University of Leeds
School of Civil Engineering **CIVE1619, Autumn 2010**

Solutions 5

Integration

1. **a)** $\int 5x^4 + 8x\,dx = x^5 + 4x^2 + c;$

b) $\int \sqrt{x} + \frac{6}{x^2}\,dx = \int x^{1/2} + 6x^{-2}\,dx = \frac{2}{3}x^{3/2} - 6x^{-1} + c;$

c) $\int (x^2+3)^2\,dx = \int x^4 + 6x^2 + 9\,dx = \frac{1}{5}x^5 + 2x^3 + 9x + c;$

d) $\int \cos x + 3\ dx = \sin x + 3x + c;$

e) $\int_0^2 5x^4 + 8x\,dx = \left[x^5 + 4x^2\right]_0^2 = (2^5 + 4\times 2^2) - (0^5 + 4\times 0^2) = 48;$

f) $\int_1^3 \sqrt{x} + \frac{6}{x^2}\,dx = \left[\frac{2}{3}x^{3/2} - 6x^{-1}\right]_1^3 = (\frac{2}{3}\times 3^{3/2} - 6\times\frac{1}{3}) - (\frac{2}{3}\times 1^{3/2} - 6) = \frac{2}{3}\times 3^{3/2} - 2 - \frac{2}{3} + 6 = 2\sqrt{3} + \frac{10}{3};$

g) $\int_{-1}^1 (x^2+3)^2\,dx = \left[x^5 + 2x^3 + 9x\right]_{-1}^1 = (1+2+9) - ((-1)^5 + 2(-1)^3 + 9(-1)) = (1+2+9) - (-1-2-9) = 24;$

h) $\int_0^{\pi/2} \cos x + 3\ dx = \left[\sin x + 3x\right]_0^{\pi/2} = 1 + \frac{3\pi}{2}.$

2. **a)** If $u = 2x$, then $x = u/2$, so $dx/du = 1/2$, i.e. $dx = \frac{1}{2}du$. Then

$$\int \sin(2x)\ dx = \int \sin u \frac{1}{2}\ du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(2x) + c.$$

b) If $u = x/2$, then $x = 2u$, so $dx/du = 2$, i.e. $dx = 2du$. Then

$$\int e^{x/2}\ dx = \int e^u 2du = 2e^u + c = 2e^{x/2} + c.$$

c) If $u = 2x+1$, then $x = (u-1)/2$, so $dx/du = 1/2$, i.e. $dx = \frac{1}{2}du$. Then

$$\int \frac{1}{2x+1}\,dx = \int \frac{1}{u}\frac{1}{2}\ du = \frac{1}{2}\int \frac{1}{u}\ du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2x+1| + c.$$

d) If $u = x^2 - 1$ then $du/dx = 2x$, so $dx/du = 1/(2x)$. Thus

$$\int x\cos(x^2-1)\,dx = \int x\cos(x^2-1) \times \frac{1}{2x} du$$
$$= \int \frac{1}{2}\cos u\,du = \frac{1}{2}\sin u + c = \frac{1}{2}\sin(x^2-1) + c.$$

e) If $u = x + 3$ then $x = u - 3$, so $dx/du = 1$. Thus

$$\int x(x+3)^8\,dx = \int (u-3)u^8\,du = \int u^9 - 3u^8\,du = \frac{u^{10}}{10} - \frac{u^9}{3} + c$$
$$= \frac{(x+3)^{10}}{10} - \frac{(x+3)^9}{3} + c.$$

f) If $u = \sin x$ then $du/dx = \cos x$, so $dx/du = 1/\cos x$. Thus

$$\int \cos x \sin^2 x\,dx = \int \cos x \sin^2 x \times \frac{1}{\cos x}\,du$$
$$= \int \sin^2 x\,du = \int u^2\,du = \frac{u^3}{3} + c = \frac{\sin^3 x}{3} + c.$$

g) If $u = \tan x$ then $du/dx = \sec^2 x$, so $dx/du = 1/\sec^2 x$. Thus the indefinite integral is

$$\int \sec^2 x \tan^3 x\,dx = \int \tan^3 x\,du = \int u^3\,du = \frac{u^4}{4} + c = \frac{\tan^4 x}{4} + c.$$

Thus the definite integral is

$$\int_0^{\pi/4} \sec^2 x \tan^3 x\,dx = \left[\frac{\tan^4 x}{4}\right]_0^{\pi/4} = \frac{\tan^4(\pi/4)}{4} = \frac{1}{4}.$$

h) If $u = 1 - x^2$ then $du/dx = -2x$, so $dx/du = -1/(2x)$. Thus the indefinite integral is

$$\int x\sqrt{1-x^2}\,dx = \int x\sqrt{1-x^2} \times \frac{-1}{2x}\,du$$
$$= -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{3}u^{3/2} + c = -\frac{1}{3}(1-x^2)^{3/2} + c.$$

Thus the definite integral is

$$\int_0^1 x\sqrt{1-x^2}\,dx = \left[-\frac{1}{3}(1-x^2)^{3/2}\right]_0^1 = (-\frac{1}{3} \times (1-1)^{3/2}) - (-\frac{1}{3} \times (1-0)^{3/2}) = \frac{1}{3}.$$

3. **a)** Let $u = x$ and $dv/dx = e^x$, so $v = e^x$ and $du/dx = 1$. Then

$$\int xe^x\,dx = \int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx = xe^x - \int e^x\,dx = xe^x - e^x + c.$$

b) Let $u = x^2$ and $dv/dx = e^x$, so $v = e^x$ and $du/dx = 2x$. Then

$$\int x^2e^x\,dx = \int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx = x^2e^x - \int 2x\,e^x\,dx$$

The integral on the right is already calculated in (a), so we get

$$\int x^2e^x\,dx = x^2e^x - 2(xe^x - e^x) + c = (x^2 - 2x + 2)e^x + c.$$

c) Let $u = \ln x$ and $dv/dx = x^3$, so $v = x^4/4$ and $du/dx = 1/x$. Then

$$\int x^3 \ln x \, dx \;=\; \int u \frac{dv}{dx} \, dx \;=\; uv - \int \frac{du}{dx} v \, dx \;=\; \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} \, dx \;=\; \frac{x^4 \ln x}{4} - \frac{x^4}{16} + c.$$

d) Let $u = \ln x$ and $dv/dx = x^{-2}$, so $v = -x^{-1}$ and $du/dx = 1/x$. Then the indefinite integral is

$$\int \frac{\ln x}{x^2} \, dx = \int u \frac{dv}{dx} \, dx = uv - \int \frac{du}{dx} v \, dx = -\frac{\ln x}{x} + \int x^{-2} \, dx = -\frac{\ln x}{x} - \frac{1}{x} + c.$$

Thus the definite integral is

$$\int_2^3 \frac{\ln x}{x^2} \, dx = \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_2^3 = (-\frac{\ln 3}{3} - \frac{1}{3}) - (-\frac{\ln 2}{2} - \frac{1}{2}) = \frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{1}{6}.$$

(In decimals this is 0.147)

e) Let $u = x$ and $dv/dx = \sin x$, so $v = -\cos x$ and $du/dx = 1$. Then the indefinite integral is

$$\int x \sin x \, dx = \int u \frac{dv}{dx} \, dx = uv - \int \frac{du}{dx} v \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + c.$$

Thus the definite integral is

$$\int_0^{\pi} x \sin x \, dx = \left[-x \cos x + \sin x \right]_0^{\pi} = (-\pi \cos \pi + \sin \pi) - (-0 \cos 0 + \sin 0) = \pi.$$

4. We try to write

$$\frac{5x + 3}{(x+1)(2x+1)} = \frac{A}{x+1} + \frac{B}{2x+1}.$$

Multiplying through, we need

$$5x + 3 = A(2x + 1) + B(x + 1).$$

Substituting $x = -1$ gives $-2 = -A$, so $A = 2$. Substituting $x = -1/2$ gives $1/2 = B/2$, so $B = 1$. Now clearly we do have

$$5x + 3 = 2(2x + 1) + (x + 1),$$

so that

$$\frac{5x + 3}{(x+1)(2x+1)} = \frac{2}{x+1} + \frac{1}{2x+1}.$$

Thus

$$\int \frac{5x + 3}{(x+1)(2x+1)} \, dx = 2 \int \frac{1}{x+1} \, dx + \int \frac{1}{2x+1} \, dx$$

Now the second integral in the sum has been worked out in question 2(a), so

$$\int \frac{5x + 3}{(x+1)(2x+1)} \, dx = 2 \ln |x + 1| + \frac{1}{2} \ln |2x + 1| + c.$$

Thus the definite integral is

$$\int_0^1 \frac{5x + 3}{(x+1)(2x+1)} \, dx = \left[2 \ln |x + 1| + \frac{1}{2} \ln |2x + 1| \right]_0^1$$

$$= (2 \ln 2 + \frac{1}{2} \ln 3) - (2 \ln 1 + \frac{1}{2} \ln 1) = 2 \ln 2 + \frac{1}{2} \ln 3.$$

Using properties of logarithms we can rewrite this as follows.

$$2 \ln 2 + \frac{1}{2} \ln 3 = \ln(2^2) + \ln(\sqrt{3}) = \ln 4 + \ln(\sqrt{3}) = \ln(4\sqrt{3}).$$

5. **a)** By division, we find that

$$\frac{x^5}{x^3-1} = x^2 + \frac{x^2}{x^3-1}.$$

Hence

$$\int \frac{x^5}{x^3-1}dx = \int (x^2 + \frac{x^2}{x^3-1})dx = \frac{x^3}{3} + \frac{1}{3}\ln|x^3-1| + c.$$

b) We use the identity

$$\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}.$$

(Prove this yourself!) Then

$$\int \cos^2 x dx = \int (\frac{1}{2} + \frac{\cos 2x}{2})dx = \frac{x}{2} + \frac{\sin 2x}{4} + c.$$

c) We have

$$\int \cos^3 x dx = \int \cos x \cos^2 x dx = \int \cos x(1-\sin^2 x)dx.$$

Let $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$. Therefore, the integral becomes

$$\int (1-u^2)du = u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c.$$

d) Let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$. Therefore, we get

$$\begin{aligned}\int \frac{\sin^3 x}{1+\cos x}dx = -\int \frac{\sin^2 x}{1+u}du &= \int \frac{u^2-1}{u+1}du \\ &= \int (u-1)du = \frac{u^2}{2} - u + c \\ &= \frac{\cos^2 x}{2} - \cos x + c.\end{aligned}$$

Solving differential equations

6. **a)** Separating the variables gives

$$\int \frac{1}{y}\,dy = \int 4\,dx$$

so

$$\ln y = 4x + c,$$

so the general solution is

$$y = e^{4x+c} = Ce^{4x}$$

writing $C = e^c$.

b) Separating the variables gives

$$\int \frac{1}{y}\,dy = \int x + 2\,dx$$

so

$$\ln y = \frac{1}{2}x^2 + 2x + c,$$

so the general solution is

$$y = e^{\frac{1}{2}x^2+2x+c} = Ce^{\frac{1}{2}x^2+2x}$$

writing $C = e^c$.

c) Separating the variables gives

$$\int e^y\,dy = \int \cos x\,dx.$$

so

$$e^y = \sin x + c.$$

Thus the general solution is

$$y = \ln(\sin x + c).$$

d) The general solution is $y = Ce^{4x}$. If $y = 3$ when $x = 0$, then $3 = Ce^0$, so $C = 3$. Thus the particular solution is

$$y = 3e^{4x}.$$

e) The general solution is $y = Ce^{\frac{1}{2}x^2+2x}$. If $y = 2$ when $x = 0$, then $2 = Ce^0$, so $C = 2$. Thus the particular solution is

$$y = 2e^{\frac{1}{2}x^2+2x}.$$

f) The general solution is $y = \ln(\sin x + c)$. If $y = 1$ when $x = \pi/2$, then $1 = \ln(1 + c)$, so $1 + c = e^1 = e$, so $c = e - 1$. Thus the particular solution is

$$y = \ln(\sin x + e - 1).$$