EXPLICIT MATHEMATICS WITH MONOTONE INDUCTIVE DEFINITIONS: A SURVEY

MICHAEL RATHJEN

Abstract. This paper provides a survey of results about the strength of Feferman's theory of explicit mathematics augmented by least fixed point principles for monotone inductive definitions. The paper gathers all the results that are known up till now and also proves a heretofore unpublished result concerning an upper bound for the strength of $\mathbf{T_0} + \mathbf{MID}$.

§1. Introduction. Prompted by the question of constructive justification of Spector's consistency proof for analysis, Kreisel initiated in 1963 the study of formal theories featuring inductive definitions (cf. [15]). Proof-theoretic investigations (cf. [1], [5], [17]) of such theories have shown that the strength of monotone inductive definitions is not greater than that of positive or even accessibility inductive definitions, and the strength is the same regardless of whether intuitionistic or classical logic is being assumed. However, the status of monotone inductive definitions in a more general constructive setting, like Feferman's explicit mathematics $\mathbf{T_0}$, remained open. $\mathbf{T_0}$ is a formal framework serving many purposes. It is suitable for representing Bishop-style constructive mathematics as well as generalized recursion, including direct expression of structural concepts which admit self-application.

Let **MID** be the axiom asserting the existence of a least fixed point for any monotone operation f on classifications (the notion of set in explicit mathematics), and let **UMID** be its uniform rendering, where a least solution $\mathbf{lfp}(f)$ is presented as a function of the operation by adjoining a new constant **lfp** to the language of $\mathbf{T_0}$.

The question of the strength of systems of explicit mathematics with **MID** and **UMID** was raised by Feferman in [5]; we quote:

> *What is the strength of* $\mathbf{T}_0 + \mathbf{MID}$*? [...] I have tried, but did not succeed, to extend my interpretation of* $\mathbf{T_0}$ *in* $\Sigma^1_2 - AC + BI$ *to include the statement* **MID**. *The theory* $\mathbf{T}_0 + \mathbf{MID}$ *includes all constructive formulations of iteration of monotone inductive definitions of which I am aware, while* $\mathbf{T_0}$ *(in its IG axiom) is based squarely on the general iteration of accessibility inductive definitions. Thus it would*

Written in honor of Professor Sol Feferman's seventieth birthday.

Meeting

be of great interest for the present subject to settle the relationship between these theories. (p. 88)

This paper provides a survey of results about the strength of Feferman's theory of explicit mathematics augmented by least fixed point principles for monotone inductive definitions.

Section 2 introduces the formal version of explicit mathematics used in this paper, and fixes the terminology for monotone fixed point principles. Section 3 provides a brief history of results about monotone fixed point principles. An upper bound for the strength of $\mathbf{T}_0 + \mathbf{MID}$ is proved in section 4.

S2. Some background on explicit mathematics.

2.1. Feferman's $\mathbf{T_0}$. The following presents the main features of $\mathbf{T_0}$; for full details cf. [3, 4]. The language of $\mathbf{T_0}$, $\mathcal{L}_{EM_0}$, is two-sorted, with individual variable $a, b, c, \ldots, x, y, z, \ldots$ and *classification* variables $A, B, C, \ldots, X, Y, Z$, however, diverging from tradition, we do not wish these sorts to be disjoint realms of objects. The indended constructive interpretation is that individual variables range over a universe V of finite symbolic expressions, and the classification variables over the subuniverse of V which define properties of individuals.

$\mathbf{N}$ is a classification constant taken to define the class of natural numbers. $\mathbf{0}$, $\mathbf{s_N}$ and $\mathbf{p_N}$ are operation constants whose intended interpretations are the natural number 0 and the successor and predecessor operations. Additional operation constants are $\mathbf{k}$, $\mathbf{s}$, $\mathbf{d}$, $\mathbf{p}$, $\mathbf{p_0}$ and $\mathbf{p_1}$ for the two basic combinators, definition by cases on $\mathbf{N}$, pairing and the corresponding two projections. Additional classification constants are generated using the axioms and the constants $\mathbf{j}$, $\mathbf{i}$ and $\mathbf{c_n}(\mathbf{n} < \omega)$ for *join*, *induction on well-founded parts* and *comprehension*.

There is no arity associated with the various constants. The *terms* of $\mathbf{T_0}$ are just the variables and constants of the two sorts. The atomic formulae of $\mathbf{T_0}$ are built up using the terms and three primitive relation symbols $=$, $\mathbf{App}$, and $\mathring{\in}$ as follows. If q, r, r_1, r_2 are terms, then $q = r$, $\mathbf{App}(q, r_1, r_2)$, and $q \mathring{\in} P$ (where P has to be a classification variable or constant) are atomic formulae. $\mathbf{App}(q, r_1, r_2)$ expresses that the operation q applied to r_1 yields the value r_2; $q \mathring{\in} P$ asserts[1] that q is in P or that q is classified under P.

We write $t_1 t_2 \simeq t_3$ for $\mathbf{App}(t_1, t_2, t_3)$.

Formulae are then generated from atomic formulae using the propositional connectives and the two quantifiers of each sort.

In order to facilitate the formulation of the axioms, the language of $\mathbf{T_0}$ is expanded definitionally with the symbol $\simeq$ and the auxiliary notion of an *application term* is introduced. The set of application terms is given by two clauses:

---

[1]We use the symbol "$\mathring{\in}$" instead of "$\in$", the latter being reserved for the set–theoretic elementhood relation.

a) all terms of $\mathbf{T_0}$ are application terms; and
b) if s and t are application terms, then (st) is an application term.

For s and t application terms, we have auxiliary, defined formulae of the form:

$$s \simeq t \ := \ \forall y (s \simeq y \leftrightarrow t \simeq y),$$

if t is not a variable. Here $s \simeq a$ (for a a free variable) is inductively defined by:

$$s \simeq a \quad \text{is} \quad \begin{cases} s = a, & \text{if } s \text{ is a variable} \\ & \text{or a constant,} \\ \exists x, y[s_1 \simeq x \wedge s_2 \simeq y \wedge \mathbf{App}(x, y, a)] & \text{if } s \text{ is an application} \\ & \text{term } (s_1 s_2). \end{cases}$$

Some abbreviations are $t_1 t_2 \dots t_n$ for $((...(t_1 t_2)...)t_n)$; $t \downarrow$ for $\exists y(t \simeq y)$ and $\phi(t)$ for $\exists y(t \simeq y \wedge \phi(y))$. If s, t are application terms, where t is not a classification constant or variable, then $s \mathrel{\mathring{\in}} t$ is short for $\exists X\, [t \simeq X \wedge s \mathrel{\mathring{\in}} X]$.

Some further conventions are useful. Systematic notation for *n-tuples* is introduced as follows: (t) is t, (s, t) is $\mathbf{p}st$, and $(t_1, \dots, t_n)$ is defined by $((t_1, \dots, t_{n-1}), t_n)$. t' is written for the term $\mathbf{s_N}t$, and $\bot$ is the atomic formula $\mathbf{0} \simeq \mathbf{0}'$. $s \mathrel{\mathring{\notin}} X$ and $s \neq t$ are short for $\neg(s \mathrel{\mathring{\in}} X)$ and $\neg(s = t)$, respectively. $\forall x \mathrel{\mathring{\in}} Y(...)$ stands for $\forall x(x \mathrel{\mathring{\in}} Y \rightarrow ...)$. Similar conventions apply to $\exists$. Variables i, j, k, n, m are supposed to range over $\mathbf{N}$, i.e. $\forall n$ and $\exists n$ are short for $\forall n \mathrel{\mathring{\in}} \mathbf{N}$ and $\exists n \mathrel{\mathring{\in}} \mathbf{N}$, respectively.

A Gödel numbering for formulae is assumed in the axioms introducing the classification constants $\mathbf{c_n}$. A formula is said to be *elementary* if it contains only free occurrences of classification variables A (i.e. only as *parameters*), and even those free occurrences of A are restricted: A must occur only to the right of $\mathring{\in}$ in atomic formulae. The Gödel number $\mathbf{c_n}$ above is is the Gödel number of an elementary formula. We assume that a standard Gödel numbering has been chosen for $\mathcal{L}_{EM_0}$; if ϕ is an elementary formula and $a, b_1, \dots, b_m, A_1, \dots, A_n$ is a list of variables which includes all parameters of ϕ, then $\{x : \phi(x, b_1, \dots, b_n, A_1, \dots, A_n)\}$ stands for $\mathbf{c_n}(b_1, \dots, b_n, A_1, \dots, A_n)$, where $\mathbf{n}$ is the numeral that codes the pair of Gödel numbers

$$\langle \ulcorner \phi \urcorner, \ulcorner (a, b_1, \dots, b_m, A_1, \dots, A_n) \urcorner \rangle;$$

$\mathbf{n}$ is called the 'index' of ϕ and the list of variables.

In this paper, the **logic** of $\mathbf{T_0}$ is assumed to be that of the classical two-sorted predicate logic with identity. $\mathbf{T_0}$'s **non-logical axioms** are the following:
I. Basic Axioms.

a) $\forall X \exists x(X = x)$
b) $\mathrm{App}(a, b, c_1) \wedge \mathrm{App}(a, b, c_2) \rightarrow c_1 = c_2$

II. Applicative Axioms. ,

a) $(\mathbf{k}ab)\downarrow \wedge\, \mathbf{k}ab \simeq a$,
b) $(\mathbf{s}ab)\downarrow \wedge\, \mathbf{s}abc \simeq ac(bc)$,
c) $(\mathbf{p}a_1a_2)\downarrow \wedge (\mathbf{p_1}a)\downarrow \wedge (\mathbf{p_2}a)\downarrow \wedge\, \mathbf{p_i}(\mathbf{p}a_1a_2) \simeq a_i$ for $i = 0, 1$,
d) $c_1 \mathring{\in} \mathbf{N} \wedge c_2 \mathring{\in} \mathbf{N} \;\rightarrow\; \big[(\mathbf{d}abc_1c_2)\downarrow \wedge (c_1 = c_2 \rightarrow \mathbf{d}abc_1c_2 \simeq a) \wedge (c_1 \neq c_2 \rightarrow \mathbf{d}abc_1c_2 \simeq b)\big]$,
e) $a \mathring{\in} \mathbf{N} \wedge b \mathring{\in} \mathbf{N} \;\rightarrow\; \big[a' \downarrow \wedge\, \mathbf{p_N}(a') \simeq a \wedge \neg(a' = 0) \wedge (a' \simeq b' \rightarrow a \simeq b)\big]$.

III. Classification Axioms.

Elementary Comprehension Axiom (ECA)
$\exists X\big[X \simeq \{x : \psi(x)\} \wedge \forall x(x \mathring{\in} X \leftrightarrow \psi(x))\big]$
for each elementary formula $\psi(a)$, which may contain additional parameters.

Natural Numbers

(N1) $\quad \mathbf{0} \mathring{\in} \mathbf{N} \wedge \forall x(x \mathring{\in} \mathbf{N} \rightarrow x' \mathring{\in} \mathbf{N})$

$(\mathbf{IND_N})$ $\quad \phi(\mathbf{0}) \wedge \forall x(\phi(x) \rightarrow \phi(x')) \rightarrow (\forall x \mathring{\in} \mathbf{N})\phi(x)$

for each formula ϕ of $\mathcal{L}_{EM_0}$.

Join (J)
$\forall x \mathring{\in} A\, \exists Y\, fx \simeq Y \;\rightarrow\; \exists X\big[X \simeq \mathbf{j}(A, f) \wedge \forall z\big(z \mathring{\in} X \leftrightarrow \exists x \mathring{\in} A \exists y(z \simeq (x, y) \wedge y \mathring{\in} fx)\big)\big]$

Inductive Generation (IG)

$$\exists X\big[X \simeq \mathbf{i}(A, B) \wedge \forall x \mathring{\in} A\big(\forall y[(y, x) \mathring{\in} B \rightarrow y \mathring{\in} X] \rightarrow x \mathring{\in} X\big)$$
$$\wedge\; \big(\forall x \mathring{\in} A\, [\forall y\, ((y, x) \mathring{\in} B \rightarrow \phi(y)) \rightarrow \phi(x)] \rightarrow \forall x \mathring{\in} X\, \phi(x)\big)\big]$$

where ϕ is an arbitrary formula of $\mathbf{T_0}$.

Frequently, when **IG** is invoked, it actually suffices to use a restricted form.

Restricted Inductive Generation IG $\restriction$

$$\exists X\big[X \simeq \mathbf{i}(A, B) \wedge \forall x \mathring{\in} A\big(\forall y[(y, x) \mathring{\in} B \rightarrow y \mathring{\in} X] \rightarrow x \mathring{\in} X\big)$$
$$\wedge\; \forall Z\big(\forall x \mathring{\in} A\, [\forall y\, ((y, x) \mathring{\in} B \rightarrow y \mathring{\in} Z) \rightarrow x \mathring{\in} Z] \rightarrow \forall a \mathring{\in} X\, x \mathring{\in} Z\big)\big].$$

By $\mathbf{T_0}\restriction$ we denote the alteration of $\mathbf{T_0}$ with $\mathbf{IG}\restriction$ in place of **IG** and with **N**-induction, i.e. $\mathbf{IND_N}$, replaced by the **N-induction axiom**

$$\forall Z\, \big[0 \mathring{\in} Z \wedge \forall x\, (x \mathring{\in} Z \rightarrow x' \mathring{\in} Z) \;\rightarrow\; \forall x \mathring{\in} \mathbf{N}\, (x \mathring{\in} Z)\big].$$

2.2. Formulation of monotone inductive definition principles. In this subsection we describe the monotone inductive definition principle and its uniform version. Several other principles considered in this paper will also be described.

Definition 2.1. For *extensional equality* of classifications we use the shorthand "$\mathring{=}$", i.e.

$$X \mathrel{\mathring{=}} Y \;:=\; \forall v(v \mathrel{\mathring{\in}} X \leftrightarrow v \mathrel{\mathring{\in}} Y).$$

Further, let

$$X \mathrel{\mathring{\subseteq}} Y \;:=\; \forall v(v \mathrel{\mathring{\in}} X \rightarrow v \mathrel{\mathring{\in}} Y).$$

To state the monotone fixed point principle for subclassifications of a given classification A we introduce the following shorthands:

$\mathbf{Clop}(f,A)$ if $\forall X \mathrel{\mathring{\subseteq}} A\, \exists Y \mathrel{\mathring{\subseteq}} A\; fX \simeq Y$

$\mathbf{Ext}(f,A)$ if $\forall X \mathrel{\mathring{\subseteq}} A\, \forall Y \mathrel{\mathring{\subseteq}} A\; [X \mathrel{\mathring{=}} Y \rightarrow fX \mathrel{\mathring{=}} fY]$

$\mathbf{Mon}(f,A)$ if $\forall X \mathrel{\mathring{\subseteq}} A\, \forall Y \mathrel{\mathring{\subseteq}} A\, [X \mathrel{\mathring{\subseteq}} Y \rightarrow fX \mathrel{\mathring{\subseteq}} fY]$.

$\mathbf{Lfp}(Y,f,A)$ if $fY \mathrel{\mathring{\subseteq}} Y \;\wedge\; Y \mathrel{\mathring{\subseteq}} A \;\wedge\; \forall X \mathrel{\mathring{\subseteq}} A\; \big[fX \mathrel{\mathring{\subseteq}} X \;\rightarrow\; Y \mathrel{\mathring{\subseteq}} X\big]$

When f satisfies $\mathbf{Clop}(f,A)$, we call f a *classification operation on* A. When f satisfies $\mathbf{Clop}(f,A)$ and $\mathbf{Ext}(f,A)$, we call f *extensional* or an *extensional operation on* A. When f satisfies $\mathbf{Clop}(f,A)$ and $\mathbf{Mon}(f,A)$, we say that f is a *monotone operation on* A. Since monotonicity entails extensionality, a monotone operation is always extensional.

Now we state $\mathbf{UMID}_A$.

$\mathbf{MID}_A$ (Monotone Inductive Definition on A)

$$\forall f\,[\mathbf{Clop}(f,A) \wedge \mathbf{Mon}(f,A) \rightarrow \exists Y\, \mathbf{Lfp}(Y,f,A)].$$

$\mathbf{UMID}_A$ (Uniform Monotone Inductive Definition on A)

$$\forall f\,[\mathbf{Clop}(f,A) \wedge \mathbf{Mon}(f,A) \rightarrow \mathbf{Lfp}(\mathbf{lfp}(f),f,A)].$$

$\mathbf{UMID}_A$ states that if f is monotone on subclassifications of A, then $\mathbf{lfp}(f)$ is a least fixed point of f.

Let $\mathbf{V}$ be the universe, i.e. $\mathbf{V} := \{x : x = x\}$. By **MID** and **UMID** we denote the principles $\mathbf{MID}_{\mathbf{V}}$ and $\mathbf{UMID}_{\mathbf{V}}$, respectively.[2]

2.3. Subsystems of second order arithmetic. The language $\mathcal{L}_2$ of second-order arithmetic contains (free and bound) number variables $a,b,c,\dots,x,y,z,\dots$, (free and bound) set variables $A,B,C,\dots,X,Y,Z,\dots$, the constant 0, function symbols $Suc,+,\cdot$, and relation symbols $=,<,\in$. Suc stands for the successor function.

Terms are built up as usual. For $n \in \mathbf{N}$, let $\bar{n}$ be the canonical term denoting n. Formulae are built from the prime formulae $s = t$, $s < t$, and $s \in A$ using $\wedge,\vee,\neg,\forall x,\exists x,\forall X$ and $\exists X$ where s,t are terms.

[2]The acronym for the principle **MID** in Feferman's paper [4], section 7 was MIG $\restriction$.

We like to point out that equality in $\mathcal{L}_2$ is only a relation on numbers. However, equality of sets will be considered a defined notion, namely

$$A = B \text{ iff } \forall x[x{\in}A \Leftrightarrow x{\in}B].$$

As usual, number quantifiers are called bounded if they occur in the context $\forall x(x < s \to \dots)$ or $\exists x(x < s \wedge \dots)$ for a term s which does not contain x. The $\mathbf{\Delta}^0_0$-formulae are those formulae in which all quantifiers are bounded number quantifiers, $\mathbf{\Sigma}^0_k$-formulae are formulae of the form $\exists x_1 \forall x_2 \dots Qx_k\phi$, where ϕ is $\mathbf{\Delta}^0_0$, $\mathbf{\Pi}^0_k$-formulae are those of the form $\forall x_1 \exists x_2 \dots Qx_k\phi$. The union of all $\mathbf{\Pi}^0_k$- and $\mathbf{\Sigma}^0_k$-formulae for all $k \in \mathbb{N}$ is the class of *arithmetical* or $\mathbf{\Pi}^0_\infty$*-formulae*. The $\mathbf{\Sigma}^1_k(\mathbf{\Pi}^1_k)$-formulae are the formulae $\exists X_1 \forall X_2 \dots QX_k\phi$ (resp. $\forall X_1 \exists X_2 \dots Qx_k\phi$) for arithmetical ϕ.

The basic axioms in all theories of second-order arithmetic are the defining axioms of $0, 1, +, \cdot, <$ and the *induction axiom*

$$\forall X(0 \in X \wedge \forall x(x \in X \to x + 1 \in X) \to \forall x(x \in X)),$$

respectively the *schema of induction*

$$\mathbf{IND} \qquad \phi(0) \wedge \forall x(\phi(x) \to \phi(x + 1)) \to \forall x\phi(x),$$

where ϕ is an arbitrary $\mathcal{L}_2$-formula.

We consider the axiom schema of $\mathcal{C}$*-comprehension* for formula classes $\mathcal{C}$ which is given by

$$\mathcal{C} - \mathbf{CA} \qquad \exists X \forall x(x \in X \leftrightarrow \phi(x))$$

for all formulae $\phi \in \mathcal{C}$.

We will only consider theories containing at least $\mathbf{\Pi}^0_\infty - \mathbf{CA}$. For each axiom schema $\mathbf{Ax}$ we denote by $(\mathbf{Ax})$ the theory consisting of the basic arithmetical axioms, the schema $\mathbf{\Pi}^0_\infty - \mathbf{CA}$, the schema of induction and the schema $\mathbf{Ax}$. If we replace the schema of induction by the induction axiom, we denote the resulting theory by $(\mathbf{Ax})\restriction$.

An example for these notations is the theory $(\mathbf{\Pi}^1_1\text{-}\mathbf{CA})$ which contains the induction schema, whereas $(\mathbf{\Pi}^1_1\text{-}\mathbf{CA})\restriction$ only contains the induction axiom in addition to the comprehension schema for $\mathbf{\Pi}^1_1$-formulae.

When arguing in a particular formal theory, we also say that a formula belongs to one of the aforementioned formula classes if it is equivalent to one formula of the class over this theory.

Any set R gives rise to a binary relation $\prec_R$ defined by $y \prec_R x := 2^y \cdot 3^x{\in}R$. We also use yRx as short for $y \prec_R x$.

Using the latter coding, one can formulate the axiom of choice for formulae ϕ in $\mathcal{C}$ by

$$\mathcal{C} - \mathbf{AC} \qquad \forall x \exists Y \phi(x, Y) \to \exists Y \forall x \phi(x, Y_x).$$

BI is the so-called principle of *Bar Induction*, i.e. the axiom schema

$$\forall X \left(\mathrm{WO}(<_X) \wedge \forall n \left[\forall m <_X n\, \Phi(m) \to \Phi(n)\right] \to \forall n \Phi(n)\right)$$

for all formulae Φ of the language of second order arithmetic.

2.4. Set theories. The axiom systems for set theory considered in this paper are formulated in the usual language of set theory containing $\in$ as the only non-logical symbol besides $=$. Formulae are built from prime formulae $a \in b$ and $a = b$ by use of propositional connectives and quantifiers $\forall x, \exists x$. Bounded quantifiers $\forall x \in a$, $\exists x \in a$ are defined as usual. Δ_0-formulae are the formulae wherein all quantifiers are bounded; Σ_1-formulae are those of the form $\exists x \varphi(x)$ where $\varphi(a)$ is a Δ_0-formula. For $n > 0$, Π_n-formulae (Σ_n-formulae) are the formulae with a prefix of n alternating unbounded quantifiers starting with a universal (existential) one followed by a Δ_0-formula. The class of Σ-formulae is the smallest class of formulae containing the Δ_0-formulae which is closed under $\wedge$, $\vee$, bounded quantification and unbounded existential quantification.

Definition 2.2. Kripke-Platek set theory, **KP**, consists of the axioms of Extensionality, Pairing, Union, Infinity and of the axiom schemata of Separation and Collection for Δ_0-formulae as well as the Foundation schema for arbitrary formulae.

$\mathbf{KP}^r$ arises from **KP** by replacing the axiom schema of Foundation by the Foundation axiom

$$\forall x(\exists y(y \in x) \to \exists y(y \in x \wedge \forall z \in x(z \notin y))).$$

$\mathbf{KP}^w$ is obtained from $\mathbf{KP}^r$ by adding the schema

$$\mathbf{IND}_\omega \;\; \forall x \in \omega(\forall y \in x \phi(y) \to \phi(x)) \to \forall x \in \omega \phi(x)$$

of induction on ω to $\mathbf{KP}^r$ (for all formulae ϕ).

A non-empty transitive set which is a model of **KP** is called an *admissible set*.

Definition 2.3. **KPi** is the theory **KP** plus an axiom asserting that any set is contained in an admissible set. The theories $\mathbf{KPi}^r$ and $\mathbf{KPi}^w$ are derived as in the case of **KP**.

Definition 2.4. Σ_n-Separation (abbreviated Σ_n-**Sep**) is the schema of axioms

$$\exists z \, \forall u \left(u{\in}z \;\leftrightarrow\; [u{\in}a \wedge \phi(u)]\right)$$

for all set-theoretic Σ_n-formulae ϕ with z not free in ϕ.

Definition 2.5. We will use Gödel's constructible hierarchy $\mathbf{L} = (\mathbf{L}_\alpha)_{\alpha \in \mathrm{On}}$ in one of its usual formulations. For definiteness let

$$\mathbf{L}_0 = \emptyset, \;\; \mathbf{L}_{\alpha+1} = \mathbf{Def}(\mathbf{L}_\alpha), \;\; \mathbf{L}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha \text{ for } \lambda \in \mathrm{Lim}.$$

Here $\mathbf{Def}(x)$ is the set of all definable subsets of x.

Definition 2.6. An ordinal κ is said to be stable if $\mathbf{L}_\kappa \prec_1 \mathbf{L}$, i.e. $\mathbf{L}_\kappa$ is a Σ_1-elementary substructure of $\mathbf{L}$.

Let $\rho > \kappa$. κ is ρ-stable if $\mathbf{L}_\kappa \prec_1 \mathbf{L}_\rho$.

Lemma 2.7. **KP** $+ \Sigma_1$-**Sep** *proves that there exist recursively inaccessible ordinals* $\kappa_1 < \kappa_2 < \kappa_3$ *such that*

$$\mathbf{L}_{\kappa_1} \prec_1 \mathbf{L}_{\kappa_2} \prec_1 \mathbf{L}_{\kappa_3}.$$

In particular, $\mathbf{L}_{\kappa_3}$ *is a model of* **KPi** *plus the assertion*

$$\exists\gamma\exists\pi\,[\gamma < \pi \ \wedge \ \mathbf{L}_\gamma \prec_1 \mathbf{L}_\pi \ \wedge \ \pi \textit{ is recursively inaccessible}].$$

Proof. See [20], Lemma 2.13. ⊣

S3. A survey. Proof-theoretic investigations of formal first order theories of positive iterated inductive definitions have shown that their strength is the same for classical and intuitionistic logic.

Theorem 3.1. (Buchholz, Pohlers, Sieg 1978), [1]

$$\mathbf{pos\text{-}ID}^c_\alpha \equiv \mathbf{pos\text{-}ID}^i_\alpha \equiv \mathbf{ID}^i_\alpha(\mathcal{O}) \equiv \mathbf{ID}^{i,acc}_\alpha,$$

where α *is an effectively presented ordinal, "c" stands for classical logic, "i" stands for intuitionistic logic,* $\mathbf{ID}^i_\alpha(\mathcal{O})$ *is the intuitionistic theory of* α *times iterated tree classes.*

Feferman also showed that iterated monotone inductive definitions do not provide more strength than positive ones.

Theorem 3.2. (Feferman 1982), [5]

$\mathbf{mon\text{-}ID}^c_\alpha$ *is conservative over* $\mathbf{pos\text{-}ID}^c_\alpha$.

The same pattern propagates when one considers theories where the iteration of inductive definitions is internalized, i.e. theories of second order arithmetic with axioms stating that first order inductive definitions (with set parameters) can be iterated along any well-ordering.

Theorem 3.3. [17]

$$\mathbf{mon\text{-}Aut\text{-}ID}^c \equiv \mathbf{pos\text{-}Aut\text{-}ID}^c$$
$$\mathbf{mon\text{-}Aut\text{-}ID}^c \equiv \mathbf{pos\text{-}Aut\text{-}ID}^i$$

where $\equiv$ *stands for proof-theoretic equivalence.*

Positive inductive definitions are captured in $\mathbf{T_0}$ by the following principle:

Definition 3.4. $\mathbf{EID}^+$ is the schema of *positive elementary inductive definitions*,

$$\exists X\left[\forall u\,(\phi(u,X) \rightarrow u \mathring{\in} X) \ \wedge \ \forall Y\,[\forall u\,(\phi(u,Y) \rightarrow u \mathring{\in} Y) \ \rightarrow \ X \subseteq Y]\right]$$

for each elementary formula $\phi(u,X)$ in which X occurs only positively. To be precise, $\phi(u,X)$ may have additional free variables of both kinds which are not exhibited.

Theorem 3.5. $\mathbf{T_0} + \mathbf{EID}^+$ *is of the same strength as* $\mathbf{T_0}$.

Proof. It is readily seen that the recursion-theoretic model for $\mathbf{T_0}$ given by Feferman (cf. [6]) also validates the principle $\mathbf{EID}^+$. Moreover, the modelling can be carried out in the theory $(\mathbf{\Delta}^1_2\text{-}\mathbf{CA}) + \mathbf{BI}$ and the latter theory is known to have the same strength as $\mathbf{T_0}$ by results of Jäger and Pohlers (cf. [13], [12]). $\dashv$

The general principle of monotone inductive definitions in explicit mathematics is captured by the principles **MID** and **UMID**. First investigations in the way of strength of $\mathbf{T}_0 + \mathbf{MID}$ were begun by Takahashi (cf. [21]). It turned out that even the construction of models of $\mathbf{T}_0 + \mathbf{MID}$ was surprisingly difficult.

Theorem 3.6. (Takahashi 1989), [21]
$\mathbf{T_0} + \mathbf{MID}$ *is interpretable in the fragment of analysis with* $\mathbf{\Pi}^1_2$*-comprehension and Bar induction.*

The question whether $\mathbf{T}_0 + \mathbf{MID}$ is stronger than $\mathbf{T_0}$ was left open by Takahashi's work. New insights into the strength of **MID** came with [18]. In [18] it was shown that $\mathbf{T_0} + \mathbf{MID}$, when based on classical logic, also proves the existence of non-monotone inductive definitions that arise from arbitrary extensional operations on classifications. From the latter, one can deduce that $\mathbf{T_0} + \mathbf{MID}$ is indeed a much stronger theory than $\mathbf{T_0}$.

Any extensional operation Θ gives rise to an inductively defined class Θ^∞ that is set-theoretically introduced via

$$\Theta^\infty := \bigcup_\alpha \Theta^\alpha,$$
$$\Theta^\alpha := \Theta\big(\bigcup_{\beta<\alpha} \Theta^\beta\big) \;\cup\; \bigcup_{\beta<\alpha} \Theta^\beta,$$

where α ranges over the ordinals.

Theorem 3.7. [18]

(i) $(\mathbf{T_0}\restriction + \mathbf{MID})$ *To any extensional operator Θ there can be associated a monotone operator Υ and a total operation $x \mapsto \Theta^x$, giving a classification Θ^x for all x, such that with $<_\Upsilon$ denoting the prewellordering pertaining to Υ (according to* [18]*, Definition 3.2),*

$$\Theta^x \mathrel{\mathring{=}} \Theta\big(\bigcup_{y<_\Upsilon x} \Theta^y\big) \;\cup\; \bigcup_{y<_\Upsilon x} \Theta^y, \tag{1}$$

and, for the classification $\mathbf{I}_\Theta$ defined by

$$\mathbf{I}_\Theta := \bigcup_{x \mathring{\in} \mathbf{V}} \Theta^x, \tag{2}$$

it holds

$$\Theta(\mathbf{I}_\Theta) \mathrel{\mathring{\subseteq}} \mathbf{I}_\Theta.$$

Put differently, $\mathbf{I}_\Theta$ is a classification that arises by iterating Θ along $<_\Upsilon$ and is closed under Θ.

(ii) $\mathbf{T_0}\!\restriction + \mathbf{MID}$ *proves that* $\mathbf{T_0}$ *has a model.*

Central technical results employed in [18] are that the stage comparison theorem for monotone operators can be proved in $\mathbf{T_0}\!\restriction + \mathbf{MID}$ and that techniques for capturing closure ordinals of non-monotone operators via stage comparison pre-wellorderings of monotone operators as developed by Harrington and Kechris (cf. [10], [11]) can be recast in $\mathbf{T_0}\!\restriction + \mathbf{MID}$.

Subsequently Glaß, Rathjen and Schlüter in [8] characterized the exact proof-theoretic strength of several variants of $\mathbf{T}_0 + \mathbf{MID}$. Roughly speaking, $\mathbf{MID}$ turned out to be related to lightface Π^1_2 comprehension. [8] utilized all the previous work on $\mathbf{MID}$, that is, Takahashi's subtle model constructions in [21], Glaß' thesis [7], and the crucial [18]. In addition, [8] used techniques from generalized recursion theory relating non-monotone inductive definitions to stability in set theory (cf. [2]).

Letting Π^1_2-CA$^-$ be the principle of Π^1_2-comprehension for Π^1_2-formulae without set parameters, the exact results obtained in [8] are:

Theorem 3.8. (Glaß, Rathjen, Schlüter 1997), [8]

(i)

$$\begin{aligned}\mathbf{T_0}\!\restriction + \mathbf{MID} &\equiv_{\Sigma^1_2} (\mathbf{\Sigma^1_2}\text{-}\mathbf{AC})\!\restriction + \Pi^1_2\text{-}\mathbf{CA}^- \\ &\equiv_{\Sigma^1_2} \mathbf{KPi}^r + \exists\beta\, \mathbf{L}_\beta \prec_1 \mathbf{L}.\end{aligned}$$

(ii)

$$\begin{aligned}\mathbf{T_0}\!\restriction + (\mathbf{IND}_{\mathbb{N}}) + \mathbf{MID} &\equiv_{\Sigma^1_2} (\mathbf{\Sigma^1_2}\text{-}\mathbf{AC}) + \Pi^1_2\text{-}\mathbf{CA}^- \\ &\equiv_{\Sigma^1_2} \mathbf{KPi}^r + (\mathbf{IND}_\omega) + \exists\beta\, \mathbf{L}_\beta \prec_1 \mathbf{L},\end{aligned}$$

where $T \equiv_{\Sigma^1_2} S$ *means that* S *and* T *prove the same* Σ^1_2*-formulae of second order arithmetic.*

The strength of $\mathbf{T_0} + \mathbf{MID}$ was not determined in [8]; this problem will be addressed in the last section.

The principles $\mathbf{UMID}_A$ are more in keeping with the spirit of explicit mathematics in that the fixed point is explicitely given as a function of the monotone operator. [18] hints at a close relationship between the principle $\mathbf{UMID}_{\mathbb{N}}$ and higher recursion theory in a functional R^* of type 3 that has been studied by Harrington in the unpublished notes [9] (but see references in [14] and full statements in [16]).[3] On account of the latter, it was conjectured that $\mathbf{UMID}_{\mathbb{N}}$ is a stronger principle than $\mathbf{MID}$. The methods of [8] are not amenable to determining the strength of $\mathbf{UMID}_{\mathbb{N}}$. Results about this principle were obtained in [19] and [20]. Thus [19], Theorem 5.3 can be sharpened as follows:

---

[3] R^* is essentially the diagonalization operator for inductive definitions (conceived as type 2 objects) on ω. There is an interesting connection to the Kolmogorov R-operator. The main result of [9] is that $1-sc({}^2E, R^*) = L_{\sigma(\rho_o)} \cap 2^\omega$, where ρ_0 is the least non-projectible ordinal and $\sigma(\rho_0)$ is the least ρ_0 stable ordinal.

THEOREM 3.9. [19]
Let ϕ be a Π^1_3 sentence of second order arithmetic.

(i) *If* $(\mathbf{\Pi^1_2 - CA})\restriction \vdash \phi$, *then* $\mathbf{T_0}\restriction + \mathbf{UMID}_{\mathbb{N}} \vdash \phi^*$.
(ii) *If* $(\mathbf{\Pi^1_2 - CA}) \vdash \phi$, *then* $\mathbf{T_0}\restriction + \mathbf{IND_N} + \mathbf{UMID}_{\mathbb{N}} \vdash \phi^*$.

A reversal was established in [20].

For theories T_1, T_2, we use the notation $T_1 \leq T_2$ to signify that T_1 is proof-theoretically reducible to T_2. $T_1 < T_2$ signifies that T_2 is proof-theoretically stronger than T_1. $T_1 \equiv T_2$ stands for proof-theoretic equivalence.

THEOREM 3.10. [20]

(i) $\mathbf{T_0}\restriction + \mathbf{UMID_N} \;\equiv\; (\mathbf{\Pi^1_2 - CA})\restriction \equiv\; \mathbf{KP}^r + \Sigma_1\text{-}\mathbf{Sep}$.
(ii) $\mathbf{T_0}\restriction + \mathbf{IND_N} + \mathbf{UMID_N} \;\equiv\; (\mathbf{\Pi^1_2 - CA}) \;\equiv\; \mathbf{KP}^w + \Sigma_1\text{-}\mathbf{Sep}$.
(iii) $(\mathbf{\Pi^1_2 - CA}) \;<\; \mathbf{T_0} + \mathbf{UMID_N} \;\leq\; (\mathbf{\Pi^1_2 - CA}) + \mathbf{BI} \;\equiv\; \mathbf{KP} + \Sigma_1\text{-}\mathbf{Sep}$.

Regarding the full system $\mathbf{T_0} + \mathbf{UMID}$, I conjecture that

$$\mathbf{T_0} + \mathbf{UMID_N} \;\equiv\; \mathbf{T_0} + \mathbf{UMID} \;\equiv\; (\mathbf{\Pi^1_2 - CA}) + \mathbf{BI}.$$

In point of fact, [20] obtained a more specific results than the previous theorem. Any sentence ϕ of second order arithmetic has a canonical translation ϕ^* in the language of $\mathbf{T_0}$ (see [19], Definition 5.1).

THEOREM 3.11. [20]
Let ϕ be a Π^1_3 sentence of second order arithmetic.

(i) $(\mathbf{\Pi^1_2 - CA})\restriction \vdash \phi$ *iff* $\mathbf{T_0}\restriction + \mathbf{UMID_N} \vdash \phi^*$.
(ii) $(\mathbf{\Pi^1_2 - CA}) \vdash \phi$ *iff* $\mathbf{T_0}\restriction + \mathbf{IND_N} + \mathbf{UMID_N} \vdash \phi^*$.
(iii) *If* $\mathbf{T_0} + \mathbf{UMID}_{\mathbb{N}} \vdash \phi^*$, *then* $(\mathbf{\Pi^1_2 - CA}) + \mathbf{BI} \vdash \phi$.

REMARK 3.12. Starting with Theorem 3.7, the results of this survey assume that the underlying logic of explicit mathematics is classical logic. At present, virtually nothing is known about the strength of the corresponding systems based on intuitionistic logic.

Further lists of open problems in this area can be found in section 6 of [19] and at the end of section 5 in [20].

§4. An upper bound for the strength of $\mathbf{T_0}$+MID. In this section we will give an upper bound for the strength of $\mathbf{T_0} + \mathbf{MID}$ from which it follows that the latter system is much weaker than $\mathbf{T_0}\restriction + \mathbf{UMID_N}$.

Let $\mathcal{K}$ be the system **KP** augmented by the axiom

$$\exists\gamma\exists\pi\,[\gamma < \pi \;\wedge\; \mathbf{L}_\gamma \prec_1 \mathbf{L} \;\wedge\; \pi \textit{ is recursively inaccessible}].$$

This is a theory slightly stronger than $(\Sigma^1_2 - \mathbf{AC}) + \Pi^1_2 - \mathbf{CA}^- + \mathbf{BI}$. I conjecture that the strength of $\mathbf{T_0} + \mathbf{MID}$ is strictly in-between that of $(\Sigma^1_2 - \mathbf{AC}) + \Pi^1_2 - \mathbf{CA}^- + \mathbf{BI}$ and $\mathcal{K}$. I actually had a proof sketch for that conjecture which convinced Andreas Schlüter. The machinery for determining the exact strength of $\mathbf{T_0} + \mathbf{MID}$ seems to be available but it also appears that the law of diminishing returns would apply here in that the full presentation of that result would very likely amount to a long and technically difficult paper.

THEOREM 4.1. *A set model for* $\mathbf{T_0}+\mathbf{MID}$ *can be constructed in* $\mathcal{K}$.

COROLLARY 4.2. $\mathbf{T_0}+\mathbf{MID}$ *is proof-theoretically weaker than* $\mathbf{T_0}\restriction+\mathbf{UMID_N}$.

PROOF. This follows from Theorem 4.1, Theorem 3.10 and Lemma 2.7. ⊣

The remainder of this section is devoted to proving Theorem 4.1. We will use the machinery of [8] and therefore assume familiarity with definitions and results of [8].

$\mathcal{K}$ will be our background theory. So fix γ and π and assume that $\gamma<\pi$, $\mathbf{L}_\gamma \prec_1 \mathbf{L}$ and π is recursively inaccessible. We may also assume that γ is the least stable ordinal, i.e. γ is the least ordinal such that $\mathbf{L}_\gamma \prec_1 \mathbf{L}$. From the latter it follows that $\mathbf{L}_\gamma$ is countable (cf. [20], Lemma 2.13). As a result, γ satisfies the conditions of [8], section 4.3, and therefore the model construction of [8], section 4.3 can be carried out in $\mathcal{K}$.

DEFINITION 4.3. For sets $X, X_1, \ldots, X_k \subseteq \omega$ with $X, X_1, \ldots, X_k \in \mathbf{L}_\pi$ let $\rho(X, X_1, \ldots, X_k)$ denote the least recursively inaccessible ordinal ν with $X, X_1, \ldots, X_k \in \mathbf{L}_\nu$. Observe that the existence of $\rho(X, X_1, \ldots, X_k)$ follows from the fact that π is recursively inaccessible.

a) An operator $\Gamma : \mathbf{Pow}(\omega)\cap\mathbf{L}_\pi \to \mathbf{Pow}(\omega)\cap\mathbf{L}$ will be called a Σ_{inac+}-operator iff

$$\forall X \in \mathbf{Pow}(\omega)\cap\mathbf{L}_\pi\, \forall n \in \omega \left(n \in \Gamma(X) \Leftrightarrow \mathbf{L}_{\rho(X,X_1,\ldots,X_k)} \models \phi[n, X, X_1, \ldots, X_k]\right)$$

holds for some Σ_1-formula ϕ and $X_1, \ldots, X_k \in \mathbf{Pow}(\omega)\cap\mathbf{L}_\pi$. The sets $X_1, \ldots, X_k \subseteq \omega$ are called the parameters of Γ.

b) The iteration stages of an operator Γ are defined as

$$I_\Gamma^\alpha = \Gamma(I_\Gamma^{<\alpha}) \cup I_\Gamma^{<\alpha} \text{ where } I_\Gamma^{<\alpha} = \textstyle\bigcup_{\beta<\alpha} I_\Gamma^\beta.$$

Note however that these iterates might not exist as $I_\Gamma^\alpha = \Gamma(I_\Gamma^{<\alpha})$ is not defined when $I_\Gamma^{<\alpha} \notin \mathbf{L}_\pi$.

c) A set $X \subseteq \omega$ is called a *sub-fixed point* of an operator Γ if $\Gamma(X) \subseteq X$.

REMARK 4.4. The definition of $Y = \Gamma(X)$ is a Σ-statement, namely it expresses that the ordinal $\rho(X, X_1, \ldots, X_k)$ exists and that a certain statement holds in $\mathbf{L}_{\rho(X,X_1,\ldots,X_k)}$.

LEMMA 4.5. *If* Γ *is a* Σ_{inac+}*-operator with parameters from* L_γ*, then* $(I_\Gamma^\alpha)_{\alpha<\gamma}$ *can be defined by* Σ*-recursion in* $\mathbf{L}_\gamma$ *and therefore is an element of* $\mathbf{L}_{\gamma+1}$.

PROOF. Note that for all $\alpha<\gamma$ there exists $\alpha<\beta<\gamma$ such that β is recursively inaccessible. The rest is standard. ⊣

LEMMA 4.6. *Let* Γ *be a monotone* Σ_{inac+}*-operator with parameters from* $\mathbf{L}_\gamma$*. Then for all* $X \in \mathbf{L}_\gamma \cap \mathbf{Pow}(\omega)$ *such that* $\Gamma(X)\subseteq X$ *and all* $\alpha<\gamma$ *we have* $I_\Gamma^\alpha \subseteq X$.

Proof. Obvious induction on α. ⊣

Proposition 4.7. *If* Γ *is a* Σ_{inac+}*-operator with parameters from* L_γ*, then* $\Gamma(I_\Gamma^{<\gamma}) \subseteq I_\Gamma^{<\gamma}$.

Proof. Assume $n \in \Gamma(I_\Gamma^{<\gamma})$. Then

$$\mathbf{L}_{\rho(I_\Gamma^{<\gamma}, X_1, \ldots, X_k)} \models \phi[n, I_\Gamma^{<\gamma}, X_1, \ldots, X_k].$$

Hence

$$\exists\eta\, \mathbf{L}_{\rho(I_\Gamma^{<\eta}, X_1, \ldots, X_k)} \models \phi[n, I_\Gamma^{<\eta}, X_1, \ldots, X_k].$$

The latter formula is Σ_1 with parameters in $\mathbf{L}_\gamma$. Hence, as $\mathbf{L}_\gamma \prec_1 \mathbf{L}$, there exists $\eta < \gamma$ such that $\mathbf{L}_{\rho(I_\Gamma^{<\eta}, X_1, \ldots, X_k)} \models \phi[n, I_\Gamma^{<\eta}, X_1, \ldots, X_k]$. This implies $n \in I_\Gamma^{\eta}$ and hence $n \in I_\Gamma^{<\gamma}$. ⊣

Corollary 4.8. *Every* Σ_{inac+}*-operator* Γ *with parameters from* $\mathbf{L}_\gamma$ *has a sub-fixed point in* L_γ*.*

Moreover, if Γ *is monotone, then the least fixed point of* Γ *is in* $\mathbf{L}_\gamma$*.*

Proof. By Proposition 4.7 we have $\Gamma(I_\Gamma^{<\gamma}) \subseteq I_\Gamma^{<\gamma}$. Thus, by $\mathbf{L}_\gamma \prec_1 \mathbf{L}$, there exists $\eta < \gamma$ such that $\Gamma(I_\Gamma^{<\eta}) \subseteq I_\Gamma^{<\eta}$. Since $I_\Gamma^{<\eta} \in \mathbf{L}_\gamma$, this proves the claim.

The "moreover" part follows from the previous together with Lemma 4.6. ⊣

Section 4.2 of [8] defines structures

$$\mathfrak{S}_{M_0,\alpha} = (S, \mathbf{CL}_{M_0,\alpha}, \mathring{\in}_{M_0,\alpha}, App, N_S, \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p_0}, \mathbf{p_1}, \mathbf{d}, \mathbf{s_N}, \mathbf{p_N}, 0, (\mathbf{c}_m)_{m\in\omega}, \mathbf{j}, \mathbf{i})$$

where M_0 is a finite subset of M.[4] $\mathfrak{S}_{M_0,\alpha}$ also depends on an assignment $j_0 : M_0 \to \mathbf{Pow}(S)$. Sometimes we prefer to make this dependence explicit by writing $\mathfrak{S}^{j_0}_{M_0,\alpha}$ instead of just $\mathfrak{S}_{M_0,\alpha}$.

Lemma 4.9. *Let* M_0 *be finite and* $j_0 : M_0 \to \mathbf{Pow}(S)$ *satisfy* $\forall b \in M_0\, j_0(b) \in \mathbf{L}_\alpha$ *where* α *is recursively inaccessible, then*

(i) $\forall\beta \geq \alpha\ \mathfrak{S}_{M_0,\alpha} = \mathfrak{S}_{M_0,\beta}$.
(ii) $\mathfrak{S}_{M_0,\alpha} \models \mathbf{T_0}$.

Proof. Ad (i): It suffices to show $\mathfrak{S}^{j_0}_{M_0,\alpha} = \mathfrak{S}^{j_0}_{M_0,\alpha+1}$ as the assertion follows then by induction on β. We have to show

$$\mathrm{Cl}^{j_0}_{M_0,\alpha} = \mathrm{Cl}^{j_0}_{M_0,\alpha+1}.$$

Let $c \in \mathrm{Cl}^{j_0}_{M_0,\alpha+1}$. If c is of the form $\mathbf{j}(a, f)$, then $a \in \mathrm{CL}^{j_0}_{M_0,\alpha}$ and $\mathfrak{S}^{j_0}_{M_0,\alpha} \models \forall x \mathring{\in} a\, \exists Y(fx \simeq Y)$. Then $a \in \mathrm{CL}^{j_0}_{M_0,\delta}$ for some $\delta < \alpha$ and

$$\forall x \in S\, \exists\beta < \alpha \left[\beta \geq \delta \wedge \mathfrak{S}^{j_0}_{M_0,\beta} \models x \mathring{\in} a \to \exists Y(fx \simeq Y)\right].$$

[4] M is a set providing names for the parameters to be coded into our model (cf. [8], Section 4.1).

As the function $(\xi \mapsto \mathfrak{S}^{j_0}_{M_0,\xi})_{\xi<\alpha}$ is Σ_1- definable in $\mathbf{L}_\alpha$ it follows by Σ_1-collection in $\mathbf{L}_\alpha$ that there exists $\delta < \nu < \alpha$ such that

$$\forall x \in S\, \exists \beta < \nu \left[\beta \geq \delta \wedge \mathfrak{S}^{j_0}_{M_0,\beta} \models x \mathrel{\mathring{\in}} a \to \exists Y(fx \simeq Y)\right].$$

The latter implies $c = \mathbf{j}(a, f) \in \mathrm{Cl}^{j_0}_{M_0,\nu}$, thus $c \in \mathrm{Cl}^{j_0}_{M_0,\alpha}$.

Finally, c is of any other form than the above, then the assertion follows by similar arguments.

(ii) is an immediate consequence of (i). ⊣

For the rest of this section we fix a surjection

$$\widehat{\ } : M \to \bigcup\{X \in L_\gamma : X \in \mathbf{Pow}(S/F) \text{ for some finite } F \subseteq \mathbf{B}\}$$

as in [8], Section 4.3 and also let $\mathbf{CL}_{M,\alpha}$ and $\mathfrak{S}_{M,\alpha}$ be defined as in [8], Section 4.3, where $\mathbf{B}$ is the set of non-pairs as defined in [8], Definition 4.1.

Lemma 4.10. *Let* $\widehat{\ } : M \to \mathbf{Pow}(S)$ *and* $\mathfrak{S}_M$ *be as above. Assume that for some* $f \in S$ *and for all* $b \in M$

$$\mathfrak{S}_{M,\gamma} \models \exists Y(fb \simeq Y)$$

as well as for all $b_1, b_2 \in M$

$$\mathfrak{S}_{M,\gamma} \models b_1 \mathrel{\mathring{=}} b_2 \to fb_1 \mathrel{\mathring{=}} fb_2.$$

Then there is a finite $F \subseteq \mathbf{B}$ *and a* Σ_{inac+}*-operator* $\Gamma : \mathbf{Pow}(\omega) \cap \mathbf{L}_\pi \to \mathbf{Pow}(\omega) \cap \mathbf{L}$ *such that for all* $b \in M$ *with* $\widehat{b} \in \mathbf{Pow}(S/F)$

$$\Gamma(\widehat{b}) \in \mathbf{Pow}(S/F) \text{ and } \mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} fb \Leftrightarrow x \in \Gamma(\widehat{b}).$$

Proof. By Lemma [8], Lemma 4.14 choose $F \supseteq \mathbf{supp_B}(f)$ such that $\widehat{\ } : F \cap M \to \mathbf{Pow}(S/F)$. Write $F \cap M = \{b_1, \ldots, b_n\}$ and choose $b_0 \in M \setminus F$. Then we have $\mathfrak{S}_{M,\gamma} \models fb_0 \simeq a$ for some $a \in \mathbf{CL}_{M,\gamma}$. Let $M_0 := \{b_0, \ldots, b_n\}$.

Then $\mathbf{supp_B}(a) \cap M \subseteq \mathbf{supp_B}(f) \cup \mathbf{supp_B}(b_0) \subseteq \{b_0, \ldots, b_n\}$ and so by [8], Proposition 4.12 (for some $M_1 \supseteq M_0$ such that $a \in \mathbf{CL}_{M_1,\gamma}$) we see that $a \in \mathbf{CL}_{M_0,\gamma}$.

Define the operator $\Gamma : \mathbf{Pow}(\omega) \cap \mathbf{L}_\pi \to \mathbf{Pow}(\omega) \cap \mathbf{L}$ by

$$\Gamma(X) = \{x : \mathfrak{S}^{j_0}_{M_0,\alpha} \models x \mathrel{\mathring{\in}} a, \text{ where } \alpha = \rho(X, \widehat{b_1}, \ldots, \widehat{b_n}) \text{ and } j_0(b_0) = X, j_0(b_1) = \widehat{b_1}, \ldots, j_0(b_n) = \widehat{b_n}\}.$$

Note that (by Lemma 4.9) $a \in \mathrm{Cl}^{j_0}_{M_0,\alpha}$ when $\alpha = \rho(X, \widehat{b_1}, \ldots, \widehat{b_n})$. Obviously, Γ is a Σ_{inac+}-operator in the parameters $\widehat{b_1}, \ldots, \widehat{b_n}$.

Claim 1: If $\widehat{b} \in \mathbf{Pow}(S/F)$ and $b \notin \{b_1, \ldots, b_n\}$, and if $\sigma \in \mathbf{Aut}(\mathbf{B}/F)$, then

$$\mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} a[b_0 := b] \Leftrightarrow \sigma(x) \mathrel{\mathring{\in}} a[b_0 := b].$$

Proof of claim 1. Let $M_0' := \{b, b_1, \ldots, b_n\}$. Then $b, b_0 \notin \mathbf{supp_B}(f)$ and therefore $fb \simeq a[b_0 := b]$, consequently $\mathbf{supp_B}(a[b_0 := b]) \cap M \subseteq M_0'$, which by [8], Lemma 4.12 gives $a[b_0 := b] \in \mathbf{CL}_{M_0',\gamma}$.

For $\sigma \in \mathbf{Aut}(\mathbf{B}/F)$ and $x \in S$ choose some $b' \in M$ such that $\widehat{b} = \widehat{b'}$ but $b' \notin \mathbf{supp_B}(x) \cup \mathbf{supp_B}(\sigma(x)) \cup \{b_1, \dots, b_n\}$. Use this to define $\sigma' \in \mathbf{Aut}(\mathbf{B}/\{b'\} \cup F)$ which agrees with σ on $\mathbf{supp_B}(x)$. Then we easily compute:

$$
\begin{aligned}
(3) \qquad \mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} a[b_0 := b] &\Leftrightarrow \mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} a[b_0 := b'] \\
&\quad \text{because } fb' \simeq a[b_0 := b'] \text{ and} \\
&\quad f \text{ is extensional on } M. \\
&\Leftrightarrow \mathfrak{S}_{M,\gamma} \models \sigma'(x) \mathrel{\mathring{\in}} a[b_0 := b'] \\
&\quad \text{by [8], Lemma 4.9} \\
&\Leftrightarrow \mathfrak{S}_{M,\gamma} \models \sigma(x) \mathrel{\mathring{\in}} a[b_0 := b] \text{ by} \\
&\quad \text{extensionality of } f \text{ on } M.
\end{aligned}
$$

Claim 2: For $\widehat{b} \in \mathbf{Pow}(S/F)$,

$$\mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} fb \text{ if and only if } x \in \Gamma(\widehat{b}).$$

Proof of claim 2. First we consider the case that $b \notin \{b_1, \dots, b_n\}$. We commence by showing

$$(4) \qquad \mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} a[b_0 := b](= fb) \Leftrightarrow \mathfrak{S}^{\sim}_{M_0,\gamma} \models x \mathrel{\mathring{\in}} a$$

where the latter model is based on $\widetilde{\ } : M_0 \to \mathbf{Pow}(S/F)$ with $\widetilde{b_0} = \widehat{b}$ and $\widetilde{b_i} = \widehat{b_i}$ for $i = 1, \dots, n$. To this end, choose $\sigma \in \mathbf{Aut}(\mathbf{B}/F)$ with $\sigma(b) = b_0$ and let $M_0' = \{b, b_1, \dots, b_n\}$ and $M_1 = \{b, b_0, b_1, \dots, b_n\}$. We can extend $\widehat{\ }$ from M_0' to a mapping $\widetilde{\ } : M_1 \to \mathbf{Pow}(S/F)$ by additionally defining $\widetilde{b_0} := \widehat{b}$. Since $a[b_0 := b]$ in $\mathbf{CL}_{M_0',\gamma}$ by [8], Lemma 4.9 we have

$$
\begin{aligned}
\mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} a[b_0 := b] &\Leftrightarrow \mathfrak{S}_{M_0',\gamma} \models x \mathrel{\mathring{\in}} a[b_0 := b] \\
&\Leftrightarrow \mathfrak{S}_{M_0',\gamma} \models \sigma^{-1}(x) \mathrel{\mathring{\in}} a[b_0 := b] \text{ by claim 1.} \\
&\Leftrightarrow \mathfrak{S}^{\sim}_{M_0,\gamma} \models x \mathrel{\mathring{\in}} \sigma(a[b_0 := b]) = a
\end{aligned}
$$

where in the final equivalence we used [8], Lemma 4.9 for $M_0' \subseteq M_1$ and the mapping $\widetilde{\ } : M_1 \to \mathbf{Pow}(S/F)$.

This finishes the case when $b \notin \{b_1, \dots, b_n\}$.

If $b \in \{b_1, \dots, b_n\}$ holds, then choose $b' \notin \{b_1, \dots, b_n\}$ such that $\widehat{b'} = \widehat{b}$. By extensionality of f on M, we conclude

$$\mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} fb \Leftrightarrow x \mathrel{\mathring{\in}} fb'.$$

Hence, using (3) and (4),

$$
\begin{aligned}
(5) \qquad \mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} a[b_0 := b](= fb) &\Leftrightarrow \mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} a[b_0 := b'](= fb) \\
&\Leftrightarrow \mathfrak{S}^{\sim}_{M_0,\gamma} \models x \mathrel{\mathring{\in}} a
\end{aligned}
$$

where the latter model is also based on $\widetilde{\ }: M_0 \to \mathbf{Pow}(S/F)$ with $\widetilde{b_0} = \widehat{b'} = \widehat{b}$ and $\widetilde{b_i} = \widehat{b}_i$ for $i = 1, \ldots, n$.

Note that (by Lemma 4.9)

$$\mathfrak{S}^{\sim}_{M_0,\gamma} \models x \mathring{\in} a \Leftrightarrow \mathfrak{S}^{j_0}_{M_0,\alpha} \models x \mathring{\in} a$$

where $\alpha = \rho(\widehat{b}, \widehat{b_1}, \ldots, \widehat{b_n})$ and $j_0(b_0) = \widehat{b}, j_0(b_1) = \widehat{b_1}, \ldots, j_0(b_n) = \widehat{b_n}$. Claim 2 therefore follows from (4) and (5).

Claim 3: If $\widehat{b} \in \mathbf{Pow}(S/F)$, then $\Gamma(\widehat{b}) \in \mathbf{Pow}(S/F)$.

Proof of claim 3. Assume $\widehat{b} \in \mathbf{Pow}(S/F)$, $\sigma \in \mathbf{Aut}(\mathbf{B}/F)$, and choose $b' \notin \{b_1, \ldots, b_n\}$ such that $\widehat{b} = \widehat{b'}$. Then we have

$$\begin{aligned} x \in \Gamma(\widehat{b}) = \Gamma(\widehat{b'}) &\Leftrightarrow \mathfrak{S}_{M,\gamma} \models x \mathring{\in} fb' \\ &\Leftrightarrow \mathfrak{S}_{M,\gamma} \models x \mathring{\in} a[b_0 := b'] \\ &\Leftrightarrow \mathfrak{S}_{M,\gamma} \models \sigma(x) \mathring{\in} a[b_0 := b'] \\ &\Leftrightarrow \mathfrak{S}_{M,\gamma} \models \sigma(x) \mathring{\in} fb' \\ &\Leftrightarrow \sigma(x) \in \Gamma(\widehat{b'}) = \Gamma(\widehat{b}). \end{aligned}$$

$\dashv$

LEMMA 4.11. *Let $F \subseteq \mathbf{B}$ be finite and Γ a Σ_{inac+}-operator as in the preceding lemma such that for all $b \in M$ with $\widehat{b} \in \mathbf{Pow}(S/F)$*

$$x \in \Gamma(\widehat{b}) \text{ and } \mathfrak{S}_{M,\gamma} \models x \mathring{\in} fb \quad \Leftrightarrow \quad x \in \Gamma(\widehat{b}).$$

Define the operator $\Gamma' : \mathbf{Pow}(\omega) \cap \mathbf{L}_\pi \to \mathbf{Pow}(\omega) \cap \mathbf{L}$ by

$$\Gamma'(X) = \Gamma(\bigcup\{\mathrm{tr}_{B/F}(x) : x \in X\}).$$

Then the following hold:

a) *Γ' is a Σ_{inac+}-operator.*
b) *If f is monotone, then Γ is monotone on $\mathcal{M}_F = \mathbf{L}_\gamma \cap \mathbf{Pow}(S/F)$ and Γ' is monotone on $\mathbf{L}_\gamma$.*
c) *Let f be monotone on M. Let $X' \subseteq \omega$ be minimal in $\mathbf{L}_\gamma$ such that $\Gamma'(X') \subseteq X'$, which exists by Corollary 4.8. Then $X = \bigcup\{\mathrm{tr}_{\mathbf{B}/F}(x) : x \in X'\} \in \mathbf{Pow}(S/F)$, therefore there is some $b \in M_F$ such that $X = \widehat{b}$. For this b*

$$\mathfrak{S}_{M,\gamma} \models fb \subseteq b.$$

Moreover, for all $a \in \mathrm{Cl}_{M,\gamma}$ we can conclude

$$\mathfrak{S}_{M,\gamma} \models fa \subseteq a \to b \subseteq a.$$

PROOF. a) follows from [8], Lemma 4.4(a).

b) The monotonicity of Γ follows from that of f using the equivalence characterizing Γ. From this, the monotonicity of Γ' is obvious since $\bigcup\{\mathrm{tr}_{\mathbf{B}/F}(x) : x \in X\} \in \mathbf{Pow}(S/F)$ by [8], Lemma 4.4(b).

c) Since $\mathrm{tr}_{\mathbf{B}/F}(x) \subseteq X'$ is arithmetical in x, X' by [8], Lemma 4.4(a), the set X is in $\mathbf{L}_\gamma$ if X' is and moreover it is in $\mathbf{Pow}(S/F)$.
Note that

$$\Gamma(X) = \Gamma'(X') \subseteq X' \subseteq X$$

from which $\mathfrak{S}_{M,\gamma} \models fb \subseteq b$ follows since $b \in M_F$.
Now assume $\mathfrak{S}_{M,\gamma} \models fa \subseteq a$. By [8], Lemma 4.15 we have

$$\mathfrak{S}_{M,\gamma} \models x \mathrel{\mathring{\in}} a \Leftrightarrow x \in Y$$

for some $Y \in \mathcal{M} = \{X \in \mathbf{L}_\gamma : X \in \mathbf{Pow}(S/F) \text{ for some finite } F \subseteq \mathbf{B}\}$. Then the set $Y' = \{x \in S : \mathrm{tr}_{\mathbf{B}/F}(x) \subseteq Y\}$ is in $\mathbf{L}_\gamma \cap \mathbf{Pow}(S/F)$ by [8], Lemma 4.4. So there is some $b' \in M_F$ such that $\widehat{b'} = Y'$. Obviously also $\mathfrak{S}_{M,\gamma} \models b' \subseteq a$ and so by monotonicity of f we have $\mathfrak{S}_{M,\gamma} \models fb' \subseteq fa \subseteq a$. Since $b' \in M_F$ this means $\Gamma(Y') \subseteq Y$ and $\Gamma(Y') \in \mathcal{M}_F$.
Therefore for all $x \in \Gamma(Y')$ and $\sigma \in \mathbf{Aut}(\mathbf{B}/F)$ we have $\sigma(x) \in \Gamma(Y') \subseteq Y$, which means $\mathrm{tr}_{\mathbf{B}/F}(x) \subseteq Y$, leading to $\Gamma(Y') \subseteq Y'$. Since $Y' \in \mathcal{M}_F$, we moreover have $\Gamma'(Y') = \Gamma(Y') \subseteq Y'$. The minimality of X' yields $X' \subseteq Y'$ and thus $X \subseteq X' \subseteq Y' \subseteq Y$. But this means $\mathfrak{S}_{M,\gamma} \models b \subseteq a$. ⊣

Theorem 4.1 is a consequence of the next result.

Corollary 4.12. $\mathfrak{S}_{M,\gamma} \models \mathbf{T_0} + \mathbf{MID}$.

Proof. From Lemma 4.10 combined with Lemma 4.11 we obtain $\mathfrak{S}_{M,\gamma} \models \mathbf{MID}$. That $\mathfrak{S}_{M,\gamma}$ is a model all axioms of $\mathbf{T_0}$ except for (Join) is immediate. To show that it models (Join) as well, assume $a \in \mathrm{Cl}_{M,\gamma}$ and $\mathfrak{S}_{M,\gamma} \models \forall x \mathrel{\mathring{\in}} a\, \exists Y\, (fx \simeq Y)$. Let $M_0 := (\mathbf{supp_B}(a) \cup \mathbf{supp_B}(f)) \cap M$. By Proposition 4.13 below we get $a \in \mathrm{Cl}_{M_0,\gamma}$ and

$$\mathfrak{S}_{M_0,\gamma} \models \forall x \mathrel{\mathring{\in}} a\, \exists Y\, (fx \simeq Y)$$

and thus $\mathbf{j}(a, f) \in \mathrm{Cl}_{M_0,\gamma}$ by Lemma 4.9. Hence $\mathbf{j}(a, f) \in \mathrm{Cl}_{M,\gamma}$, verifying that $\mathfrak{S}_{M,\gamma}$ is a model of (Join). ⊣

Proposition 4.13. *Let M_0 be finite subsets of M. If $\widehat{\ } \restriction M_0 : M_0 \to \mathbf{Pow}(S/F)$ for a finite set F such that $F \cap M \subseteq M_0$, then we have for all $c \in \mathbf{CL}_{M,\alpha}$*

$$\mathbf{supp_B}(c) \cap M \subseteq M_0 \Rightarrow c \in \mathbf{CL}_{M_0,\alpha}.$$

Proof. We use induction on α. In the most important case, that is $c = \mathbf{j}(a, f)$, we use Lemma 4.14 below. The induction hypothesis guarantees the assumption of that lemma. ⊣

Lemma 4.14. *Let M_0, F be as in Proposition 4.13. If $a \in \mathbf{CL}_{M_0,\alpha}$,*

$$(\mathbf{supp_B}(a) \cup \mathbf{supp_B}(f)) \cap M \subseteq M_0 \text{ and } \forall x \mathrel{\mathring{\in}} a \exists y \mathrel{\mathring{\in}} \mathbf{CL}_{M,\alpha}[fx \simeq y]$$

and furthermore

$$\forall x \mathrel{\mathring{\in}} a \forall y \in \mathbf{CL}_{M,\alpha}[fx \simeq y \wedge \mathbf{supp_B}(y) \cap M \subseteq M_0 \Rightarrow y \in \mathbf{CL}_{M_0,\alpha}],$$

then, for $x \mathrel{\mathring{\in}} a$ *and* $fx \simeq y$ *we have* $y \in \mathbf{CL}_{M_0,\alpha}$.

Proof. Put $G := \mathbf{supp_B}(a) \cup \mathbf{supp_B}(f)$ in [8], Corollary 4.11. Then the hypotheses of [8], Corollary 4.11 are satisfied and so there is an automorphism

$$\sigma \in \mathbf{Aut}(\mathbf{B}/M_0 \cup F \cup \mathbf{supp_B}(a) \cup \mathbf{supp_B}(f))$$

such that $\sigma(x) \in a$ and $\mathbf{supp_B}(\sigma(x)) \cap M \subseteq M_0$.

From $fx \simeq y$ we therefore conclude

$$\mathbf{supp_B}(\sigma(y)) \subseteq \mathbf{supp_B}(\sigma(f)) \cup \mathbf{supp_B}(\sigma(x)) \subseteq \mathbf{supp_B}(f) \cup \mathbf{supp_B}(\sigma(x))$$

since $\sigma(f) = f$. This gives $\mathbf{supp_B}(\sigma(y)) \cap M \subseteq M_0$. Using $\sigma(y)$ instead of y in the additional hypothesis of the lemma we get $\sigma(y) \in \mathbf{CL}_{M_0,\alpha}$ and from this $y \in \mathbf{CL}_{M_0,\alpha}$ applying [8], Corollary 4.10 to σ^{-1}. $\dashv$

References.

[1] W. Buchholz, S. Feferman, W. Pohlers, W. Sieg: *Iterated inductive definitions and subsystems of analysis*, Lecture Notes in Math. 897 (Springer, Berlin, 1981) 78–142.

[2] D. Cenzer: *Ordinal recursion and inductive definitions* in: E. Fenstad and P. Hinman: *Generalized Recursion Theory I* (North-Holland, Amsterdam, 1974) 221-264.

[3] S. Feferman: *A language and axioms for explicit mathematics*, in: J.N. Crossley (ed.): *Algebra and Logic*, Lecture Notes in Math. 450 (Springer, Berlin 1975) 87–139.

[4] S. Feferman: *Constructive theories of functions and classes* in: Boffa, M., van Dalen, D., McAloon, K. (eds.), *Logic Colloquium '78* (North-Holland, Amsterdam 1979) 159–224.

[5] S. Feferman: *Monotone inductive definitions* in: Troelstra, A. S., van Dalen, D. (eds), *The L.E.J. Brouwer Centenary Symposium* (North–Holland, Amsterdam, 1982) 77–89.

[6] S. Feferman and W. Sieg: *Proof-theoretic Equivalences between classical and constructive theories of analysis*, in: W. Buchholz, S. Feferman, W. Pohlers, W. Sieg: *Iterated inductive definitions and subsystems of analysis*, Lecture Notes in Math. 897 (Springer, Berlin, 1981) 78–142.

[7] T. Glaß: *Standardstrukturen für Systeme Expliziter Mathematik*, Inaugural–Dissertation (Münster, 1993).

[8] T. Glaß, M. Rathjen, A. Schlüter: *The strength of monotone inductive definitions in explicit mathematics*, Annals of Pure and Applied Logic 85 (1997) 1–46.

[9] L.A. Harrington: *Kolmogorov's R-operator and the first nonprojectible ordinal,* mimeographed notes (1975) 13 pages.

[10] L.A. Harrington and A.S. Kechris: *On monotone versus nonmonotone induction*, Bull. Am. Math. Soc. 82, 888–890 (1976).

[11] L.A. Harrington and A.S. Kechris: *Inductive definability*, unpublished, typewritten manuscript, 8 pages.

[12] G. Jäger: *A well-ordering proof for Feferman's theory* T_0, Archiv f. Math. Logik 23 (1983) 65–77.

[13] G. Jäger and W. Pohlers: *Eine beweistheoretische Untersuchung von* $\mathbf{\Delta^1_2\text{-}\ CA + BI}$ *und verwandter Systeme*, Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathematisch–Naturwissenschaftliche Klasse (1982).

[14] A.S. Kechris: *Spector second order classes and reflection*, in: J.E. Fenstad, R.O. Gandy, G.E. Sacks: *Generalized recursion theory II* (North-Holland, Amsterdam, 1978) 147–183.

[15] G. Kreisel: *Generalized inductive definitions*, in: Stanford Report on the Foundations of Analysis (mimeographed), CH. III, Stanford 1963.

[16] T. John: *Recursion in Kolmogorov's R-operator and the ordinal* σ_3, Journal of Symbolic Logic 51 (1986) 1–11.

[17] M. Rathjen: *Untersuchungen zu Teilsystemen der Zahlentheorie zweiter Stufe und der Mengenlehre mit einer zwischen* $\Delta^1_2 - CA$ *und* $\Delta^1_2 - CA + BI$ *liegenden Beweisstärke* (Publication of the Institute for Mathematical Logic and Foundational Research of the University of Münster, 1989).

[18] M. Rathjen: *Monotone inductive definitions in explicit mathematics.* Journal of Symbolic Logic 61 (1996) 125–146.

[19] M. Rathjen: *Explicit mathematics with the monotone fixed point principle.* Journal of Symbolic Logic 63 (1998) 509–542.

[20] M. Rathjen: *Explicit mathematics with the monotone fixed point principle. II: Models.* Journal of Symbolic Logic 64 (1999) 517–550.

[21] S. Takahashi: *Monotone inductive definitions in a constructive theory of functions and classes*, Ann. Pure Appl. Logic 42 (1989) 255-279.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF LEEDS
LEEDS LS2 9JT, ENGLAND
E-mail: rathjen@amsta.leeds.ac.uk