Choice principles in constructive and classical set theories

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Abstract

The objective of this paper is to assay several forms of the axiom of choice that have been deemed constructive. In addition to their deductive relationships, the paper will be concerned with metamathematical properties effected by these choice principles and also with some of their classical models.

1 Introduction

Among the axioms of set theory, the axiom of choice is distinguished by the fact that it is the only one that one finds ever mentioned in workaday mathematics. In the mathematical world of the beginning of the 20th century, discussions about the status of the axiom of choice were important. In 1904 Zermelo proved that every set can be well-ordered by employing the axiom of choice. While Zermelo argued that it was self-evident, it was also criticized as an excessively non-constructive principle by some of the most distinguished analysts of the day. At the end of a note sent to the *Mathematische Annalen* in December 1905, Borel writes about the axiom of choice:

> *It seems to me that the objection against it is also valid for every reasoning where one assumes an arbitrary choice made an uncountable number of times, for such reasoning does not belong in mathematics.* ([10], pp. 1251-1252; translation by H. Jervell, cf. [22], p. 96.)

Borel canvassed opinions of the most prominent French mathematicians of his generation - Hadamard, Baire, and Lebesgue - with the upshot that Hadamard sided with Zermelo whereas Baire and Lebesgue seconded Borel. At first blush Borel's strident reaction against the axiom of choice utilized in Cantor's new theory of sets is surprising as the French analysts had used and continued to use choice principles routinely in their work. However, in the context of 19th century classical analysis

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only the Axiom of Dependent Choices, **DC**, is invoked and considered to be natural, while the full axiom of choice is unnecessary and even has some counterintuitive consequences.

Unsurprisingly, the axiom of choice does not have a unambiguous status in constructive mathematics either. On the one hand it is said to be an immediate consequence of the constructive interpretation of the quantifiers. Any proof of $\forall x \in A\, \exists y \in B\, \phi(x, y)$ must yield a function $f : A \to B$ such that $\forall x \in A\, \phi(x, f(x))$. This is certainly the case in Martin-Löf's intuitionistic theory of types. On the other hand, it has been observed that the full axiom of choice cannot be added to systems of extensional constructive set theory without yielding constructively unacceptable cases of excluded middle (see [11] and Proposition 3.2). In extensional intuitionistic set theories, a proof of a statement $\forall x \in A\, \exists y \in B\, \phi(x, y)$, in general, provides only a function F, which when fed a proof p witnessing $x \in A$, yields $F(p) \in B$ and $\phi(x, F(p))$. Therefore, in the main, such an F cannot be rendered a function of x alone. Choice will then hold over sets which have a canonical proof function, where a constructive function h is a canonical proof function for A if for each $x \in A$, $h(x)$ is a constructive proof that $x \in A$. Such sets having natural canonical proof functions "built-in" have been called *bases* (cf. [40], p. 841).

The objective of this paper is to assay several forms of the axiom of choice that have been deemed constructive. In addition to their deductive relationships, the paper will be concerned with metamathematical properties effected by these choice principles and also with some of their classical models. The particular form of constructivism adhered to in this paper is Martin-Löf's intuitionistic type theory (cf. [23, 24]). Set-theoretic choice principles will be considered as constructively justified if they can be shown to hold in the interpretation in type theory. Moreover, looking at set theory from a type-theoretic point of view has turned out to be valuable heuristic tool for finding new constructive choice principles.

The plan for the paper is as follows: After a brief review of the axioms of constructive Zermelo-Fraenkel set theory, **CZF**, in the second section, the third section studies the implications of full **AC** on the basis of **CZF**. A brief section 4 addresses two choice principles which have always featured prominently in constructive accounts of mathematics, namely the axioms of countable choice and dependent choices. A stronger form of choice is the *presentation axiom*, **PAx** (also known as the *existence of enough projective sets*). **PAx** is the topic of section 5. It asserts that every set is the surjective image of a set over which the axiom of choice holds. It implies countable choice as well as dependent choices. **PAx** is validated by various realizability interpretations and also by the interpretation of **CZF** in Martin-Löf type theory. On the other hand, **PAx** is usually not preserved under sheaf constructions. Moerdijk and Palmgren in their endeavour to find a categorical counterpart for constructive type theory, formulated a categorical form of an axiom of choice which they christened the *axiom of multiple choice*. The pivotal properties of this axiom are that it is preserved under the construction of sheaves and that it encapsulates "enough choice" to allow for the construction of categorical models of **CZF** plus the regular extension axiom, **REA**. Section 6 explores a purely set-theoretic version of the axiom of multiple choice, notated **AMC**, due to Peter Aczel and Alex Simpson. The main purpose of this section is to show that "almost all" models of **ZF** satisfy **AMC**. Furthermore, it is shown that in **ZF**, **AMC** implies the existence of arbitrarily large regular cardinals.

In the main, the corroboration for the constructivenes of **CZF** is owed to its interpretation in Martin-Löf type theory given by Aczel (cf. [1, 2, 3]). This interpretation is in many ways canonical and can be seen as providing **CZF** with a standard model in type theory. It will be recalled in section 7. Except for the general axiom of choice, all the foregoing choice principles are validated in this model and don't add any proof-theoretic strength. In section 8 it will be shown that an axiom of subcountability, which says that every set is the surjective image of a subset of ω, is also validated by this type of interpretation.

It is a natural desire to explore whether still stronger version of choice can be validated through this interpretation. Aczel has discerned several new principles in this way, among them are the $\mathbf{\Pi\Sigma{-}AC}$ and $\mathbf{\Pi\Sigma W{-}AC}$. In joint work with S. Tupailo we have shown that these are the strongest choice principles validated in type theory in the sense that they imply all the "mathematical" statements that are validated in type theory. Roughly speaking, the "mathematical statements" encompass all statements of workaday mathematics, or more formally, they are statements wherein the quantifiers are bounded by sets occurring in the cumulative hierarchy at levels $< \omega + \omega$. Section 9 reports on these findings. They also have metamathematical implications for the theories $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC}$ and $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC}$ such as the disjunction property and the existence property for mathematical statements, as will be shown in section 10.

2 Constructive Zermelo-Fraenkel Set Theory

Constructive set theory grew out of Myhill's endeavours (cf. [28]) to discover a simple formalism that relates to Bishop's constructive mathematics as **ZFC** relates to classical Cantorian mathematics. Later on Aczel modified Myhill's set theory to a system which he called *Constructive Zermelo-Fraenkel Set Theory*, **CZF**.

Definition: 2.1 (Axioms of **CZF**) The language of **CZF** is the first order language of Zermelo-Fraenkel set theory, LST, with the non logical primitive symbol $\in$. **CZF** is based on intuitionistic predicate logic with equality. The set theoretic axioms of axioms of **CZF** are the following:

1. **Extensionality** $\forall a\,\forall b\,(\forall y\,(y \in a \leftrightarrow y \in b) \;\rightarrow\; a = b)$.

2. **Pair** $\forall a\,\forall b\,\exists x\,\forall y\,(y \in x \leftrightarrow y = a \lor y = b)$.

3. **Union** $\forall a\,\exists x\,\forall y\,(y \in x \leftrightarrow \exists z \in a\, y \in z)$.

4. **Restricted Separation scheme** $\forall a\,\exists x\,\forall y\,(y \in x \leftrightarrow y \in a \,\land\, \varphi(y))$,

for every *restricted* formula $\varphi(y)$, where a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are restricted, i.e. of the form $\forall x \in b$ or $\exists x \in b$.

5. **Subset Collection scheme**

$$\begin{aligned}&\forall a\, \forall b\, \exists c\, \forall u\ (\forall x \in a\, \exists y \in b\ \varphi(x,y,u) \ \rightarrow \\ &\exists d \in c\ (\forall x \in a\, \exists y \in d\ \varphi(x,y,u)\ \wedge\ \forall y \in d\, \exists x \in a\ \varphi(x,y,u)))\end{aligned}$$

for every formula $\varphi(x,y,u)$.

6. **Strong Collection scheme**

$$\begin{aligned}&\forall a\ (\forall x \in a\, \exists y\ \varphi(x,y) \ \rightarrow \\ &\exists b\ (\forall x \in a\, \exists y \in b\ \varphi(x,y)\ \wedge\ \forall y \in b\, \exists x \in a\ \varphi(x,y)))\end{aligned}$$

for every formula $\varphi(x,y)$.

7. **Infinity**

$$\exists x \forall u[u \in x\ \leftrightarrow\ (0 = u\ \vee\ \exists v \in x(u = v \cup \{v\}))]$$

where $y+1$ is $y \cup \{y\}$, and 0 is the empty set, defined in the obvious way.

8. **Set Induction scheme**

$$(IND_{\in}) \qquad \forall a\ (\forall x \in a\ \varphi(x)\ \rightarrow\ \varphi(a))\ \rightarrow\ \forall a\ \varphi(a),$$

for every formula $\varphi(a)$.

From Infinity, Set Induction, and Extensionality one can deduce that there exists exactly one set x such that $\forall u[u \in x\ \leftrightarrow\ (0 = u\ \vee\ \exists v \in x(u = v \cup \{v\}))]$; this set will be denoted by ω.

The Subset Collection scheme easily qualifies for the most intricate axiom of **CZF**. It can be replaced by a single axiom in the presence of Strong Collection.

Definition: 2.2 ([1]) For sets A, B let ${}^{A}B$ be the class of all functions with domain A and with range contained in B. Let $\mathbf{mv}({}^{A}B)$ be the class of all sets $R \subseteq A \times B$ satisfying $\forall u \in A\, \exists v \in B\, \langle u, v\rangle \in R$. A set C is said to be **full in** $\mathbf{mv}({}^{A}B)$ if $C \subseteq \mathbf{mv}({}^{A}B)$ and

$$\forall R \in \mathbf{mv}({}^{A}B)\, \exists S \in C\ S \subseteq R.$$

The expression $\mathbf{mv}({}^{A}B)$ should be read as the class of **multi-valued functions** (or **multi functions**) from the set A to the set B.

Additional axioms we consider are:

Fullness: For all sets A, B there exists a set C such that C is full in $\mathbf{mv}({}^{A}B)$.

Exponentiation Axiom This axiom (abbreviated **Exp**) postulates that for sets A, B the class of all functions from A to B forms a set.

$$\forall a \forall b \exists c \forall f\, [f \in c \;\leftrightarrow\; (f : a \to b)].$$

Theorem: 2.3 Let $\mathbf{CZF}^-$ be **CZF** without Subset Collection.

(i) ($\mathbf{CZF}^-$) Subset Collection and Fullness are equivalent.

(ii) ($\mathbf{CZF}^-$) Fullness implies Exponentiation.

(iii) ($\mathbf{CZF}^-$) The Power Set axiom implies Subset Collection.

Proof: (ii) is obvious. (iii) is obvious in view of (i). For (i) see [1] or [4] Theorem 3.12. □

On the basis of classical logic and basic set-theoretic axioms, **Exp** implies the Power Set Axiom. However, the situation is radically different when the underlying logic is intuitionistic logic.

In what follows we shall use the notions of proof-theoretic equivalence of theories and proof-theoretic strength of a theory whose precise definitions can be found in [32].

Theorem: 2.4 *Let* **KP** *be Kripke-Platek Set Theory (with the Infinity Axiom) (see [5]).*

(i) **CZF** *and* $\mathbf{CZF}^-$ *are of the same proof-theoretic strength as* **KP** *and the classical theory* $\mathbf{ID}_1$ *of non-iterated positive arithmetical inductive definitions. These systems prove the same* Π^0_2 *statements of arithmetic.*

(ii) The system **CZF** *augmented by the Power Set axiom is proof-theoretically stronger than classical Zermelo Set theory,* **Z** *(in that it proves the consistency of* **Z***).*

(iii) **CZF** *does not prove the Power Set axiom.*

Proof: Let **Pow** denote the Power Set axiom. (i) follows from [33] Theorem 4.14. Also (iii) follows from [33] Theorem 4.14 as one easily sees that 2-order Heyting arithmetic has a model in $\mathbf{CZF}+\mathbf{Pow}$. Since second-order Heyting arithmetic is of the same strength as classical second-order arithmetic it follows that $\mathbf{CZF}+\mathbf{Pow}$ is stronger than classical second-order arithmetic (which is much stronger than **KP**). But more than that can be shown. Working in $\mathbf{CZF} + \mathbf{Pow}$ one can iterate the power set operation $\omega + \omega$ times to arrive at the set $V_{\omega+\omega}$ which is readily seen to be a model of intuitionistic Zermelo Set Theory, $\mathbf{Z}^i$. As **Z** can be interpreted in $\mathbf{Z}^i$ by means of a double negation translation as was shown in [15] Theorem 2.3.2, we obtain (ii). □

In view of the above, a natural question to ask is whether **CZF** proves that $\mathbf{mv}({}^A B)$ is a set for all sets A and B. This can be answered in the negative as the following the result shows.

Proposition: 2.5 *Let* $\mathcal{P}(x) := \{u : u \subseteq x\}$, *and* **Pow** *be the Power Set axiom, i.e.,* $\forall x \exists y\, y = \mathcal{P}(x)$.

(i) $(\mathbf{CZF}^-)$ $\forall A \forall B\, (\mathbf{mv}({}^A B)$ is a set$) \leftrightarrow \mathbf{Pow}$.

(ii) **CZF** *does not prove that* $\mathbf{mv}({}^A B)$ *is set for all sets* A *and* B.

Proof: (i): We argue in $\mathbf{CZF}^-$. It is obvious that Power Set implies that $\mathbf{mv}({}^A B)$ is a set for all sets A, B. Henceforth assume the latter. Let C be an arbitrary set and $D = \mathbf{mv}({}^C \{0,1\})$, where $0 := \emptyset$ and $1 := \{0\}$. By our assumption, D is a set. To every subset X of C we assign the set $X^* := \{\langle u, 0\rangle \mid u \in X\} \cup \{\langle z, 1\rangle \mid z \in C\}$. As a result, $X^* \in D$. For every $S \in D$ let $pr(S)$ be the set $\{u \in C \mid \langle u, 0\rangle \in S\}$. We then have $X = pr(X^*)$ for every set $X \subseteq C$, and thus

$$\mathcal{P}(C) \quad = \quad \{pr(S) \mid S \in D\}.$$

Since $\{pr(S) \mid S \in D\}$ is a set by Strong Collection, $\mathcal{P}(S)$ is a set as well.
(ii) follows from (i) and Theorem 2.4(iii). □

The first large set axiom proposed in the context of constructive set theory was the *Regular Extension Axiom*, **REA**, which was introduced to accommodate inductive definitions in **CZF** (cf. [1], [3]).

Definition: 2.6 A set c is said to be *regular* if it is transitive, inhabited (i.e. $\exists u\, u \in c$) and for any $u \in c$ and set $R \subseteq u \times c$ if $\forall x \in u\, \exists y\, \langle x, y\rangle \in R$ then there is a set $v \in c$ such that

$$\forall x \in u\, \exists y \in v\, \langle x, y\rangle \in R \wedge \forall y \in v\, \exists x \in u\, \langle x, y\rangle \in R.$$

We write $\mathbf{Reg}(a)$ for 'a is regular'.
REA is the principle

$$\forall x\, \exists y\, (x \in y \wedge \mathbf{Reg}(y)).$$

Theorem: 2.7 *Let* **KPi** *be Kripke-Platek Set Theory plus an axiom asserting that every set is contained in an admissible set (see [5]).*

(i) **CZF** + **REA** *is of the same proof-theoretic strength as* **KPi** *and the subsystem of second-order arithmetic with* $\mathbf{\Delta}^1_2$*-comprehension and Bar Induction.*

(ii) **CZF** + **REA** *does not prove the Power Set axiom.*

Proof: (i) follows from [33] Theorem 5.13. (ii) is a consequence of (i) and Theorem 2.4. □

Definition: 2.8 Another familiar intuitionistic set theory is *Intuitionistic Zermelo-Fraenkel Set Theory*, **IZF**, which is obtained from **CZF** by adding the Power Set Axiom and the scheme of Separation for all formulas.

What is the constructive notion of set that constructive set theory claims to be about? An answer to this question has been provided by Peter Aczel in a series of three papers on the type-theoretic interpretation of **CZF** (cf. [1, 2, 3]). These papers are based on taking Martin-Löf's predicative type theory as the most acceptable foundational framework of ideas to make precise the constructive approach to mathematics. The interpretation shows how the elements of a particular type **V** of the type theory can be employed to interpret the sets of set theory so that by using the Curry-Howard 'formulae as types' paradigm the theorems of constructive set theory get interpreted as provable propositions. This interpretation will be recalled in section 6.

3 CZF plus general choice

The **Axiom of Choice**, **AC**, asserts that for all sets I, whenever $(A_i)_{i\in I}$ is family of inhabited sets (i.e., $\forall i \in I\, \exists y \in A_i$), then there exists a function f with domain I such that $\forall i \in I\, f(i) \in A_i$.

A set I is said to be a **base** if the axioms of choice holds over I, i.e., whenever $(A_i)_{i\in I}$ is family of inhabited sets (indexed) over I, then there exists a function f with domain I such that $\forall i \in I\, f(i) \in A_i$.

Diaconescu [11] showed that the full Axiom of Choice implies certain forms of excluded middle. On the basis of **IZF**, **AC** implies excluded middle for all formulas, and hence $\mathbf{IZF} + \mathbf{AC} = \mathbf{ZFC}$. As will be shown shortly that, on the basis of **CZF**, **AC** implies the restricted principle of excluded middle, **REM**, that is the scheme $\theta \vee \neg\theta$ for all restricted formulas. Note also that, in the presence of Restricted Separation, **REM** is equivalent to the decidability of $\in$, i.e., the axiom $\forall x \forall y\, (x \in y \ \vee\ x \notin y)$. To see that the latter implies **REM** let $a = \{x \in 1 : \phi\}$, where ϕ is Δ_0. Then $0 \in a \ \vee\ 0 \notin a$ by decidability of $\in$. $0 \in a$ yields ϕ while $0 \notin a$ entails $\neg\phi$.

Lemma: 3.1 $\mathbf{CZF} + \mathbf{REM} \vdash \mathbf{Pow}$.

Proof: Let $0 := \emptyset$, $1 := \{0\}$, and $2 := \{0, 1\}$. Let B be a set. In the presence of **REM**, the usual proof that there is a one-to-one correspondence between subsets of B and the functions from B to 2 works. Thus, utilizing Exponentiation and Strong Collection, $\mathcal{P}(B)$ is a set. □

Proposition: 3.2 *(i)* $\mathbf{CZF}$ + full Separation + $\mathbf{AC} = \mathbf{ZFC}$.

(ii) $\mathbf{CZF} + \mathbf{AC} \vdash \mathbf{REM}$.

(iii) $\mathbf{CZF} + \mathbf{AC} \vdash \mathbf{Pow}$.

Proof: (i) and (ii) follow at once from Diaconescu [11] but for the readers convenience proofs will be given below. The proofs also slightly differ from [11] in that they are phrased in terms of equivalence classes and quotients.

(i): Let ϕ be an arbitrary formula. Define an equivalence relation $\sim_\phi$ on 2 by

$$\begin{array}{rcl} a \sim_\phi b & :\Leftrightarrow & a = b \vee \phi \\ [a]_{\sim_\phi} & := & \{b \in 2 \,:\, a \sim_\phi b\} \\ 2/_{\sim_\phi} & := & \{[0]_{\sim_\phi}, [1]_{\sim_\phi}\}. \end{array}$$

Note that $[0]_{\sim_\phi}$ and $[1]_{\sim_\phi}$ are sets by full Separation and thus $2/_{\sim_\phi}$ is a set, too. One easily verifies that $\sim_\phi$ is an equivalence relation.

We have

$$\forall z \in 2/_{\sim_\phi} \exists k \in 2\,(k \in z).$$

Using $\mathbf{AC}$, there is a choice function f defined on $2/_{\sim_\phi}$ such that

$$\forall z \in 2/_{\sim_\phi} [f(z) \in 2 \wedge f(z) \in z],$$

in particular, $f([0]_{\sim_\phi}) \in [0]_{\sim_\phi}$ and $f([1]_{\sim_\phi}) \in [1]_{\sim_\phi}$. Next, we are going to exploit the important fact

$$\forall n, m \in 2\,(n = m \vee n \neq m). \tag{1}$$

As $\forall z \in 2/_{\sim_\phi} [f(z) \in 2]$, we obtain

$$f([0]_{\sim_\phi}) = f([1]_{\sim_\phi}) \vee f([0]_{\sim_\phi}) \neq f([1]_{\sim_\phi})$$

by (1). If $f([0]_{\sim_\phi}) = f([1]_{\sim_\phi})$, then $0 \sim_\phi 1$ and hence ϕ holds. So assume $f([0]_{\sim_\phi}) \neq f([1]_{\sim_\phi})$. As ϕ would imply $[0]_{\sim_\phi} = [1]_{\sim_\phi}$ (this requires Extensionality) and thus $f([0]_{\sim_\phi}) = f([1]_{\sim_\phi})$, we must have $\neg\phi$. Consequently, $\phi \vee \neg\phi$.

(i) follows from the fact that $\mathbf{CZF}$ plus the schema of excluded middle for all formulas has the same provable formulas as $\mathbf{ZF}$.

(ii): If ϕ is restricted, then $[0]_{\sim_\phi}$ and $[1]_{\sim_\phi}$ are sets by Restricted Separation. The rest of the proof of (i) then goes through unchanged.

(iii) follows from (ii) and Lemma 3.1,(i). □

Remark 3.3 It is interesting to note that the form of $\mathbf{AC}$ responsible for $\mathbf{EM}$ is reminiscent of that used by Zermelo in his well-ordering proof of $\mathbb{R}$. $\mathbf{AC}$ enables one to pick a representative from each equivalence class in $2/_{\sim_\phi}$. Being finitely enumerable and consisting of subsets of $\{0, 1\}$, $2/_{\sim_\phi}$ is a rather small set, though. Adopting a pragmatic constructive stance on **AC**, one might say that choice principles are benign as long as they don't imply the decidability of $\in$ and don't destroy computational information. From this point of view, Borel's objection against Zermelo's usage of **AC** based on the size of the index set of the family is a non sequitur. As we shall see later, it makes constructive sense to assume that ${}^{\omega}\omega$ is a base, i.e., that inhabited families of sets indexed over the set of all functions from ω to ω possess a choice function. Indeed, as will be detailed in section 9, this applies to any index set generated by the set formation rules of Martin-Löf type theory. The axiom of choice is (trivially) provable in Martin-Löf type theory on account of the propositions-as-types interpretation. (Allowing for quotient types, though, would destroy this feature.) The interpretation of set theory in Martin-Löf type theory provides an illuminating criterion for singling out the sets for which the axiom of choice is validated. Those are exactly the sets which have an *injective presentation* (see Definition 7.4 and section 9) over a type. The canonical and, in general, non-injective presentation of $2/_{\sim_\phi}$ is the function $\wp$ with domain $\{0,1\}$, $\wp(0) = [0]_{\sim_\phi}$, and $\wp(1) = [1]_{\sim_\phi}$.

What is the strength of $\mathbf{CZF}+\mathbf{AC}$? From Theorem 2.4 and Proposition 3.2 it follows that $\mathbf{CZF}+\mathbf{AC}$ and $\mathbf{CZF}+\mathbf{REM}$ are hugely more powerful than **CZF**. In $\mathbf{CZF}+\mathbf{AC}$ one can show the existence of a model of $\mathbf{Z}+\mathbf{AC}$. Subset Collection is crucial here because $\mathbf{CZF}^-+\mathbf{AC}$ is not stronger than $\mathbf{CZF}^-$. To characterize the strength of $\mathbf{CZF}+\mathbf{AC}$ we introduce an extension of Kripke-Platek set theory.

Definition: 3.4 Let $\mathbf{KP}(\mathcal{P})$ be Kripke-Platek Set Theory (with Infinity Axiom) formulated in a language with a primitive function symbol $\mathcal{P}$ for the power set operation. The notion of Δ_0 formula of $\mathbf{KP}(\mathcal{P})$ is such that they may contain the symbol $\mathcal{P}$. In addition to the Δ_0-Separation and Δ_0-Collection schemes for this expanded language, $\mathbf{KP}(\mathcal{P})$ includes the defining axiom

$$\forall x\,\forall y\,[y \subseteq x \;\leftrightarrow\; y \in \mathcal{P}(x)]$$

for $\mathcal{P}$.

Theorem: 3.5 *1.* $\mathbf{CZF}$, $\mathbf{CZF}^-$, $\mathbf{CZF}^-+\mathbf{REM}$, $\mathbf{CZF}^-+\mathbf{AC}$, *and* **KP** *are of the same proof-theoretic strength. These systems prove the same* Π^0_2 *statements of arithmetic.*

2. $\mathbf{CZF}+\mathbf{REM}$ *and* $\mathbf{KP}(\mathcal{P})$ *are of the same proof-theoretic strength, while* $\mathbf{CZF}+\mathbf{AC}$ *is proof-theoretically reducible to* $\mathbf{KP}(\mathcal{P})+V=L$.

3. The strength of $\mathbf{CZF}+\mathbf{AC}$ *and* $\mathbf{CZF}+\mathbf{REM}$ *resides strictly between Zermelo Set Theory and* **ZFC**.

Proof: (1): In view of Theorem 2.4 it suffices to show that $\mathbf{CZF}^-+\mathbf{AC}$ can be reduced to **KP**. This can be achieved by making slight changes to the formulae-as-classes interpretation of **CZF** in **KP** as presented in [33] Theorem 4.11. The latter is actually a realizability interpretation, where the underlying computational structure (aka partial combinatory algebra or applicative structure) is the familiar Kleene structure, where application is defined in terms of indices of partial recursive functions, i.e., $App_{Kl}(e,n,m) := \{e\}(n) \simeq m$. Instead of the Kleene structure, one can use the applicative structure of Σ_1 partial (class) functions. By a Σ_1 partial function we mean an operation (not necessarily everywhere defined) given by relations of the form $\exists z\, \phi(e,x,y,z)$ where e is a set parameter and ϕ is a bounded formula (of set theory) not involving other free variables. It is convenient to argue on the basis of $\mathbf{KP} + V = L$ which is a theory that is Σ_1-conservative over $\mathbf{KP}$. Then there is a universal Σ_1 relation that parametrizes all Σ_1 relations. We the help of the universal Σ_1 relation and the Δ_1 wellordering $<_L$ of the constructible universe one defines the Σ_1 partial recursive set function with index e (see [5]). The formulae-as-classes interpretation of $\mathbf{CZF}^- + \mathbf{AC}$ in $\mathbf{KP} + V = L$ is obtained by using indices of Σ_1 partial recursive set functions rather than partial recursive functions on the integers. Since these indices form a proper class it is no longer possible to validate Subset Collection. All the axioms of $\mathbf{CZF}^-$ can still be validated, and in addition, $\mathbf{AC}$ is validated because if an object with a certain property exists, the hypothesis $V = L$ ensures that a search along the ordering $<_L$ finds the $<_L$-least such. Space limitations (and perhaps laziness) prevent us from giving all the details.

(2): Again, the complete proof is too long to be included in this paper. However, a sketch may be sufficient. Let us first address the reduction of $\mathbf{CZF} + \mathbf{REM}$ to $\mathbf{KP}(\mathcal{P})$. In $\mathbf{KP}(\mathcal{P})$ one can develop the theory of power E-recursive functions, which are defined by the same schemata as the E-recursive functions except that the function $\mathcal{P}(x) := \{u : u \subseteq x\}$ is thrown in as an initial function (cf. [27]). The next step is to mimic the recursive realizability interpretation of $\mathbf{CZF}$ in $\mathbf{KP}$ as given in [33]. In that interpretation, formulas of $\mathbf{CZF}$ were interpreted as types (mainly) consisting of indices of partial recursive functions and realizers were indices of partial recursive functions, too. Crucially, for the realizability interpretation at hand one has to use classes of indices of power E-recursive functions for the modelling of types and indices of power E-recursive functions as realizers. The details of the interpretation can be carried out in $\mathbf{KP}(\mathcal{P})$, and are very similar to the E-realizability techniques employed in [36]. Thereby it is important that the types associated to restricted set-theoretic formulas are interpreted as sets.

The interpretation of $\mathbf{CZF} + \mathbf{AC}$ in $\mathbf{KP}(\mathcal{P}) + V = L$ proceeds similarly to the foregoing, however, here we use indices of partial functions Σ_1-definable in the power set function as realizers. To realize $\mathbf{AC}$ we need this collection of functions to be closed under a search operator. This is were the hypothesis $V = L$ is needed.

For the reduction of $\mathbf{CZF} + \mathbf{REM}$ to $\mathbf{KP}(\mathcal{P})$ one can use techniques from the ordinal analysis of $\mathbf{KP}$. First note that the ordinals are linearly ordered in $\mathbf{CZF} + \mathbf{REM}$. The ordinal analysis of $\mathbf{KP}$ requires an ordinal representation system in which one can express the Bachmann-Howard ordinal. For reference purposes let this this be the representation system $T(\Omega)$ of [31]. The first step is develop a class size analogue of this representation system, notated OR, where the role of Ω is being played by the class of ordinals. Such a class size ordinal representation system has been developed in [30], section 4. The next step consists in finding an analogue $RS(OR)$ of the infinitary proof system $RS(\Omega)$ of [30]. Let's denote the element of OR which has all ordinals as predecessors by $\hat{\Omega}$. Similarly as the case of $RS(\Omega)$ one uses the ordinals (i.e. the elements of OR preceding $\hat{\Omega}$) to build a hierarchy of set terms. The main difference here is that rather than modelling this hierarchy on the constructible hierarchy, one uses the cumulative hierarchy V_α to accommodate the power set function $\mathcal{P}$. The embedding of $\mathbf{KP}(\mathcal{P}) + \mathbf{REM}$ into $RS(OR)$, the cut elimination theorems and the collapsing theorem for $RS(OR)$ are proved in much the same way as for $RS(\Omega)$ in [30]. Finally one has to code infinitary $RS(OR)$ derivations in $\mathbf{CZF} + \mathbf{REM}$ and prove a soundness theorem similar to [30], theorem 3.5 with L_α being replaced by V_α.

(2): We have already indicated that $V_{\omega+\omega}$ provides a set model for $\mathbf{Z}$ in $\mathbf{CZF}$+$\mathbf{REM}$. Using the reflection theorem of **ZF** one can show in **ZF** that there exists a cardinal κ such that $L_\kappa \models \mathbf{KP}(\mathcal{P}) + V = L$. □

Remark 3.6 Employing Heyting-valued semantics, N. Gambino also showed (cf. [16], Theorem 5.1.4) that $\mathbf{CZF}^- + \mathbf{REM}$ is of the same strength as $\mathbf{CZF}^-$.

The previous results show that the combination of **CZF** and the general axiom of choice has no constructive justification in Martin-Löf type theory.

4 Old acquaintances

In many a text on constructive mathematics, axioms of countable choice and dependent choices are accepted as constructive principles. This is, for instance, the case in Bishop's constructive mathematics (cf. [8] as well as Brouwer's intuitionistic analysis (cf. [40], Chap. 4, Sect. 2). Myhill also incorporated these axioms in his constructive set theory [28].

The weakest constructive choice principle we shall consider is the *Axiom of Countable Choice*, $\mathbf{AC}_\omega$, i.e. whenever F is a function with domain ω such that $\forall i \in \omega\, \exists y \in F(i)$, then there exists a function f with domain ω such that $\forall i \in \omega\, f(i) \in F(i)$.

Let xRy stand for $\langle x, y\rangle \in R$. A mathematically very useful axiom to have in set theory is the **Dependent Choices Axiom**, **DC**, i.e., for all sets a and (set) relations $R \subseteq a \times a$, whenever

$$(\forall x \in a)\,(\exists y \in a)\, xRy$$

and $b_0 \in a$, then there exists a function $f : \omega \to a$ such that $f(0) = b_0$ and

$$(\forall n \in \omega)\, f(n)Rf(n+1).$$

Even more useful in constructive set theory is the *Relativized Dependent Choices Axiom*, **RDC**.[1] It asserts that for arbitrary formulae ϕ and ψ, whenever

$$\forall x[\phi(x) \rightarrow \exists y(\phi(y) \wedge \psi(x,y))]$$

and $\phi(b_0)$, then there exists a function f with domain ω such that $f(0) = b_0$ and

$$(\forall n \in \omega)[\phi(f(n)) \wedge \psi(f(n), f(n+1))].$$

A restricted form of **RDC** where ϕ and ψ are required to be Δ_0 will be called Δ_0-**RDC**.

The *Bounded Relativized Dependent Choices Axiom*, **bRDC**, is the following schema: For all Δ_0-formulae θ and ψ, whenever

$$(\forall x \in a)\,[\theta(x) \rightarrow (\exists y \in a)\,(\theta(y) \wedge \psi(x,y))]$$

and $b_0 \in a \wedge \phi(b_0)$, then there exists a function $f : \omega \rightarrow a$ such that $f(0) = b_0$ and

$$(\forall n \in \omega)[\theta(f(n)) \wedge \psi(f(n), f(n+1))].$$

Letting $\phi(x)$ stand for $x \in a \wedge \theta(x)$, one sees that **bRDC** is a consequence of Δ_0-**RDC**.

Here are some other well known relationships.

Proposition: 4.1 (**CZF**$^-$)

(i) **DC** *implies* $\mathbf{AC}_\omega$.

(ii) **bRDC** *and* **DC** *are equivalent.*

(iii) **RDC** *implies* **DC**.

Proof: (i): If z is an ordered pair $\langle x, y\rangle$ let $1^{st}(z)$ denote x and $2^{nd}(z)$ denote y.

Suppose F is a function with domain ω such that $\forall i \in \omega\, \exists x \in F(i)$. Let $A = \{\langle i, u\rangle \mid\ i \in \omega \wedge u \in F(i)\}$. A is a set by Union, Cartesian Product and restricted Separation. We then have

$$\forall x \in A\, \exists y \in A\ xRy,$$

where $R = \{\langle x, y\rangle \in A \times A \mid 1^{st}(y) = 1^{st}(x) + 1\}$. Pick $x_0 \in F(0)$ and let $a_0 = \langle 0, x_0\rangle$. Using **DC** there exists a function $g : \omega \rightarrow A$ satisfying $g(0) = a_0$ and

$$\forall i \in \omega\,[g(i) \in A \wedge 1^{st}(g(i+1)) = 1^{st}(g(i)) + 1].$$

Letting f be defined on ω by $f(i) = 2^{nd}(g(i))$ one gets $\forall i \in \omega\ f(i) \in F(i)$.

[1] In [2], **RDC** is called the dependent choices axiom and **DC** is dubbed the axiom of limited dependent choices. We deviate from the notation in [2] as it deviates from the usage in classical set theory texts.

(ii) We argue in $\mathbf{CZF}^- + \mathbf{DC}$ to show **bRDC**. Assume

$$\forall x \in a[\phi(x) \rightarrow \exists y \in a(\phi(y) \wedge \psi(x,y))]$$

and $\phi(b_0)$, where ϕ and ψ are Δ_0. Let $\theta(x,y)$ be the formula $\phi(x) \wedge \phi(y) \wedge \psi(x,y)$ and $A = \{x \in a | \ \phi(x)\}$. Then θ is Δ_0 and A is a set by Δ_0 Separation. From the assumptions we get $\forall x \in A \, \exists y \in A \, \theta(x,y)$ and $b_0 \in A$. Thus, using **DC**, there is a function f with domain ω such that $f(0) = b_0$ and $\forall n \in \omega \, \theta(f(n), f(n+1))$. Hence we get $\forall n \in \omega \, [\phi(n) \wedge \psi(f(n), f(n+1))]$. The other direction is obvious.

(iii) is obvious. □

$\mathbf{AC}_\omega$ does not imply **DC**, not even on the basis of **ZF**.

Proposition: 4.2 $\mathbf{ZF} + \mathbf{AC}_\omega$ *does not prove* **DC**.

Proof: This was shown by Jensen [21]. □

An interesting consequence of **RDC** which is not implied by **DC** is the following:

Proposition: 4.3 ($\mathbf{CZF}^- + \mathbf{RDC}$) *Suppose* $\forall x \exists y \phi(x,y)$. *Then for every set* d *there exists a transitive set* A *such that* $d \in A$ *and*

$$\forall x \in A \, \exists y \in A \, \phi(x,y).$$

Moreover, for every set d *there exists a transitive set* A *and a function* $f : \omega \rightarrow A$ *such that* $f(0) = d$ *and* $\forall n \in \omega \, \phi(f(n), f(n+1))$.

Proof: The assumption yields that $\forall x \in b \, \exists y \phi(x,y)$ holds for every set b. Thus, by Collection and the existence of the transitive closure of a set, we get

$$\forall b \, \exists c \, [\theta(b,c) \wedge Tran(c)],$$

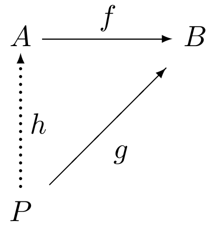
where $Tran(c)$ means that c is transitive and $\theta(b,c)$ is the formula $\forall x \in b \, \exists y \in c \, \phi(x,y)$. Let B be a transitive set containing d. Employing **RDC** there exists a function g with domain ω such that $g(0) = B$ and $\forall n \in \omega \, \theta(g(n), g(n+1))$. Obviously $A = \bigcup_{n \in \omega} g(n)$ satisfies our requirements.

The existence of the function f follows from the latter since **RDC** entails **DC**.□

5 The Presentation Axiom

The *Presentation Axiom*, **PAx**, is an example of a choice principle which is validated upon interpretation in type theory. In category theory it is also known as the *existence of enough projective sets*, **EPsets** (cf. [9]). In a category $\mathbb{C}$, an object P in $\mathbb{C}$ is *projective* (in $\mathbb{C}$) if for all objects A, B in $\mathbb{C}$, and morphisms $A \xrightarrow{f} B$,

$P \xrightarrow{g} B$ with f an epimorphism, there exists a morphism $P \xrightarrow{h} A$ such that the following diagram commutes



It easily follows that in the category of sets, a set P is projective if for any P-indexed family $(X_a)_{a \in P}$ of inhabited sets X_a, there exists a function f with domain P such that, for all $a \in P$, $f(a) \in X_a$.

PAx (or **EPsets**), is the statement that every set is the surjective image of a projective set.

Alternatively, projective sets have also been called *bases*, and we shall follow that usage henceforth. In this terminology, $\mathbf{AC}_\omega$ expresses that ω is a base whereas **AC** amounts to saying that every set is a base.

Proposition: 5.1 ($\mathbf{CZF}^-$) **PAx** *implies* **DC***.*

Proof: See [1] or [9], Theorem 6.2. □

The preceding implications cannot be reversed, not even on the basis of **ZF**.

Proposition: 5.2 $\mathbf{ZF} + \mathbf{DC}$ *does not prove* **PAx***.*

Proof: To see this, note that there are symmetric models $\mathcal{M}$ of $\mathbf{ZF}+\mathbf{DC}+\neg\mathbf{AC}$, as for instance the one used in the proof of [19], Theorem 8.3. A *symmetric model* (cf. [19], chapter 5) of **ZF** is specified by giving a ground model M of **ZFC**, a complete Boolean algebra B in M, an M-generic filter G in B, a group $\mathcal{G}$ of automorphisms of B, and a normal filter of subgroups of $\mathcal{G}$. The symmetric model consists of the elements of $M[G]$ that hereditarily have symmetric names. By [9] Theorem 4.3 and Theorem 6.1, $\mathcal{M}$ is not a model of **PAx**. □

Remark 5.3 Very little is known about the classical strength of **PAx**. It is an open problem whether **ZF** proves that **PAx** implies **AC** (cf. [9], p. 50). In P. Howard's and J.E. Rubin's book on consequences of the axiom of choice [18], this problem appears on page 322. Intuitionistically, however, **PAx** is much weaker than **AC**. In **CZF**, **AC** implies restricted excluded middle and thus the power set axiom. Moreover, in **IZF**, **AC** implies full excluded middle, whereas **PAx** does not yield any such forms of excluded middle.

Proposition: 5.4 $\mathbf{IZF} + \mathbf{PAx}$ *does not prove* **AC***. More specifically,* $\mathbf{IZF} + \mathbf{PAx}$ *does not prove that* ${}^{\omega}\omega$ *is a base.*

Proof: The realizability model $V(Kl)$ of [25] validates **PAx** ([25], Theorem 7.6) as well as Church's thesis ([25], Theorem 3.1). It is well known that ${}^{\omega}\omega$ being a basis is incompatible with Church's thesis (cf. [40], 5.7). □

6 The Axiom of Multiple Choice

A special form of choice grew out of Moerdijk's and Palmgren's endeavours to find a categorical counterpart for constructive type theory. In [26] they introduced a candidate for a notion of "predicative topos", dubbed *stratified pseudotopos*. The main results of [26] are that any stratified pseudotopos provides a model for **CZF** and that the sheaves on an internal site in a stratified pseudotopos again form a stratified pseudotopos. Their method of obtaining **CZF** models from stratified pseudotoposes builds on Aczel's original interpretation of **CZF** in type theory. They encountered, however, a hindrance which stems from the fact that Aczel's interpretation heavily utilizes that Martin-Löf type theory satisfies the axiom of choice for types. In categorical terms, the latter amounts to the principle of the existence of enough projectives. As this principle is usually not preserved under sheaf constructions, Moerdijk and Palmgren altered Aczel's construction and thereby employed a category-theoretic choice principle dubbed the *axiom of multiple choice*, **AMC**. Rather than presenting the categorical axiom of multiple choice, the following will be concerned with a purely set-theoretic version of it which was formulated by Peter Aczel and Alex Simpson (cf. [4]).

Definition: 6.1 If X is a set let $\mathbf{MV}(X)$ be the class of all multi-valued functions R with domain X, i.e., the class of all sets R of ordered pairs such that $X = \{x : \exists y \langle x, y\rangle \in R\}$. A set C *covers* $R \in \mathbf{MV}(X)$ if

$$\forall x \in X \exists y \in C[(x, y) \in R] \quad \wedge \quad \forall y \in C \exists x \in X[(x, y) \in R].$$

A class $\mathcal{Y}$ is a *cover base* for a set X if every $R \in \mathbf{MV}(X)$ is covered by an image of a set in $\mathcal{Y}$. If $\mathcal{Y}$ is a set then it is a *small cover base* for X.

We use the arrow $\twoheadrightarrow$ in $g : A \twoheadrightarrow B$ to convey that g is a surjective function from A to B. If $g : A \twoheadrightarrow B$ we also say that g is an *epi*.

Proposition: 6.2 *$\mathcal{Y}$ is a cover base for X iff for every epi $f : Z \twoheadrightarrow X$ there is an epi $g : Y \twoheadrightarrow X$, with $Y \in \mathcal{Y}$, that factors through $f : Z \to X$.*

Proof: This result is due to Aczel and Simpson; see [4]. □

Definition: 6.3 $\mathcal{Y}$ is a *(small) collection family* if it is a (small) cover base for each of its elements.

Definition: 6.4 Axiom of Multiple Choice (AMC): Every set is in some small collection family.

H-axiom: For every set A there is a smallest set $\mathbf{H}(A)$ such that if $a \in A$ and $f : a \to \mathbf{H}(A)$ then $\mathbf{ran}(f) \in \mathbf{H}(A)$.

Theorem: 6.5 (**CZF**)

1. **PAx** *implies* **AMC**.

2. **AMC** *plus* **H**-*axiom implies* **REA**.

Proof: These results are due to Aczel and Simpson; see [4]. □

Proposition: 6.6 **ZF** *does not prove that* **AMC** *implies* **PAx**.

Proof: This will follow from Corollary 6.11. □

ZF models of AMC

Definition: 6.7 There is a weak form of the axiom of choice, which holds in a plethora of **ZF** universes. The *axiom of small violations of choice*, **SVC**, has been studied by A. Blass [9]. It says in some sense, that all failure of choice occurs within a single set. **SVC** is the assertion that there is a set S such that, for every set a, there exists an ordinal α and a function from $S \times \alpha$ onto a.

Lemma: 6.8 *(i) If X is transitive and $X \subseteq B$, then $X \subseteq \mathbf{H}(B)$.*
(ii) If $2 \in B$ and $x, y \in \mathbf{H}(B)$, then $\langle x, y \rangle \in \mathbf{H}(B)$.

Proof: (i): By Set Induction on a one easily proves that $a \in X$ implies $a \in H(B)$.

(ii): Suppose $2 \in B$ and $x, y \in \mathbf{H}(B)$. Let f be the function $f : 2 \to \mathbf{H}(B)$ with $f(0) = x$ and $f(1) = y$. Then $\mathbf{ran}(f) = \{x, y\} \in \mathbf{H}(B)$. By repeating the previous procedure with $\{x\}$ and $\{x, y\}$ one gets $\langle x, y \rangle \in \mathbf{H}(B)$. □

Theorem: 6.9 (**ZF**) **SVC** *implies* **AMC** *and* **REA**.

Proof: Let M be a ground model that satisfies $\mathbf{ZF} + \mathbf{SVC}$. Arguing in M let S be a set such that, for every set a, there exists an ordinal α and a function from $S \times \alpha$ onto a.

Let $\mathbb{P}$ be the set of finite partial functions from ω to S, and, stepping outside of M, let G be an M-generic filter in $\mathbb{P}$. By the proof of [9], Theorem 4.6, $M[G]$ is a model of **ZFC**.

Let A be an arbitrary set in M. Let $B = \bigcup_{n\in\omega} F(n)$, where

$$\begin{aligned} F(0) &= \mathbf{TC}(A \cup \mathbb{P}) \cup \omega \cup \{A, \mathbb{P}\} \\ F(n+1) &= \{b \times \mathbb{P} : \ b \in \bigcup_{k \le n} F(k)\}. \end{aligned}$$

Then $B \in M$. Let $Z = (\mathbf{H}(B))^M$. Then $A \in Z$. First, we show by induction on n that $F(n) \subseteq Z$. As $F(0)$ is transitive, $F(0) \subseteq Z$ follows from Lemma 6.8, (i). Now suppose $\bigcup_{k\le n} F(k) \subseteq Z$. An element of $F(n+1)$ is of the form $b \times \mathbb{P}$ with $b \in \bigcup_{k\le n} F(k)$. If $x \in b$ and $p \in \mathbb{P}$ then $x, p \in Z$, and thus $\langle x, p\rangle \in Z$ by Lemma 6.8 since $2 \in B$. So, letting id be the identity function on $b \times \mathbb{P}$, we get $id : b \times \mathbb{P} \to Z$, and hence $\mathbf{ran}(id) = b \times \mathbb{P} \in Z$. Consequently we have $F(n+1) \subseteq Z$. It follows that $B \subseteq Z$.

We claim that

$$M \models Z \textit{ is a small collection family.} \tag{2}$$

To verify this, suppose that $x \in Z$ and $R \in M$ is a multi-valued function on x. x being an element of $(H(B))^M$, there exists a function $f \in M$ and $a \in B$ such that $f : a \to x$ and $\mathbf{ran}(f) = x$. As $M[G]$ is a model of **AC**, we may pick a function $\ell \in M[G]$ such that $\mathbf{dom}(\ell) = x$ and $\forall v \in x \; uR\ell(v)$. We may assume $x \neq \emptyset$. So let $v_0 \in x$ and pick d_0 such that $v_0 R d_0$. Let $\ddot{\ell}$ be a name for ℓ in the forcing language. For any $z \in M$ let $\check{z}$ be the canonical name for z in the forcing language. Define $\chi : a \times \mathbb{P} \to M$ by

$$\chi(u,p) := \begin{cases} w & \text{iff } f(u)Rw \text{ and} \\ & \quad p \Vdash [\ddot{\ell} \text{ is a function } \wedge \ \ddot{\ell}(\check{f}(\check{u})) = \check{w}] \\ d_0 & \text{otherwise.} \end{cases}$$

For each $u \in a$, there is a $w \in Z$ such that $f(u)Rw$ and $\ell(f(u)) = w$, and then there is a $p \in G$ that forces that $\ddot{\ell}$ is a function and $\ddot{\ell}(\check{f}(\check{u})) = \check{w}$, so w is in the range of χ. χ is a function with domain $a \times \mathbb{P}$, $\chi \in M$, and $\mathbf{ran}(\chi) \subseteq \mathbf{ran}(R)$. Note that $a \times \mathbb{P} \in B$, and thus we have $a \times \mathbb{P} \in Z$. As a result, with $C = \mathbf{ran}(\chi)$ we have $\forall v \in x \, \exists y \in C \; vRy \ \wedge \ \forall y \in C \, \exists v \in x \; vRy$, confirming the claim. □

From the previous theorem and results in [9] it follows that **AMC** and **REA** are satisfied in all permutation models and symmetric models. A *permutation model* (cf. [19], chapter 4) is specified by giving a model V of **ZFC** with atoms in which the atoms form a set A, a group $\mathcal{G}$ of permutations of A, and a normal filter $\mathcal{F}$ of subgroups of $\mathcal{G}$. The permutation model then consists of the hereditarily symmetric elements of V.

A *symmetric model* (cf. [19], chapter 5), is specified by giving a ground model M of **ZFC**, a complete Boolean algebra B in M, an M-generic filter G in B, a group $\mathcal{G}$ of automorphisms of B, and a normal filter of subgroups of $\mathcal{G}$. The symmetric model consists of the elements of $M[G]$ that hereditarily have symmetric names.

If B is a set then $\mathrm{HOD}(B)$ denotes the class of sets hereditarily ordinal definable over B.

Corollary: 6.10 *The usual models of classical set theory without choice satisfy* **AMC** *and* **REA**. *More precisely, every permutation model and symmetric model satisfies* **AMC** *and* **REA**. *Furthermore, if V is a universe that satisfies* **ZF**, *then for every transitive set $A \in V$ and any set $B \in V$ the submodels $L(A)$ and $HOD(B)$ satisfy* **AMC** *and* **REA**.

Proof: This follows from Theorem 6.9 in conjunction with [9], Theorems 4.2, 4.3, 4.4, 4.5. □

Corollary: 6.11 (**ZF**) **AMC** *and* **REA** *do not imply the countable axiom of choice,* $\mathbf{AC}_\omega$. *Moreover,* **AMC** *and* **REA** *do not imply any of the mathematical consequences of* **AC** *of [19], chapter 10. Among those consequences are the existence of a basis for any vector space and the existence of the algebraic closure of any field.*

Proof: This follows from Corollary 6.9 and [19], chapter 10. □

There is, however, one result known to follow from **AMC** that is usually considered a consequence of the axiom of choice.

Proposition: 6.12 (**ZF**) **AMC** *implies that, for any set X, there is a cardinal κ such that X cannot be mapped onto a cofinal subset of κ.*

Proof: From [35], Proposition 4.1 it follows that $\mathbf{ZF} \vdash \mathbf{H}$-*axiom*, and thus, by Theorem 6.5, we get $\mathbf{ZF} + \mathbf{AMC} \vdash \mathbf{REA}$. The assertion then follows from [35], Corollary 5.2. □

Corollary: 6.13 (**ZF**) **AMC** *implies that there are class many regular cardinals.*

Proof: If α is an ordinal then by the previous result there exists a cardinal κ such that α cannot be mapped onto a cofinal subset of κ. Let π be the cofinality of κ. Then π is a regular cardinal $> \alpha$. □

The only models of $\mathbf{ZF} + \neg\mathbf{AMC}$ known to the author are the models of

$$\mathbf{ZF} + \texttt{All uncountable cardinals are singular}$$

given by Gitik [17] who showed the consistency of the latter theory from the assumption that

$$\mathbf{ZFC} + \forall\alpha\,\exists\kappa > \alpha\,(\kappa \texttt{ is a strongly compact cardinal})$$

is consistent. This large cardinal assumption might seem exaggerated, but it is known that the consistency of all uncountable cardinals being singular cannot be proved without assuming the consistency of the existence of some large cardinals. For instance, it was shown in [12] that if $\aleph_1$ and $\aleph_2$ are both singular one can obtain an inner model with a measurable cardinal.

It would be very interesting to construct models of $\mathbf{ZF} + \neg\mathbf{AMC}$ that do not hinge on large cardinal assumptions.

7 Interpreting set theory in type theory

The basic idea of the interpretation **CZF** in Martin-Löf type theory is easily explained. The type **V** that is to be the universe of sets in type theory consists of elements of the form $\sup(A, f)$, where $A : \mathbf{U}$ and $f : A \to \mathbf{U}$. $\sup(A, f)$ may be more suggestively written as $\{f(x) : x \in A\}$. The elements of **V** are constructed inductively as families of sets indexed by the elements of a small type.

$\mathbf{ML_1V}$ is the extension of Martin-Löf type theory with one universe, $\mathbf{ML_1}$, by Aczel's set of iterative sets **V** (cf. [1]). To be more precise, $\mathbf{ML_1}$ is a type theory which has one universe **U** and all the type constructors of [24] except for the **W**-type. Indicating discharged assumptions by putting brackets around them, the natural deduction rules pertaining to **V** are as follows:[2]

$$(\mathbf{V}\text{–formation}) \quad \mathbf{V} \text{ set}$$

$$(\mathbf{V}\text{–introduction}) \quad \frac{A : \mathbf{U} \quad f : A \to \mathbf{V}}{\sup(A, f) : \mathbf{V}}$$

$$(\mathbf{V}\text{–elimination}) \quad \frac{c : \mathbf{V} \quad \begin{array}{c}[A : \mathbf{U}, f : A \to \mathbf{V}] \\ [z : (\mathbf{\Pi} v : A) C(f(v))] \\ d(A, f, z) : C(\sup(A, f))\end{array}}{\mathbf{T_V}(c, (A, f, z)d) : C(c)}$$

$$\frac{B : \mathbf{U} \quad g : B \to \mathbf{V} \quad \begin{array}{c}[A : \mathbf{U}, f : A \to \mathbf{V}] \\ [z : (\mathbf{\Pi} v : A) C(f(v))] \\ d(A, f, z) : C(\sup(A, f))\end{array}}{\mathbf{T_V}(\sup(B, g), (A, f, z)d) \ = \ d(B, g, (\lambda v)\mathbf{T_V}(g(v), (A, f, z)d)) : C(\sup(B, g))},$$

where the last rule is called (**V**–equality).

In $\mathbf{ML_1V}$ there are one-place functions assigning $\bar{\alpha} : \mathbf{U}$ and $\tilde{\alpha} : \bar{\alpha} \to \mathbf{V}$ to $\alpha : \mathbf{V}$ such that if $\alpha = \sup(A, f) : \mathbf{V}$, where $A : \mathbf{U}$ and $f : A \to \mathbf{V}$, then $\bar{\alpha} = A : \mathbf{U}$ and $\tilde{\alpha} = f : A \to \mathbf{V}$. Moreover, if $\alpha : \mathbf{V}$ then

$$\sup(\bar{\alpha}, \tilde{\alpha}) = \alpha : \mathbf{V}.$$

In the formulae-as-types interpretation of **CZF** in $\mathbf{ML_1V}$ we shall assume that **CZF** is formalized with primitive bounded quantifiers $(\forall x \in y)\phi(x)$ and $(\exists x \in y)\phi(x)$. Each formula $\psi(x_1, \ldots, x_n)$ of **CZF** (whose free variables are among $x_1, \ldots, x_n$) will be interpreted as a dependent type $\|\psi(a_1, \ldots, a_n)\|$ for $a_1 : \mathbf{V}, \ldots, a_n : \mathbf{V}$, which is also small, i.e. in **U**, if ψ contains only bounded quantifiers. The definition of $\|\psi(\alpha_1, \ldots, \alpha_n)\|$ for a non-atomic formula ψ proceeds by recursion on the build-up

[2]To increase readability and intelligibility, the formulation á la Russel is used here for type theories with universes (cf. [24]). Rather than '$a \in A$', we shall write '$a : A$' for the type-theoretic elementhood relation to distinguish it from the set-theoretic one.

of ψ and is as follows:[3]

$$\begin{array}{rcl} \|\varphi(\vec{\alpha}) \wedge \vartheta(\vec{\alpha})\| & \text{is} & \|\varphi(\vec{\alpha})\| \times \|\vartheta(\vec{\alpha})\|, \\ \|\varphi(\vec{\alpha}) \vee \vartheta(\vec{\alpha})\| & \text{is} & \|\varphi(\vec{\alpha})\| + \|\vartheta(\vec{\alpha})\|, \\ \|\varphi(\vec{\alpha}) \supset \vartheta(\vec{\alpha})\| & \text{is} & \|\varphi(\vec{\alpha})\| \to \|\vartheta(\vec{\alpha})\|, \\ \|\neg\varphi(\vec{\alpha})\| & \text{is} & \|\varphi(\vec{\alpha})\| \to N_0, \\ \|(\forall x{\in}\alpha_k)\vartheta(\vec{\alpha},x)\| & \text{is} & (\mathbf{\Pi}\, i : \overline{\alpha_k})\|\vartheta(\vec{\alpha},\widetilde{\alpha_k}(i))\|, \\ \|(\exists x{\in}\alpha_k)\vartheta(\vec{\alpha},x)\| & \text{is} & (\mathbf{\Sigma}\, i : \overline{\alpha_k})\|\vartheta(\vec{\alpha},\widetilde{\alpha_k}(i))\|, \\ \|\forall x\vartheta(\vec{\alpha},x)\| & \text{is} & (\mathbf{\Pi}\, \beta : \mathbf{V})\|\vartheta(\vec{\alpha},\beta)\|, \\ \|\exists x\vartheta(\vec{\alpha},x)\| & \text{is} & (\mathbf{\Sigma}\, \beta : \mathbf{V})\|\vartheta(\vec{\alpha},\beta)\|. \end{array}$$

To complete the above interpretation it remains to provide the types $\|\alpha = \beta\|$ and $\|\alpha \in \beta\|$ for $\alpha, \beta : \mathbf{V}$. To this end one defines by recursion on $\mathbf{V}$ a function $\doteq : \mathbf{V} \times \mathbf{V} \to \mathbf{U}$ such that

$$\alpha \doteq \beta \quad \text{is} \quad [(\mathbf{\Pi}\, i : \bar{\alpha})(\mathbf{\Sigma}\, j : \bar{\beta})\ \tilde{\alpha}(i) \doteq \tilde{\beta}(j)] \times [(\mathbf{\Pi}\, j : \bar{\beta})(\mathbf{\Sigma}\, i : \bar{\alpha})\ \tilde{\alpha}(i) \doteq \tilde{\beta}(j)].$$

Finally let $\|\alpha = \beta\|$ be $\alpha \doteq \beta$ and let $\|\alpha \in \beta\|$ be $(\mathbf{\Sigma}\, j : \bar{\beta})\ \alpha \doteq \tilde{\beta}(j)$.

The interpretation theorem can now be stated in a concise form.

Theorem: 7.1 *Let θ be a sentence of set theory. If* **CZF** $\vdash \theta$ *then there exists a term t_θ such that* $\mathbf{ML_1V} \vdash t_\theta : \|\theta\|$.

Proof: [1], Theorem 4.2. □

A strengthening of $\mathbf{ML_1V}$ is the type theory $\mathbf{ML_{1W}V}$ in which the universe $\mathbf{U}$ is also closed under the $\mathbf{W}$-type but the $\mathbf{W}$-type constructor cannot be applied to families of types outside of $\mathbf{U}$.

Theorem: 7.2 *Let θ be a sentence of set theory. If* **CZF + REA** $\vdash \theta$ *then there exists a term t_θ such that* $\mathbf{ML_{1W}V} \vdash t_\theta : \|\theta\|$.

Proof: [3]. □

A further strengthening of the properties of $\mathbf{U}$ considered by Aczel is $\mathbf{U}$-induction which asserts that $\mathbf{U}$ is inductively defined by its closure properties (cf.[2], 1.10).

Theorem: 7.3 *(i)* **CZF**, $\mathbf{ML_1V}$*, and* $\mathbf{ML_1V}$ + **U***-induction are of the same strength.*

(ii) **CZF+REA**, $\mathbf{ML_{1W}V}$*, and* $\mathbf{ML_{1W}V}$+**U***-induction are of the same strength.*

Proof: this follows from [33], Theorem 4.14 and Theorem 5.13. □

Several choice principles are also validated by the interpretations of Theorems 7.1 and 7.2. The sets that are bases in the interpretation can be characterized by the following notion.

[3]Below $\supset$ denotes the conditional and N_0 denotes the empty type.

Definition: 7.4 $\alpha : \mathbf{V}$ is *injectively presented* if for all $i, j : \bar{\alpha}$, whenever $\|\tilde{\alpha}(i) = \tilde{\alpha}(j)\|$ is inhabited, then $i = j : \bar{\alpha}$.

Section 9 will describe large collections of sets that have injective presentations.

8 Subcountability

Certain classical set theories such as Kripke-Platek set theory possess models wherein all sets are (internally) countable, and thus a particular strong form of the axiom of choice obtains. Although **CZF** has the same proof-theoretic strength as **KP**, **CZF** refutes the statement that every set is countable. However, a weaker form of countability, dubbed *subcountability*, is not only compatible with **CZF** and **CZF** + **REA** but doesn't increase the proof-theoretic strength of these theories.

Definition: 8.1 We use the arrow $\twoheadrightarrow$ in $g : A \twoheadrightarrow B$ to convey that g is a surjective function from A to B.

$$\begin{array}{rcl} X \text{ is } \textit{countable} & \text{iff} & \exists u\, u \in X \;\rightarrow\; \exists f\, (f : \omega \twoheadrightarrow X), \\ X \text{ is } \textit{subcountable} & \text{iff} & \exists A \subseteq \omega\, \exists f\, (f : A \twoheadrightarrow X). \end{array}$$

Let **EC** be the statement that every set is countable and let **ESC** be the statement that every set is subcountable. **ESC** has also been called the *Axiom of Enumerability* by Myhill (cf. [28]).

Obviously, in (rather weak) classical set theories countability and subcountability amount to the same. This is, however, far from being provable in intuitionistic set theories. Letting ${}^{\omega}\omega$ be the set of functions from ω to ω, it is, for instance, known that the theories **IZF** and **IZF** + ${}^{\omega}\omega$ *is subcountable* are equiconsistent, while **CZF** refutes the countability of ${}^{\omega}\omega$. The former fact is an immediate consequence of the equiconsistency of **IZF** and **IZF** augmented by Church's Thesis (cf. [25], Theorem 3.1).

Proposition: 8.2 **CZF** $\vdash \neg$**EC** *and* **IZF** $\vdash \neg$**ESC**.

Proof: These facts are well-known, but it won't do much harm to repeat them here. To refute **EC** in **CZF** suppose $g : \omega \twoheadrightarrow {}^{\omega}\omega$. Define $h : \omega \to \omega$ by $h(n) = (g(n)(n))+1$. Then $h = g(m)$ for some $m \in \omega$. As a result, $g(m)(m) = h(m) = (g(m)(m)) + 1$, which is impossible.

To refute **ESC** in **IZF** suppose that $A \subseteq \omega$ and $f : A \twoheadrightarrow \mathcal{P}(\omega)$. Then let $D = \{n \in A : n \notin f(n)\}$. As $D \subseteq \omega$ there exists $k \in A$ such that $D = f(k)$. But this is absurd since $k \in D$ iff $k \notin D$. □

Theorem: 8.3 ESC *may be consistently added to* **CZF** *and* **CZF** + **REA**. *More precisely,* **CZF** *and* **CZF** + **ESC** *are of the same proof-theoretic strength. The same holds for* **CZF** + **REA** *and* **CZF** + **REA** + **ESC**. *Furthermore, the previous assertions allow for improvements in that one can strengthen* **ESC** *to the following statement:*

$$\text{Every set is the surjective image of a base which is also a subset of } \omega. \tag{3}$$

Proof: This follows from close inspection of the proofs of [33] Theorem 4.14 and 5.13. The essence of those proofs is that the systems $\mathbf{ML_1V}$ and $\mathbf{ML_{1W}V}$ of Martin-Löf type theory (of the foregoing section) with one universe **U** and the type **V** can be interpreted in **KP** and **KPi**, respectively. Recall that **KPi** denotes Kripke-Platek set theory plus an axiom saying that every set is contained in an admissible set. The theories **CZF** and **CZF** + **REA** can in turn be interpreted in $\mathbf{ML_1V}$ and $\mathbf{ML_{1W}V}$, respectively, conceiving of **V** as a universe of sets and using the formulae-as-types interpretation. **U** is modelled in **KP** via an inductively defined subclass of ω whose elements code subsets of ω. Each element α of **V** is a pair of natural numbers n and e, denoted $\sup(n,e)$, with $n \in \mathbf{U}$ and e being the index of a partial recursive function $\tilde{\alpha}$ total on $\bar{\alpha}$ such that $\tilde{\alpha}(u) \in \mathbf{V}$ for all $u \in \bar{\alpha}$, where $\bar{\alpha}$ denotes the subset of ω coded by n. Let $\omega_V \in \mathbf{V}$ be the internalization of ω in **V**. Then $\overline{\omega_V} = \omega$. Also, for $\alpha, \beta \in \mathbf{V}$ let $\langle \alpha, \beta \rangle_V \in \mathbf{V}$ denote the internal ordered pair. The assignment $\alpha, \beta \mapsto \langle \alpha, \beta \rangle_V$ is a partial recursive function on $\mathbf{V} \times \mathbf{V}$.

Now, given any $\alpha \in \mathbf{V}$ with $\alpha = \sup(n,e)$, let $\beta := \sup(n, e^*)$, where e^* is an index of the partial recursive function $\tilde{\beta}$ with $\tilde{\beta}(u) := \widetilde{\omega_V}(u)$ for $u \in \bar{\alpha}$. Also, define $\gamma := \sup(n, e^{\#})$, where $e^{\#}$ is an index of the partial recursive function $\tilde{\gamma}$ with

$$\tilde{\gamma}(u) \quad := \quad \langle \widetilde{\omega_V}(u), \tilde{\alpha}(u) \rangle_V$$

for $u \in \bar{\alpha}$. e^* and $e^{\#}$ are both effectively computable from α and so are β and γ. One then has to verify that, internally in **V**, β is a subset of ω_V and γ is a function that maps β onto α (in the sense of **V**). Here one utilizes that $\omega_V \in \mathbf{V}$ is injectively represented, that is, whenever $\widetilde{\omega_V}(k) \doteq \widetilde{\omega_V}(k')$ (where $\doteq$ stands for the bi-simulation relation on **V** which interprets set-theoretic equality defined in the previous section) then $k = k'$.

To show the stronger statement (3), note first that the interpretations of [33] Theorem 4.14 and 5.13 also validate the presentation axiom **PAx**. Further note that every set that is in one-to-one correspondence with a base is a base, too. Thus, arguing in **CZF** + **PAx** + **ESC**, it suffices to show that every base is in one-to-one correspondence with a subset of ω. Let B be a base. Then there exist $A \subseteq \omega$ and $f : A \twoheadrightarrow B$. But B being a base, f can be inverted, that is, there exists $g : B \to A$ such that $f(g(u)) = u$ for all $u \in B$. g is an injective function and thus B is in one-to-one correspondence with $\{g(u) : u \in B\} \subseteq \omega$. □

9 "Maximal" choice principles

The interpretation of constructive set theory in type theory not only validates all the theorems of **CZF** (resp. **CZF** + **REA**) but many other interesting set-theoretic statements. Ideally, one would like to have a characterization of these statements and determine an extension $\mathbf{CZF}^*$ of **CZF** (resp. **CZF** + **REA**) which deduces exactly the set-theoretic statements validated in $\mathbf{ML_1V}$ (resp. $\mathbf{ML_{1W}V}$). It will turn out that the search for $\mathbf{CZF}^*$ amounts to finding the "strongest" version of the axiom of choice that is validated in $\mathbf{ML_1V}$.

The interpretation of set theory in type theory gave rise to a plethora of new choice principles which will be described next.

Definition: 9.1 (**CZF**) If A is a set and B_x are classes for all $x \in A$, we define a class $\prod_{x\in A} B_x$ by:

$$\prod_{x\in A} B_x := \{f : A \to \bigcup_{x\in A} B_x \mid \forall x\in A(f(x) \in B_x)\}. \tag{4}$$

If A is a class and B_x are classes for all $x \in A$, we define a class $\sum_{x\in A} B_x$ by:

$$\sum_{x\in A} B_x := \{\langle x, y\rangle \mid x \in A \wedge y \in B_x\}. \tag{5}$$

If A and B are classes, we define a class $\mathbf{I}(A, B)$ by:

$$\mathbf{I}(A, B) := \{z \in 1 \mid A = B\}. \tag{6}$$

If A is a class and for each $a \in A$, B_a is a set, then

$$\mathbf{W}_{a\in A} B_a$$

is the smallest class Y such that whenever $a \in A$ and $f : B_a \to Y$, then $\langle a, f\rangle \in Y$.

Lemma: 9.2 (**CZF**) *If A and B are sets and B_x is a set for all $x \in A$, then $\prod_{x\in A} B_x$, $\sum_{x\in A} B_x$ and $\mathbf{I}(A, B)$ are sets.*

Proof: □

Lemma: 9.3 (**CZF** + **REA**) *If A is a set and B_x is a set for all $x \in A$, then $\mathbf{W}_{a\in A} B_a$ is a set.*

Proof: This follows from [3], Corollary 5.3. □

Lemma: 9.4 (**CZF**) *There exists a smallest* $\Pi\Sigma$*-closed class (*$\Pi\Sigma I$*-closed class), i.e. a smallest class* $\mathbf{Y}$ *such that the following holds:*
(i) $n \in \mathbf{Y}$ *for all* $n \in \omega$*;*
(ii) $\omega \in \mathbf{Y}$*;*
(iii) $\prod_{x \in A} B_x \in \mathbf{Y}$ *and* $\sum_{x \in A} B_x \in \mathbf{Y}$ *whenever* $A \in \mathbf{Y}$ *and* $B_x \in \mathbf{Y}$ *for all* $x \in A$*.*
((iv) If $A \in \mathbf{Y}$ *and* $a, b \in A$*, then* $\mathbf{I}(a, b) \in \mathbf{Y}$*.)*

Proof: See [37], Lemma 1.2. $\square$

Lemma: 9.5 (**CZF + REA**) *There exists a smallest* $\Pi\Sigma W$*-closed class (*$\Pi\Sigma WI$*-closed class), i.e. a smallest class* $\mathbf{Y}$ *such that the following holds:*
(i) $n \in \mathbf{Y}$ *for all* $n \in \omega$*;*
(ii) $\omega \in \mathbf{Y}$*;*
(iii) $\prod_{x \in A} B_x \in \mathbf{Y}$ *and* $\sum_{x \in A} B_x \in \mathbf{Y}$ *whenever* $A \in \mathbf{Y}$ *and* $B_x \in \mathbf{Y}$ *for all* $x \in A$*.*
(iv) $\mathbf{W}_{a \in A} B_a \in \mathbf{Y}$ *whenever* $A \in \mathbf{Y}$ *and* $B_x \in \mathbf{Y}$ *for all* $x \in A$*.*
((v) If $A \in \mathbf{Y}$ *and* $a, b \in A$*, then* $\mathbf{I}(a, b) \in \mathbf{Y}$*.)*

Proof: See [3] $\square$

Definition: 9.6 *The* $\Pi\Sigma$*-generated sets are the sets in the smallest* $\Pi\Sigma$*-closed class. Similarly one defines the* $\Pi\Sigma I$*,* $\Pi\Sigma W$ *and* $\Pi\Sigma WI$*-generated sets.*

$\mathbf{\Pi\Sigma}-\mathbf{AC}$ *is the statement that every* $\Pi\Sigma$*-generated set is a base. Similarly one defines the axioms* $\mathbf{\Pi\Sigma I}-\mathbf{AC}$*,* $\mathbf{\Pi\Sigma WI}-\mathbf{AC}$*, and* $\mathbf{\Pi\Sigma W}-\mathbf{AC}$*.*

Corollary: 9.7 *(i)* (**CZF**) $\mathbf{\Pi\Sigma}-\mathbf{AC}$ *and* $\mathbf{\Pi\Sigma I}-\mathbf{AC}$ *are equivalent.*

(ii) (**CZF + REA**) $\mathbf{\Pi\Sigma W}-\mathbf{AC}$ *and* $\mathbf{\Pi\Sigma WI}-\mathbf{AC}$ *are equivalent.*

Proof: [3], Theorem 3.7 and Theorem 5.9. $\square$

The axioms $\mathbf{\Pi\Sigma}-\mathbf{AC}$ and $\mathbf{\Pi\Sigma W}-\mathbf{AC}$ may be added to the theories on the left hand side in Theorems 7.1 and 7.2, respectively. Below we shall show that these are in some sense the strongest axioms of choice that may be added.

Definition: 9.8 The *mathematical set terms* are a collection of class terms inductively defined by the following clauses:

1. ω is a mathematical set term.

2. If S and T are mathematical set terms then so are

$$\begin{aligned}\bigcup S &:= \{u : \exists x \in S\ u \in x\},\\ \{S, T\} &:= \{u : u = S \vee u = T\}\end{aligned}$$

3. If S and T are mathematical set terms then so are

$$\begin{aligned} S + T &:= \{\langle 0, x\rangle : x \in S\} \cup \{\langle 1, x\rangle : x \in T\}, \\ S \times T &:= \{\langle x, y\rangle : x \in S \wedge y \in T\}, \\ S \to T &:= \{f : f : S \to T\}. \end{aligned}$$

4. If $S, T_1, \ldots, T_n$ are mathematical set terms and $\psi(x, y_1, \ldots, y_n)$ is a restricted formula (of set theory) then

$$\{x \in S : \psi(x, T_1, \ldots, T_n)\}$$

is a mathematical set term.

5. If $S, T_1, \ldots, T_n, P_1, \ldots, P_k$ are mathematical set terms and $\psi(x, y_1, \ldots, y_n, z_1, \ldots, z_k)$ is a bounded formula (of set theory) then

$$\{u : u = \{x \in S : \psi(x, y_1, \ldots, y_n, \vec{P})\} \wedge y_1 \in T_1 \wedge \ldots \wedge y_n \in T_n\}$$

is a mathematical set term, where $\vec{P} = P_1, \ldots, P_k$.

The *generalized mathematical set terms* are defined by the clauses for mathematical set terms plus the following clause:

6. If T is a generalized mathematical set term then so is $\mathbf{H}(A)$, where $\mathbf{H}(A)$ denotes the smallest class Y such that $\mathbf{ran}(f) \in Y$ whenever $a \in A$ and $f : a \to Y$.

A *mathematical formula* (*generalized mathematical formula*) is a formula of the form $\psi(T_1, \ldots, T_n)$, where $\psi(x_1, \ldots, x_n)$ is bounded and $T_1, \ldots, T_n$ are mathematical set terms (generalized mathematical set terms). A *mathematical sentence* (*generalized mathematical sentence*) is a mathematical formula (generalized mathematical formula) without free variables.

Remark 9.9 1. From the point of view of **ZFC**, the mathematical set terms denote sets of rank $< \omega + \omega$ in the cumulative hierarchy while the generalized mathematical set terms denote sets of rank $< \aleph_\omega$.

2. The idea behind mathematical set terms is that they comprise all sets that one is interested in in ordinary mathematics. E.g., with the help of Definition 9.8, clauses (1) and (3) one constructs the set of natural numbers, integers, rationals, and arbitrary function space as, e.g, $\mathbb{N} \to \mathbb{Q}$.

   The main applications of clause (5) are made in constructing quotients: if $R \subseteq S \times S$ are set terms and R is an equivalence relation on S, then (5) permits to form the set term

$$S/_R \;=\; \{[a]_R : a \in S\},$$

where $[a]_R = \{x \in S : aRx\}$.

Using clause (4) one obtains the set of Cauchy sequences of rationals from $\mathbb{N} \to \mathbb{Q}$, and finally by employing clause (5) one can define the set of equivalence classes of such Cauchy sequences, i.e., the set of reals.

3. Definition 9.8 clause (5) is related to the abstraction axiom of Friedman's system **B** in [13].

Lemma: 9.10 *1.* (**CZF**) *Every mathematical set term is a set.*

2. (**CZF** + **REA**) *Every generalized mathematical set term is a set.*

Proof: [37]. □

Theorem: 9.11 *Let ψ be a mathematical sentence and let θ be a generalized mathematical sentence. Then there are closed terms t_ψ and t_θ of* $\mathbf{ML_1V}$ *and* $\mathbf{ML_{1W}V}$*, respectively such that*

(i) $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \psi$ *if and only if* $\mathbf{ML_1V} \vdash t_\psi : \|\psi\|$.

(ii) $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W{-}AC} \vdash \theta$ *if and only if* $\mathbf{ML_{1W}V} \vdash t_\theta : \|\theta\|$.

(iii) The foregoing results also hold if one adds **U***-induction to the type theories.*

Proof: The "only if" parts are due to Rathjen and Tupailo [37] Theorem 7.6. The "if" parts are due to [2, 3] and are proved by showing that the $\mathbf{\Pi\Sigma}$- and $\mathbf{\Pi\Sigma W}$-generated sets are injectively presentable (see Definition 7.4). □

[2, 3] feature several more choice principles. The main reason for their omission is that these axioms have no impact on the preceding result. This will be made precise below.

Definition: 9.12 *Let* $\mathbf{\Pi\Sigma{-}PAx}$ *be the assertion that every $\Pi\Sigma$-generated set is a base and every set is an image of a $\Pi\Sigma$-generated set. Similarly, one defines* $\mathbf{\Pi\Sigma W{-}PAx}$.

Let $\mathbf{BCA}_\Pi$ *be the statement that whenever A is a base and B_a is a base for each $a \in A$, then $\prod_{x\in A} B_x$ is a base.*

Let $\mathbf{BCA}_I$ *be the statement that whenever A is a base then $\mathbf{I}(b, c)$ is a base for all $b, c \in A$.*

Theorem: 9.13 *Let ψ be a mathematical sentence and let θ be a generalized mathematical sentence. Then the following obtain:*

(i) $\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} \vdash \psi$ *if and only if*
$\mathbf{CZF} + \mathbf{\Pi\Sigma{-}AC} + \mathbf{\Pi\Sigma{-}PAx} + \mathbf{BCA}_\Pi + \mathbf{BCA}_I + \mathbf{DC} \vdash \psi$.

(ii) $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W}-\mathbf{AC} \vdash \theta$ *if and only if* $\mathbf{CZF} + \mathbf{REA} + \mathbf{\Pi\Sigma W}-\mathbf{PAx} + \mathbf{BCA}_{\Pi} + \mathbf{BCA}_{I} + \mathbf{DC} \vdash \theta$.

Proof: [37]. □

10 The existence property

It is often considered a hallmark of intuitionistic systems that they possess the disjunction and existential definability properties.

Definition: 10.1 Let T be a theory whose language, $L(T)$, encompasses the language of set theory. Moreover, for simplicity, we shall assume that $L(T)$ has a constant ω denoting the set of von Neumann natural numbers and for each n a constant $\bar{n}$ denoting the n-th natural number.

1. T has the *disjunction property*, **DP**, if whenever $T \vdash \psi \vee \theta$ then $T \vdash \psi$ or $T \vdash \theta$.
2. T has the *numerical existence property*, **NEP**, if whenever $T \vdash (\exists x{\in}\omega)\phi(x)$ then $T \vdash \phi(\bar{n})$ for some n.
3. T has the *existence property*, **EP**, if whenever $T \vdash \exists x\phi(x)$ then $T \vdash \exists! x\, [\vartheta(x) \wedge \phi(x)]$ for some formula ϑ.

Of course, above we assume that the formulas have no other free variables than those exhibited.

Slightly abusing terminology, we shall also say that T enjoys any of these properties if this holds only for a definitional extension of T rather than T.

ZF and **ZFC** are known not to have the existence property. But even classical set theories can have the **EP**. Kunen observed that an extension of **ZF** has the **EP** if and only if it proves that all sets are ordinal definable, i.e., $V = OD$. Going back to intuitionistic set theories, let $\mathbf{IZF}_R$ result from **IZF** by replacing Collection with Replacement, and let **CST** be Myhill's constructive set theory of [28]. $\mathbf{CST}^-$ denotes Myhill's **CST** without the axioms of countable and dependent choice.

Theorem: 10.2 (Myhill) $\mathbf{IZF}_R$ *and* $\mathbf{CST}^-$ *have the* **DP**, **NEP**, *and the* **EP**. **CST** *has the* **DP** *and the* **NEP**.

Proof: [28]. □

Theorem: 10.3 (Beeson) **IZF** *has the* **DP** *and the* **NEP**.

Proof: [6]. □

Theorem: 10.4 **CZF** *and* **CZF** + **REA** *have the* **DP** *and the* **NEP**. *This also holds when one adds any combination of the choice principles* $\mathbf{AC}_\omega$, **DC**, **RDC**, *or* **PAx** *to these theories.*

Proof: [38] and [39]. $\square$

Theorem: 10.5 (Friedman,Ščedrov) **IZF** *does not have the* **EP**.

Proof: [14]. $\square$

The question of whether **CZF** enjoys the **EP** is currently unsolved. The proof of the failure of **EP** for **IZF** given in [14] seems to single out Collection as the culprit. It appears unlikely that that proof can be adapted to **CZF** because the refutation utilizes existential statements of the form

$$\exists b\,[\forall u{\in}a\,\exists y\,\varphi(u,y)\;\rightarrow\;\forall u{\in}a\,\exists y{\in}b\,\varphi(u,y)],$$

that are always deducible in **IZF** by employing first full Separation and then Collection, but, in general, are not not deducible in **CZF**. We conjecture that the **EP** fails for **CZF** on account of Subset Collection (and maybe Collection). There are, however, positive answers available for **CZF**+$\mathbf{\Pi\Sigma}-$**AC** and **CZF**+**REA**+$\mathbf{\Pi\Sigma W}-$**AC**. These theories have the pertaining properties for mathematical and generalized mathematical statements, respectively.

Theorem: 10.6 *Let* θ_1,θ_2 *be mathematical sentences and let* $\psi(x)$ *be a mathematical formula with at most x free. Then we have the following:*

(i) If $\mathbf{CZF}+\mathbf{\Pi\Sigma}-\mathbf{AC}\vdash\theta_1\vee\theta_2$ *then* $\mathbf{CZF}+\mathbf{\Pi\Sigma}-\mathbf{AC}\vdash\theta_1$ *or* $\mathbf{CZF}+\mathbf{\Pi\Sigma}-\mathbf{AC}\vdash\theta_2$.

(ii) If $\mathbf{CZF}+\mathbf{\Pi\Sigma}-\mathbf{AC}\vdash\exists x\psi(x)$ *then there is a formula* $\vartheta(x)$ *(with at most x free) such that* $\mathbf{CZF}+\mathbf{\Pi\Sigma}-\mathbf{AC}\vdash\exists!x[\vartheta(x)\wedge\psi(x)]$.

Proof: (i) and (ii) are stated in [37] as Theorem 8.2. (i) and (ii) follow from results in [36] and [38]. $\square$

Theorem: 10.7 *Let* θ_1,θ_2 *be generalized mathematical sentences and let* $\psi(x)$ *be a generalized mathematical formula with at most x free. Then we have the following:*

(i) If $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W}-\mathbf{AC}\vdash\theta_1\vee\theta_2$ *then* $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W}-\mathbf{AC}\vdash\theta_1$ *or* $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W}-\mathbf{AC}\vdash\theta_2$.

(ii) If $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W}-\mathbf{AC}\vdash\exists x\psi(x)$ *then there exists a formula* $\vartheta(x)$ *(with at most x free) such that* $\mathbf{CZF}+\mathbf{REA}+\mathbf{\Pi\Sigma W}-\mathbf{AC}\vdash\exists!x[\vartheta(x)\wedge\psi(x)]$.

Proof: (i) and (ii) follow from results in [36] and [38]. They are stated in [37], Theorem 8.4. $\square$

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