak{a}\, [(\bar{n})_0 = 0 \ \wedge\ \psi^\circ(\mathfrak{a}^\circ)] \ \ \vee\ \ [(\bar{n})_0 \neq 0 \ \wedge\ \neg\psi^\circ(\mathfrak{a}^\circ)].$$

Now, either $(n)_0 = 0$ or $(n)_0 \neq 0$. In the first case we have $\mathbf{CZF} \vdash (\bar{n})_0 = 0$ and therefore $\mathbf{CZF} \vdash \forall \mathfrak{a}\, \psi^\circ(\mathfrak{a}^\circ)$, while in the second case we have $\mathbf{CZF} \vdash (\bar{n})_0 \neq 0$ and therefore $\mathbf{CZF} \vdash \forall \mathfrak{a}\, \neg\psi^\circ(\mathfrak{a}^\circ)$. Thus, using Lemma 5.5, we either have $\mathbf{CZF} \vdash \forall x\, \psi(x)$ or $\mathbf{CZF} \vdash \forall x\, \neg\psi(x)$.

(**UR**): Suppose $\mathbf{CZF} \vdash \forall x\, (\exists y \in \omega)\theta(x, y)$. Consequently, there exists a closed application term t such that $\mathbf{CZF} \vdash t \Vdash_{rt} \forall x\, (\exists y \in \underline{\omega})\theta(x, y)$, and therefore

$$\mathbf{CZF} \vdash \forall \mathfrak{a}\ \ t \Vdash_{rt} (\exists y \in \underline{\omega})\theta(\mathfrak{a}, y).$$

Owing to Lemma 5.8 and the definition of $\underline{\omega}$, we arrive at

$$\mathbf{CZF} \vdash (t)_0 \in \omega \ \wedge\ \forall \mathfrak{a}\, \theta^\circ(\mathfrak{a}^\circ, (t)_0),$$

so that, by Lemma 5.5, we conclude $\mathbf{CZF} \vdash (\exists y \in \omega)\forall x\, \theta(x, y)$.

The proofs for $\mathbf{CZF} + \mathbf{REA}$ are identical, except that this time we use Theorem 7.2 rather than Theorem 6.1 . ☐

9 Variations

In this section we address several extensions of earlier results. We show that **CZF** can be replaced by **IZF** and also that Markov's principle can be added.

Theorem 9.1 *For every theorem θ of* **IZF**, *there exists an application term t such that*

$$\mathbf{IZF} \vdash\ (t \Vdash_{rt} \theta).$$

Moreover, the proof of this soundness theorem is effective in that the application term t can be effectively constructed from the **IZF** *proof of θ.*

Proof: In view of Theorem 6.1 we only need to show that **IZF** proves that the Power Set Axiom and the full Separation Axiom are realized with respect to $\Vdash_{rt}$.

(Full Separation): Let $\varphi(x)$ be an arbitrary formula with parameters in $\mathtt{V}^*$. We want to find $e, e' \in \omega$ such that for all $\mathfrak{a} \in \mathtt{V}^*$ there exists a $\mathfrak{b} \in \mathtt{V}^*$ such that

$$(e \Vdash_{rt} \forall x \in \mathfrak{b}\, [x \in \mathfrak{a} \wedge \varphi(x)]) \ \wedge\ (e' \Vdash_{rt} \forall x \in \mathfrak{a}[\varphi(x) \rightarrow x \in \mathfrak{b}]). \tag{58}$$

For $\mathfrak{a} \in \mathtt{V}^*$, define

$$\begin{aligned} Sep(\mathfrak{a}, \varphi) &= \{\langle \mathbf{p} f g, \mathfrak{c}\rangle :\ f, g \in \omega \ \wedge\ \langle g, \mathfrak{c}\rangle \in \mathfrak{a}^* \ \wedge\ f \Vdash_{rt} \varphi[x/\mathfrak{c}]\}, \\ \mathfrak{b} &= \langle \{x \in \mathfrak{a}^\circ :\ \varphi^\circ(x)\}, Sep(\mathfrak{a}, \varphi)\rangle. \end{aligned}$$

$Sep(\mathfrak{a}, \varphi)$ is a set by full Separation, and hence $\mathfrak{b}$ is a set. To ensure that $\mathfrak{b} \in \mathtt{V}^*$ let $\langle h, \mathfrak{c}\rangle \in Sep(\mathfrak{a}, \varphi)$. Then $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}^*$ and $f \Vdash_{rt} \varphi[x/\mathfrak{c}]$ for some $f, g \in \omega$. Thus $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$ and, by Lemma 5.7, $\varphi^\circ[x/\mathfrak{c}^\circ]$, yielding $\mathfrak{c}^\circ \in \{x \in \mathfrak{a}^\circ :\ \varphi^\circ(x)\}$. Therefore, by Lemma 4.2, we have $\mathfrak{b} \in \mathtt{V}^*$.

To verify (58), first assume $\langle h, \mathfrak{c}\rangle \in \mathfrak{b}^*$ and $\mathfrak{c}^\circ \in \mathfrak{b}^\circ$. Then $h = \mathbf{p}fg$ for some $f, g \in \omega$ and $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}^*$ and $f \Vdash_{rt} \varphi[x/\mathfrak{c}]$. Since $\mathfrak{c}^\circ \in \mathfrak{b}^\circ$ holds, it follows that $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$. As a result, $\mathfrak{c}^\circ \in \mathfrak{a}^\circ \ \wedge \ \langle g, \mathfrak{c}\rangle \in \mathfrak{a}^* \ \wedge \ \mathbf{i}_r \Vdash_{rt} \mathfrak{c} = \mathfrak{c}$, and consequently we have $\mathbf{p}(h)_1\mathbf{i}_r \Vdash_{rt} \mathfrak{b} \in \mathfrak{a}$ and $(h)_0 \Vdash_{rt} \varphi[x/\mathfrak{c}]$. Moreover, we have $(\forall x \in \mathfrak{b}^\circ)(x \in \mathfrak{a}^\circ \ \wedge \ \varphi^\circ(x))$. Therefore with $e = \mathbf{p}(\mathbf{p}(\lambda u.(u)_1)\mathbf{i}_r)(\lambda u.(u)_0)$, we get $e \Vdash_{rt} \forall x \in \mathfrak{b}\,[x \in \mathfrak{a} \wedge \varphi(x)]$.

Now assume $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}$, $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$ and $f \Vdash_{rt} \varphi[x/\mathfrak{c}]$. Then $\langle \mathbf{p}fg, \mathfrak{c}\rangle \in \mathfrak{b}^*$ and also $\mathfrak{c}^\circ \in \mathfrak{b}^\circ$ as $\varphi^\circ[x/\mathfrak{c}^\circ]$ is a consequence of $f \Vdash_{rt} \varphi[x/\mathfrak{c}]$ by Lemma 5.7. Therefore $\mathbf{p}(\mathbf{p}fg)\mathbf{i}_r \Vdash_{rt} \mathfrak{c} \in \mathfrak{b}$. Finally, by the very definition of $\mathfrak{b}$ we have $(\forall x \in \mathfrak{a}^\circ)[\varphi^\circ(x) \to x \in \mathfrak{b}^\circ]$, and hence with $e' = \lambda u.\lambda v.\mathbf{p}(\mathbf{p}vu)\mathbf{i}_r$ we get $e' \Vdash_{rt} (\forall x \in \mathfrak{a})[\varphi(x) \to x \in \mathfrak{b}]$.

(Powerset): It suffices to find a realizer for the formula $\forall x\,\exists y\,\forall z\,[z \subseteq x \to z \in y]$ as it implies the Powerset Axiom with the aid of Separation. Let $\mathfrak{a} \in \mathtt{V}^*$. Put $\mathcal{A} = \{\mathfrak{d} : \ \exists g\ \langle g, \mathfrak{d}\rangle \in \mathfrak{a}^*\}$. For $y \subseteq \omega \times \mathcal{A}$ let

$$\mathfrak{a}_y \quad := \quad \langle \{\mathfrak{c}^\circ : \ \exists f\ \langle f, \mathfrak{c}\rangle \in y\}, y\rangle.$$

Note that $\mathfrak{a}_y \in \mathtt{V}^*$. The role of a set large enough to comprise the powerset of $\mathfrak{a}$ in $\mathtt{V}^*$ will be played by the following set

$$\mathfrak{p} \quad := \quad \langle \mathcal{P}(\mathfrak{a}^\circ), \{\langle 0, \mathfrak{a}_y\rangle : \ y \subseteq \omega \times \mathcal{A}\}\rangle.$$

$\mathfrak{p}$ is a set in our background theory **IZF**. For $\langle 0, \mathfrak{a}_y\rangle \in \mathfrak{p}^*$ we have $\mathfrak{a}_y^\circ \subseteq \mathfrak{a}^\circ$, and thus $\mathfrak{a}_y^\circ \in \mathcal{P}(\mathfrak{a}^\circ)$, so it follows that $\mathfrak{p} \in \mathtt{V}^*$.

Now suppose $e \Vdash_{rt} \mathfrak{b} \subseteq \mathfrak{a}$. Put

$$y_{\mathfrak{b}} \quad := \quad \{\langle (d,f), \mathfrak{x}\rangle : \ d, f \in \omega \ \wedge \ \langle (df)_0, \mathfrak{x}\rangle \in \mathfrak{a}^* \ \wedge \ \exists \mathfrak{c}\,[\langle d, \mathfrak{c}\rangle \in \mathfrak{b}^* \ \wedge \ (df)_1 \Vdash_{rt} \mathfrak{x} = \mathfrak{c}]\}. \tag{59}$$

(Recall that (x, y) stands for $\mathbf{p}xy$.) By definition of $y_{\mathfrak{b}}$, $y_{\mathfrak{b}} \subseteq \omega \times \mathcal{A}$, and therefore $\langle 0, \mathfrak{a}_{y_{\mathfrak{b}}}\rangle \in \mathfrak{p}^*$.

If $\langle f, \mathfrak{c}\rangle \in \mathfrak{b}^*$ it follows that $ef \Vdash_{rt} \mathfrak{c} \in \mathfrak{a}$ since $e \Vdash_{rt} \mathfrak{b} \subseteq \mathfrak{a}$; and hence there exists $\mathfrak{x}$ such that $\langle (ef)_0, \mathfrak{x}\rangle \in \mathfrak{a}^*$ and $(ef)_1 \Vdash_{rt} \mathfrak{x} = \mathfrak{c}$; whence $\langle (e, f), \mathfrak{x}\rangle \in y_{\mathfrak{b}}$ and therefore $((e, f), (ef)_1) \Vdash_{rt} \mathfrak{c} \in \mathfrak{a}_{y_{\mathfrak{b}}}$. Thus we can infer that $\lambda f.((e, f), (ef)_1) \Vdash_{rt} \mathfrak{b} \subseteq \mathfrak{a}_{y_{\mathfrak{b}}}$.

Conversely, if $\langle g, \mathfrak{x}\rangle \in \mathfrak{a}^*_{y_{\mathfrak{b}}} = y_{\mathfrak{b}}$, then there exist d, f and $\mathfrak{c}$ such that $g = (d, f)$, $\langle d, \mathfrak{c}\rangle \in \mathfrak{b}^*$, and $(df)_1 \Vdash_{rt} \mathfrak{c} = \mathfrak{x}$, which entails that $((g)_0, ((g)_0(g)_1)_1) \Vdash_{rt} \mathfrak{x} \in \mathfrak{b}$. As a result, $\eta(e) \Vdash_{rt} \mathfrak{b} = \mathfrak{a}_{y_{\mathfrak{b}}}$, where $\eta(e) = (\lambda f.((e, f), (ef)_1), \lambda g.((g)_0, ((g)_0(g)_1)_1))$. Hence $(0, \eta(e)) \Vdash_{rt} \mathfrak{b} \in \mathfrak{p}$, so that

$$\lambda e.(0, \eta(e)) \ \Vdash_{rt} \ \forall y\,[y \subseteq \mathfrak{a} \to y \in \mathfrak{p}],$$

and therefore, by the genericity of quantifiers,

$$\lambda e.(0, \eta(e)) \quad \Vdash_{rt} \quad \forall x\,\exists y\,\forall z\,[z \subseteq x \to z \in y]. \tag{60}$$

□

Theorem 9.2 **IZF** *has the* **DP** *and* **NEP** *and* **IZF** *is closed under* **CR**, **ECR**, $\mathbf{CR_1}$, **UZR**, *and* **UR**, *too.*

Proof: This follows from Theorem 9.1 and the proof of Theorem 1.2. □

Remark 9.3 Theorems 1.2 and 9.2 allow for generalizations to extensions of **CZF**, **CZF** + **REA**, and **IZF** via "true" axioms that are of the form $\neg\psi$. This follows easily from the proofs of these theorems and the fact that negated statements are self-realizing (see Lemma 5.10). As a consequence we get, for example, that if $\neg\vartheta$ is a true sentence and $\mathbf{CZF} \vdash \neg\vartheta \rightarrow (\phi \vee \psi)$, then $\mathbf{CZF} \vdash \neg\vartheta \rightarrow \phi$ or $\mathbf{CZF} \vdash \neg\vartheta \rightarrow \psi$. Likewise, $\mathbf{CZF} \vdash \neg\vartheta \rightarrow (\exists x \in \omega)\theta(x)$ implies $\mathbf{CZF} \vdash (\exists x \in \omega)[\neg\vartheta \rightarrow \theta(x)]$.

Next we extended our results to theories a classically valid principle. *Markov's Principle*, **MP**, is closely associated with the work of the school of Russian constructivists. The version of **MP** most appropriate to the set-theoretic context is the schema

$$\forall n \in \omega\, [\varphi(n) \vee \neg\varphi(n)] \;\wedge\; \neg\neg\exists n \in \omega\, \varphi(n) \;\rightarrow\; \exists n \in \omega\varphi(n).$$

The variant

$$\neg\neg\exists n \in \omega\, R(n) \rightarrow \exists n \in \omega R(n),$$

with R being a primitive recursive predicate, will be denoted by $\mathbf{MP_{PR}}$. Obviously, $\mathbf{MP_{PR}}$ is implied by **MP**.

Theorem 9.4 *Let T be any of the theories* **CZF**, **CZF** + **REA**, **IZF**, *or* **IZF** + **REA**. *For every theorem θ of $T + \mathbf{MP}$, there exists an application term t such that*

$$T + \mathbf{MP} \vdash \; (t \Vdash_{rt} \theta).$$

Moreover, the proof of this soundness theorem is effective in that the application term t can be effectively constructed from the $T + \mathbf{MP}$ proof of θ.

Proof: Arguing in $T + \mathbf{MP}$, it remains to find realizing terms for **MP**. We assume that

$$(e)_0 \;\; \Vdash_{rt} \;\; (\forall x{\in}\underline{\omega})\, [\varphi(x) \;\vee\; \neg\varphi(x)] \tag{61}$$

$$(e)_1 \;\; \Vdash_{rt} \;\; \neg\neg(\exists x{\in}\underline{\omega})\, \varphi(x). \tag{62}$$

Let $e' = (e)_0$. Unravelling the definition of $\Vdash_{rt}$ for negated formulas, it is a consequence of (62) that $(\forall d{\in}\omega)\, \neg\, (\forall f{\in}\omega)\, \neg f \Vdash_{rt} (\exists x{\in}\underline{\omega})\varphi(x)$, and hence $\neg\, (\forall f{\in}\omega)\, \neg f \Vdash_{rt} (\exists x{\in}\underline{\omega})\varphi(x)$, which implies $\neg\neg(\exists f \in \omega) f \Vdash_{rt} (\exists x{\in}\underline{\omega})\varphi(x)$ (just using intuitionistic logic), and hence

$$\neg\neg(\exists f \in \omega)(f)_1 \Vdash_{rt} \varphi[x/\underline{(f)_0}]. \tag{63}$$

(61) yields that $(\forall n{\in}\omega)e'n \downarrow$ and

$$(\forall n{\in}\omega)([(e'n)_0 = 0 \;\wedge\; (e'n)_1 \Vdash_{rt} \varphi[x/\underline{n}]] \;\vee\; [(e'n)_0 \neq 0 \;\wedge\; (e'n)_1 \Vdash_{rt} \neg\varphi[x/\underline{n}]]).$$

Since $(e'n)_1 \Vdash_{rt} \neg\varphi(\underline{n})$ entails that $\neg(e'n)_1 \Vdash_{rt} \varphi(\underline{n})$ we arrive at

$$(\forall n{\in}\omega)[\psi(n) \vee \neg\psi(n)], \tag{64}$$

where $\psi(n)$ is the formula $(e'n)_0 = 0 \wedge (e'n)_1 \Vdash_{rt} \varphi[x/\underline{n}]$. Utilizing that **MP** holds in the background theory, from (63) and (64) we can deduce that there exists a natural number m such that $\psi(m)$ is true, i.e., $(e'm)_0 = 0$ and $(e'm)_1 \Vdash_{rt} \varphi[x/\underline{m}]$. Then, with $r := \mu n.(e'n)_0 = 0$,

$$(e'r)_1 \Vdash_{rt} \varphi[x/\underline{m}].$$

r can be computed by a partial recursive function ζ from e'. Taking into account that for any instance θ of **MP** with parameters in $\mathtt{V}^*$, θ° is an instance of **MP**, too, the upshot of the foregoing is that $\lambda e.(\zeta((e)_0), ((e)_0\zeta((e)_0))_1)$ is a realizer for **MP**.

□

Theorem 9.5 *If is T any of the theories* **CZF**, **CZF + REA**, **IZF**, *or* **IZF + REA**, *then* $T +$ **MP** *has the* **DP** *and the* **NEP**, *and* $T +$ **MP** *is closed under* **CR**, **ECR**, $\mathbf{CR_1}$, **UZR**, *and* **UR**.

Proof: This follows from Theorem 9.5 and the proof of Theorem 1.2. □

10 Final remarks

1. In a sequel to this paper it will be shown that Theorem 1.2 also obtains when one augments **CZF**, **CZF + REA**, or **IZF** by the principles of Countable Choice, Dependent Choices, and the Presentation Axiom (or any subcollection thereof).

2. Theorem 1.2 has been put to use in [29], where it has been applied to the extension of **CZF** by the **ΠΣ** Axiom of Choice and that of **CZF + REA** by the **ΠΣW** Axiom of Choice. [29] Theorem 8.3 and Theorem 8.4 yield that the **DP**, **NEP** and **EP** hold for these theories, not for all formulas, but for a significant collection of formulas which comprises all statements of concern to ordinary mathematics.

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