Kripke-Platek Set Theory and the Anti-Foundation Axiom

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Abstract. The paper investigates the strength of the Anti-Foundation Axiom, \textbf{AFA}, on the basis of Kripke-Platek set theory without Foundation. It is shown that the addition of \textbf{AFA} considerably increases the proof theoretic strength.

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1. Introduction

Intrinsically circular phenomena have come to the attention of researchers in differing fields such as mathematical logic, computer science, artificial intelligence, linguistics, cognitive science, and philosophy. Logicians first explored set theories whose universe contains what are called non-wellfounded sets, or hypersets (cf. [6], [2]). But the area was considered rather exotic until these theories were put to use in developing rigorous accounts of circular notions in computer science (cf. [4]). Instead of the Foundation Axiom these set theories adopt the so-called \textit{Anti-Foundation Axiom, AFA}, which gives rise to a rich universe of sets. \textbf{AFA} provides an elegant tool for modeling all sorts of circular phenomena. The application areas range from knowledge representation and theoretical economics to the semantics of natural language and programming languages.

This paper investigates the strength of the Anti-Foundation Axiom, \textbf{AFA}, on the basis of Kripke-Platek set theory without Foundation. This system is dubbed \textbf{KPA}. It is shown that the addition of \textbf{AFA} considerably increases the proof theoretic strength. Indeed, \textbf{KPA} has the same strength as the subsystem of second order arithmetic based on $\Delta^1_2$ Comprehension.

2. The Anti-foundation Axiom

Definition 2.1. A \textit{graph} will consist of a set of \textit{nodes} and a set of \textit{edges}, each edge being an ordered pair $(x, y)$ of nodes. If $(x, y)$ is an edge then we’ll write $x \rightarrow y$ and say that $y$ is a \textit{child} of $x$.

A \textit{path} is a finite or infinite sequence $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots$ of nodes $x_0, x_1, x_2, \ldots$ linked by edges $(x_0, x_1), (x_1, x_2), \ldots$.

A \textit{pointed graph} is a graph together with a distinguished node $x_0$ called its \textit{point}. A pointed graph is \textit{accessible} if for every node $x$ there is a path $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x$ from the point $x_0$ to $x$.

A \textit{decoration} of a graph is an assignment $d$ of a set to each node of the graph in such a way that the elements of the set assigned to a node are the sets assigned to \ldots
the children of that node, i.e.
\[ d(a) = \{d(x) : a \to x \}. \]

**Definition 2.2.** The *Anti-Foundation Axiom*, **AFA**, is the statement that every graph has a unique decoration.

### 3. **AFA** in Kripke-Platek set theory

*Kripke-Platek set theory*, **KP**, is a truly remarkable subsystem of Zermelo-Fraenkel set theory. Since its beginnings in the early sixties, it has become a major source of interaction between model theory, recursion theory, set theory, and proof theory (see [3] and [10]).

This section is concerned with the strength of Kripke-Platek set theory without foundation but augmented by **AFA**. The axioms of **KP** comprise the axioms of Extensionality, Pair, Union, and Infinity\(^1\) of **ZF** and the following axioms:

- **Foundation**: \( \exists x \phi(x) \to \exists x[\phi(x) \land (\forall y \in x) \neg \phi(y)] \) for all formulae \( \phi \).
- **\( \Delta_0 \) Separation**: \( \exists x (x = \{y \in a : \psi(y)\}) \) for all \( \Delta_0 \)-formulas \( \psi \) in which \( x \) does not occur free.
- **\( \Delta_0 \) Collection**: \( (\forall x \in a) \exists y \theta(x, y) \to \exists z (\forall x \in a)(\exists y \in z) \theta(x, y) \) for all \( \Delta_0 \)-formulas \( \theta \) (in which \( z \) does not occur free).

By a \( \Delta_0 \) formula we mean a formula of set theory in which all the quantifiers appear restricted, that is have one of the forms \((\forall x \in b)\) or \((\exists x \in b)\).

**KP** arises from **ZF** by completely omitting the Power Set Axiom and restricting separation and collection to absolute predicates (cf. [3]), i.e. \( \Delta_0 \) formulas. These alterations are suggested by the informal notion of ‘predicative’. It is known from [7],[8], and [9] that **KP** proves the same arithmetic sentences as Feferman’s system \( \text{ID}_1 \) of positive, non-iterated inductive definitions (cf. [5]). Its proof-theoretic ordinal is the so-called Bachmann-Howard ordinal \( \theta_{\varepsilon_{\Omega+1}} \).

**Definition 3.1.** By **KP**\(^-\) we shall denote Kripke-Platek set theory without Foundation. On account of the lack of Foundation, the axiom of Infinity is formalized with a constant \( \omega \) for the least limit ordinal via the following axioms:

\[ (\omega_1) \quad 0 \in \omega \land (\forall y (y \in \omega \to y + 1 \in \omega)), \]
\[ (\omega_2) \quad \forall x (0 \in x \land (\forall y (y \in x \to y + 1 \in x) \to \omega \subseteq x)), \]

where \( y + 1 \) is \( y \cup \{y\} \), and \( 0 \) is the empty set, defined in the obvious way.

Unfortunately, the above two axioms don’t seem to suffice for proving the existence of the usual primitive recursive functions on \( \omega \). Therefore we shall add a third axiom,

\[ (\omega_3) \quad \text{the functions of addition and multiplication on } \omega \text{ exist.} \]

\(^1\)This contrasts with Barwise [3] where Infinity is not included in **KP**.
Suppose (3.1) were false. Let

Proof. Let

by letting

then there exists

definition of

Lemma 3.3. (KPA) If \( \prec \) is well-founded, i.e.

\[
(3.1) \quad (\forall x \in a_\prec) \ x \not\in x.
\]

Proof. Suppose (3.1) were false. Let \( n \in \omega \) be \( \prec \)-minimal with \( d_\prec(n) \in d_\prec(n) \). Then, due to the equation

\[
d_\prec(n) = \{d_\prec(m) : n \rightarrow m\} = \{d_\prec(m) : m \prec n\},
\]

there exists \( m_0 \prec n \) such that \( d_\prec(m_0) = d_\prec(n) \in d_\prec(n) \), yielding \( d_\prec(m_0) \in d_\prec(m_0) \).

This, however, contradicts the choice of \( n \).

Lemma 3.4. (KPA) If \( \prec \) is not well-founded, then there exists \( x \in a_\prec \) such that \( x \in x \).

Proof. Suppose \( \prec \) is not well-founded. Then there exists a non-empty set \( x \subseteq \omega \) such that \( (\forall n \in x)(\exists k \in x) (k \prec n) \). Let’s call a finite sequence \( (n_0, \ldots, n_k) \) of elements of \( x \) bad if \( n_0 \succ n_1 \succ n_2 \succ \ldots \succ n_k \) and for all \( i < k \), \( n_{i+1} \) is the smallest number \( j \in x \) such that \( j \prec n_i \). All primitive recursive functions on \( \omega \) are available in KPA. They include in particular pairing and projection functions, which allow the canonical coding of finite sequences. Therefore finite bad sequences can be coded as elements of \( \omega \) and the set \( B \) of finite bad sequences exists by \( \Delta_0 \) Separation. Using induction on \( \omega \) one shows that \( B \) contains arbitrarily long sequences. Thus one can select an infinite descending sequence

\[
n_0 \succ n_1 \succ n_2 \succ n_3 \succ \ldots
\]

by letting \( n_i \) be the \( i \)-th element of a bad sequence of length \( i + 1 \).

Put \( Z := \{k \in \omega : (\exists i \in \omega) n_i \prec k\} \). Define a graph \( G^* \) by letting

\[
\text{nodes}(G^*) := \omega \setminus Z \cup \{b\},
\]

where \( b := \{d_\prec(l) : l \in Z\} \), and

\[
x \ast \rightarrow y = \begin{cases} 
\text{if } x, y \in \omega \setminus Z \text{ and } y \prec x \\
\text{if } y \in \omega \setminus Z \text{ and } x = b \\
\text{if } x = y = b.
\end{cases}
\]
Note that $b \notin \omega$ since otherwise there would exist $i < j$ with $d_\prec(n_i) = d_\prec(n_j) \in d_\prec(n_i) \in b \in \omega$.

Let $d^*$ be a decoration of $G^*$.

Define a function $f$ with domain $\omega$ via
\[
 f(n) = \begin{cases} 
 d^*(n) & \text{if } n \notin Z \\
 d^*(b) & \text{otherwise.}
\end{cases}
\]

**Claim** $f$ is a decoration of $G_\prec$.

If $m \notin Z$ and $m \rightarrow l$ then $l \notin Z$ and $l \prec m$. Therefore one gets

\[
 f(m) = d^*(m) = \{d^*(l) : m \rightarrow l\} \quad (d^* \text{ is a decoration})
\]
\[
 = \{f(l) : l \prec m\}.
\]

If $m \in Z$ then there exists $n \in Z$ such that $n \prec m$, and hence
\[
 \{d^*(b)\} = \{f(n) : n \in Z \land n \prec m\}.
\]

As a result,

\[
 f(m) = d^*(b) = \{d^*(x) : b \rightarrow x\} \quad (d^* \text{ is a decoration})
\]
\[
 = \{d^*(n) : n \in \omega \setminus Z\} \cup \{d^*(b)\}
\]
\[
 = \{f(n) : n \in \omega \setminus Z \land n \prec m\} \cup \{f(n) : n \in Z \land n \prec m\}
\]
\[
 = \{f(n) : n \prec m\}.
\]

The claim now follows from (3.2) and (3.3). Since decorations are unique, $f = d_\prec$.

As $Z \neq \emptyset$, there are $m, n \in Z$ with $n \prec m$ and thus $f(m) = d^*(b) = f(n) \in f(m)$.

It follows that there exists $x$ in the range of $d_\prec$ such that $x \in x$. $\square$

**Proposition 3.5.** $(\text{KPA}) \prec$ is well-founded if and only if there exists a decoration $f$ of $G_\prec$ such that
\[
 (\forall n \in \omega)(f(n) \not\in f(n)).
\]

**Proof.** This is an immediate consequence of Lemma 3.3 and Lemma 3.4. $\square$

**Corollary 3.6.** The notion of being a well-ordering on $\omega$ is $\Delta_1$ in KPA.

**Proof.** Immediate by Proposition 3.5. $\square$
interpreted as ranging over subsets of $\omega$. For the interpretation we need to recall a
fact about $\Pi^1_1$ normal forms.

Lemma 3.7. Let $\phi(u, \vec{v}, \vec{X})$ be a $\Pi^1_1$ formula of second order arithmetic with all
free variables exhibited. Then there is an arithmetical formula $\vartheta(x, y, u, \vec{v}, \vec{X})$ such
that with $x \prec \vec{v}, \vec{X} u y$ abbreviating $\vartheta(x, y, u, \vec{v}, \vec{X})$ the theory $\text{ACA}_0$ proves
$$\forall \vec{v}, \vec{X} \forall u [\phi(u, \vec{v}, \vec{X}) \iff \text{WO}(\prec_{\vec{v}, \vec{X}} u)]$$
where $\text{WO}(\prec_{\vec{v}, \vec{X}} u)$ expresses that the relation $\prec_{\vec{v}, \vec{X}} u$ is a well-ordering on the natural
numbers.


Theorem 3.8. On account of the above interpretation, the language of second
order arithmetic can be viewed as a sublanguage of $\text{KPA}$.

(i) $\Delta^1_2\text{-CA}_0$ is a subtheory of $\text{KPA}$.

(ii) $\Delta^1_2\text{-CA}$ is a subtheory of $\text{KPA} + \text{IND}_\omega$.

Proof. It is straightforward to show that $\text{ACA}_0$ is a subtheory of $\text{KPA}$. Thus
it remains to show that (the translation of) $\Delta^1_2$ Comprehension is provable in
$\text{KPA}$. Owing to Corollary 3.6 and Lemma 3.7, every $\Pi^1_2$ formula is equivalent to a
set-theoretic $\Pi_1$-formula and every $\Sigma^1_2$ formula is equivalent to a set-theoretic $\Sigma_1$-
formula in $\text{KPA}$; hence $\Delta^1_2$ Comprehension follows from $\Delta_1$ Separation in $\text{KPA}$.

Theorem 3.9.  

(i) $\text{KPA}$ can be interpreted in $\Delta^1_2\text{-CA}_0$.

(ii) $\text{KPA} + \text{IND}_\omega$ can be interpreted in $\Delta^1_2\text{-CA}$.

Proof. The interpretation we have in mind is derived from the one that was used
for interpreting set theory with $\text{AFA}$ in standard set theory with Foundation. Sets
are interpreted as equivalence classes on graphs on the natural numbers, whereby
two graphs are equivalent if they are bisimulable. The notion of bisimulation is
$\Sigma^1_1$. As a result, a set-theoretic $\Pi_1$ formula gets translated into a $\Pi^1_1$ formula and
a set-theoretic $\Sigma_1$ formula gets translated into a $\Sigma^1_2$ formula. Under this interpre-
tation, $\Delta_1$ Separation is provable using $\Delta^1_2$ Comprehension and $\Delta_0$ Collection is a
consequence of the $\Sigma^1_2$ Axiom of Choice which is provable in $\Delta^1_2\text{-CA}_0$ (cf. [11]).

Corollary 3.10.  

(i) $\text{KPA}$ and $\Delta^1_2\text{-CA}_0$ have the same proof-theoretic strength.

(ii) $\text{KPA} + \text{IND}_\omega$ and $\Delta^1_2\text{-CA}$ have the same proof-theoretic strength.

Proof. This follows from Theorem 3.8 and Theorem 3.9.

REFERENCES

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