Reverse Mathematics and Well-ordering Principles

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Abstract

The paper is concerned with generally Π₁² sentences of the form “if $X$ is well ordered then $f(X)$ is well ordered”, where $f$ is a standard proof theoretic function from ordinals to ordinals. It has turned out that a statement of this form is often equivalent to the existence of countable coded $\omega$-models for a particular theory $T_f$ whose consistency can be proved by means of a cut elimination theorem in infinitary logic which crucially involves the function $f$. To illustrate this theme, we shall focus on the well-known $\varphi$-function which figures prominently in so-called predicative proof theory. However, the approach taken here lends itself to generalization in that the techniques we employ can be applied to many other proof-theoretic functions associated with cut elimination theorems. In this paper we show that the statement “if $X$ is well ordered then $\varphi X0$ is well ordered” is equivalent to $\text{ATR}_0$. This was first proved by Friedman (see [8]) using recursion-theoretic and combinatorial methods. The proof given here is proof-theoretic, the main techniques being Schütte’s method of proof search (deduction chains) [15], generalized to $\omega$ logic, and cut elimination for infinitary ramified analysis.

Key words: reverse mathematics, well ordering principles, Schütte deduction chains, countable coded $\omega$-model, $\text{ATR}_0$

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1 Introduction

The larger project broached in this paper is to present a general proof-theoretic machinery for investigating a special kind of Π₁² statements about well-orderings from a reverse mathematics point of view. These Π₁² statements are of the form

\[ \text{WOP}(f) \quad \text{“if } X \text{ is well ordered then } f(X) \text{ is well ordered”} \quad (1) \]

where $f$ is a standard proof theoretic function from ordinals to ordinals. There are by now several examples of functions $f$ where the statement $\text{WOP}(f)$ has turned out to be equivalent to one of the theories of reverse mathematics over a weak base theory (usually $\text{RCA}_0$). The first example is due to Girard [9].

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**Theorem 1.1** (Girard 1987) Let $\text{WO}(X)$ express that $X$ is a well ordering. Over $\text{RCA}_0$ the following are equivalent:

(i) Arithmetic Comprehension

(ii) $\forall X \ [\text{WO}(X) \rightarrow \text{WO}(2^X)]$.

Recently two new theorems appeared in preprints [11, 8]. These results give characterizations of the form (1) for the theories $\text{ACA}_0^+$ and $\text{ATR}_0$, respectively, in terms of familiar proof-theoretic functions. $\text{ACA}_0^+$ denotes the theory $\text{ACA}_0$ augmented by an axiom asserting that for any set $X$ the $\omega$-th jump in $X$ exists while $\text{ATR}_0$ asserts the existence of sets constructed by transfinite iterations of arithmetical comprehension. $\alpha \mapsto \varepsilon_\alpha$ denotes the usual $\varepsilon$ function while $\varphi$ stands for the two-place Veblen function familiar from predicative proof theory (cf. [15]). More detailed descriptions of $\text{ATR}_0$ and the function $X \mapsto \varphi_X0$ will be given shortly. Definitions of the familiar subsystems of reverse mathematics can be found in [17].

**Theorem 1.2** (Montalban, Marcone, 2007) Over $\text{RCA}_0$ the following are equivalent:

(i) $\text{ACA}_0^+$

(ii) $\forall X \ [\text{WO}(X) \rightarrow \text{WO}(\varepsilon_X)]$.

**Theorem 1.3** (Friedman) Over $\text{RCA}_0$ the following are equivalent:

(i) $\text{ATR}_0$

(ii) $\forall X \ [\text{WO}(X) \rightarrow \text{WO}(\varphi_X0)]$.

The proof of Theorem 1.3 uses rather sophisticated recursion-theoretic results about linear orderings and is quite combinatorial. Theorem 1.3 builds on a result from [6] to the effect that there is no arithmetic sequence of degrees descending by $\omega$-jumps. The latter result was then improved by Steel [18] to descent by Turing jumps: If $Q \subseteq \text{Pow}(\omega) \times \text{Pow}(\omega)$ is arithmetic, then there is no sequence $\{A_n \mid n \in \omega\}$ such that (a) for every $n$, $A_{n+1}$ is the unique set such that $Q(A_n, A_{n+1})$, (b) for every $n$, $A_{n+1} \leq_T A_n$.

For a proof theorist, theorems 1.2 and 1.3 bear a striking resemblance to cut elimination theorems for infinitary logics. This prompted the first author of this paper to look for proof-theoretic ways of proving these results. The hope was that this would also unearth a common pattern behind them and possibly lead to more results of this kind. The project commenced with [2], where a purely proof-theoretic proof of Theorem 1.2 was presented. In this paper we shall give a new proof of Theorem 1.3. It is principally proof-theoretic, the main techniques being Schütte’s method of proof search (deduction chains) [15] and cut elimination for ramified analysis. The general pattern, of which this paper provides a second example, is that a statement $\text{WOP}(f)$ is often equivalent to the assertion that every set is contained in an $\omega$-model of a certain theory $T_f$.

To guide the reader through the paper we shall briefly sketch the main parts of the proof of Theorem 1.3, i.e., that (ii) implies (i). We start with the observation that $\text{ATR}_0$ can be be axiomatized over $\text{ACA}_0$ via a single sentence of the form $\forall X (\text{WO}(X) \rightarrow \forall Y B_0(X, Y, Z))$ where $B_0(X, Y, Z)$ is an arithmetical formula (cf. Lemma 3.2). Thus to verify $\text{ATR}_0$ it suffices to show that for every well-ordering $<_Q$ there exists an $\omega$-model of $M$ of $\text{ACA}_0$ which contains $Q$ such that $M \models \forall Y B_0(Q, Y, Z)$. To find $M$ we employ Schütte’s method of proof search from
which he used to prove the completeness theorem for first order logic (cf. [15, Theorem 5.7]). The method has to be extended to ω-logic, though. Rather than working in the Schütte calculus of positive and negative forms we work in a Gentzen sequent calculus with finite sets of formulas, called sequents. Let \( C \) be a sentence that axiomatizes arithmetic comprehension and let \( D_\varphi(n) \) be the formula \( n \in Q \) if the latter formula is true and \( n \notin Q \) otherwise. The main idea is to start with the sequent \( \{ \lnot \forall Z \exists Y B_0(Q, Y, Z), \lnot C, \lnot D_\varphi(0) \} \) and systematically apply the rules of ω-logic for the second order sequent calculus backwards, giving rise to a tree of sequents \( D_\varphi \). One also has to add the formula \( \lnot D_\varphi(n) \) to all sequents generated in this way after \( n \) steps.

There are two possible outcomes. If the tree \( D_\varphi \) is not well-founded then it contains an infinite path \( \mathbb{P} \). Now define a set \( M \) via

\[
(M)_i = \{ n \mid n \notin U_i \text{ occurs in } \mathbb{P} \}
\]

and let \( \mathbb{M} = (\mathbb{N}; \{(M)_i \mid i \in \mathbb{N}\}, +, \cdot, 0, 1, <) \). For a formula \( F \), let \( F \in \mathbb{P} \) mean that \( F \) occurs in \( \mathbb{P} \), i.e. \( F \in \Gamma \) for some \( \Gamma \in \mathbb{P} \). Let \( U_0, U_1, U_2, \ldots \) be an enumeration of the free set variables. For the assignment \( U_i \mapsto (M)_i \), one can then show that \( F \in \mathbb{P} \Rightarrow \mathbb{M} \models \lnot F \). Whence \( \mathbb{M} \) is an \( \omega \)-model of \( \text{ACA} \) and \( \mathbb{M} \models \forall Z \exists Y B_0(Q, Y, Z) \). Also note that \( (M)_0 = Q \), thus \( Q \) is in \( \mathbb{M} \).

The other conceivable outcome is that \( D_\varphi \) is well-founded, i.e. all paths in \( D_\varphi \) are finite, and thus every maximal path ends in a sequent which contains a basic axiom. In other words \( D_\varphi \) is a proof tree and the Kleene-Brouwer ordering of this tree is some well-ordering \( \tau \). The crucial step to perform next consists in envisaging \( D_\varphi \) as a skeleton of a proof tree in infinitary ramified analysis, dubbed \( \text{RA}^* \) in [15]. In actuality \( D_\varphi \) can be viewed as the skeleton of a proof of the empty sequent in \( \text{RA}^* \). As we can remove all cuts in this proof we end up with a cut free proof of the empty sequent. But this is impossible, and therefore \( D_\varphi \) cannot be well-founded. To be able to carry out the removal of all cuts we have to enlist the help of arithmetical transfinite induction, roughly up to the ordinal \( \varphi \tau 0 \). Hence this is the step where the principle \( \forall x (\text{WO}(x) \rightarrow \text{WO}(\varphi x 0)) \) makes its appearance in showing the direction (ii) \( \Rightarrow \) (i) of Theorem 1.3.

2 The ordering \( \varphi x 0 \)

Via simple coding procedures, countable well-orderings and functions on them can be expressed in the language of second order arithmetic, \( \text{L}_2 \). Variables \( X, Y, Z, \ldots \) are supposed to range over subsets of \( \mathbb{N} \). Using an elementary injective pairing function \( \langle, \rangle \) (e.g. \( \langle n, m \rangle := (n + m)^2 + n + 1 \)), every set \( X \) encodes a sequence of sets \( \langle X \rangle_i \), where \( \langle X \rangle_i := \{ m \mid \langle i, m \rangle \in X \} \). We also adopt from [17], II.2 the method of encoding a finite sequence \( (n_0, \ldots, n_{k-1}) \) of natural numbers as a single number \( \langle n_0, \ldots, n_{k-1} \rangle \).

**Definition 2.1** Every set of natural numbers \( Q \) can be viewed as encoding a binary relation \( \triangleleft \) on \( \mathbb{N} \) such that \( n \triangleleft m \) iff \( \langle n, m \rangle \in Q \). The field of \( Q \), \( \text{fld}(Q) \) is the set \( \{ n \mid \exists m \langle n, m \rangle \in Q \wedge m \triangleleft n \} \).

We say that \( Q \) is a well-ordering if \( \triangleleft \) is a well-ordering, that is \( \triangleleft \) is a linear ordering of its field and every non-empty subset \( U \) of \( \text{fld}(Q) \) has a \( \triangleleft \)-least element.

**Definition 2.2** Let \( Q \) be a linear ordering with least element \( 0_Q \). Let \( \varphi u a := \langle 0, \langle u, a \rangle \rangle \), \( \mathbf{H} := \{ \varphi u a \mid u, a \in \mathbb{N} \} \), \( \mathbf{h}(\varphi u a) = u \) and \( \mathbf{h}(b) = 0_Q \) if \( b \notin \mathbf{H} \).

We introduce the ordering \( \varphi Q 0 \) by inductively defining its field \( \text{fld}(\varphi Q 0) \) and the ordering \( \triangleleft_{\varphi Q 0} : \)

1. \( 0 \in \text{fld}(\varphi Q 0) \).
2. \( 0 <_{\varphi Q 0} \alpha \) if \( \alpha \in \text{fld}(\varphi Q 0) \) and \( \alpha \neq 0 \).
3. $\varphi u \alpha \in \text{fld}(\varphi Q0)$ if $u \in \text{fld}(Q)$, $\alpha \in \text{fld}(\varphi Q0)$ and $h(\alpha) \leq_{Q} u$.

4. If $\alpha_1, \ldots, \alpha_n \in \text{fld}(\varphi Q0) \cap H$, $n > 1$ and $\alpha_n \leq_{\varphi Q0} \ldots \leq_{\varphi Q0} \alpha_1$, then

$$\alpha_1 + \ldots + \alpha_n \in \text{fld}(\varphi Q0)$$

where $\alpha_1 + \ldots + \alpha_n := \langle 1, \langle \alpha_1, \ldots, \alpha_n \rangle \rangle$.

5. If $\alpha_1 + \ldots + \alpha_n, \beta_1 + \ldots + \beta_m \in \text{fld}(\varphi Q0)$, then

$$\alpha_1 + \ldots + \alpha_n <_{\varphi Q0} \beta_1 + \ldots + \beta_m \text{ iff } n < m \land \forall i \leq n \alpha_i = \beta_i \text{ or }$$

$$\exists i \leq \min(n, m)[\alpha_i <_{\varphi Q0} \beta_i \land \forall j < i \alpha_j = \beta_j].$$

6. If $\alpha_1 + \ldots + \alpha_n \in \text{fld}(\varphi Q0)$, $\varphi u \beta \in \text{fld}(\varphi Q0)$ and $\varphi u \beta \leq_{\varphi Q0} \alpha_1$ then $\varphi u \beta <_{\varphi Q0} \alpha_1 + \ldots + \alpha_n$.

7. If $\alpha_1 + \ldots + \alpha_n \in \text{fld}(\varphi Q0)$, $\varphi u \beta \in \text{fld}(\varphi Q0)$ and $\alpha_1 <_{\varphi Q0} \varphi u \beta$ then $\alpha_1 + \ldots + \alpha_n <_{\varphi Q0} \varphi u \beta$.

8. If $\varphi u \alpha, \varphi v \beta \in \text{fld}(\varphi Q0)$, then

$$\varphi u \alpha <_{\varphi Q0} \varphi v \beta \text{ iff } u <_{Q} v \land \alpha <_{\varphi Q0} \varphi v \beta \text{ or }$$

$$u = v \land \alpha <_{\varphi Q0} \beta \text{ or }$$

$$v <_{Q} u \land \varphi u \alpha <_{\varphi Q0} \beta.$$

Lemma 2.3 (RCA$_0$)

(i) If $Q$ is a linear ordering then so is $\varphi Q0$.

(ii) $\varphi Q0$ is elementary recursive in $Q$.

3 The theory ATR$_0$

Definition 3.1 Let $A(u, Y)$ be any formula. Define $H_A(X, Y)$ to be the formula which says that $<_X$ is a linear ordering and that $Y$ is equal to the set of pairs $\langle n, j \rangle$ such that $j$ is in the field of $<_X$ and $A(n, Y^j)$ where $Y^j = \{ \langle m, i \rangle \mid i <_X j \land \langle m, i \rangle \in Y \}$. Intuitively $H_A(X, Y)$ says that $Y$ is the result of iterating $A$ along $<_X$.

$\text{ATR}_0$ is the formal system in the language of second order arithmetic whose axioms consist of $\text{ACA}_0$ plus all instances of

$$\forall X (\text{WO}(<_X) \rightarrow \exists Y H_A(X, Y))$$

where $A$ is arithmetical.

Lemma 3.2 $\text{ATR}_0$ can be axiomatized over $\text{ACA}_0$ via a single sentence

$$\forall X (\text{WO}(<_X) \rightarrow \forall Z \exists Y B_0(X, Y, Z)) \quad (2)$$

where $B_0(X, Y, Z)$ is of the form $H_A(X, Y)$ for some arithmetical formula $A(u, Y, Z)$ with all free variables exhibited.
Proof: This is a standard result. One could for instance take $B_0(X,Y,Z)$ to mean that $Y$ is obtained from $Z$ by iterating the Turing jump operation along $X$ starting with $Z$; so $A(u,Y,Z)$ would actually be a $\Sigma^0_1$ (complete) formula. Another (shorter and citable) way of showing this is to use the fact that $\text{ATR}_0$ is equivalent over $\text{RCA}_0$ to the statement that every two well-orderings are comparable (see [17], Theorem V.6.8). The proof of the latter statement in $\text{ATR}_0$ just requires an instance $H_A$ of said form (see the proof of [17] Lemma V.2.9).

Definition 3.3 Let $T$ be a theory in the language of second order arithmetic, $L_2$. A countable coded $\omega$-model of $T$ is a set $W \subseteq \mathbb{N}$, viewed as encoding the $L_2$-model

$$M = (\mathbb{N}, S, +, \cdot, 0, 1, <)$$

with $S = \{(W)_n | n \in \mathbb{N}\}$ such that $M \models T$.

This definition can be made in $\text{RCA}_0$ (see [17], Definition VII.2).

We write $X \in W$ if $\exists n X = (W)_n$.

4 Main Theorem

The main result we want to prove is the following.

Theorem 4.1 $\text{RCA}_0 + \forall X [\text{WO}(X) \rightarrow \text{WO}(\varphi_X)]$ proves $\text{ATR}_0$.

A central ingredient of the proof will be a method of proof search (deduction chains) pioneered by Schütte [15].

4.1 Deduction chains in $\omega$-logic

Definition 4.2 (i) Let $U_0, U_1, U_2, \ldots$ be an enumeration of the free set variables of $L_2$. For a closed term $t$, let $t^N$ be its numerical value. We shall assume that all predicate symbols of the language $L_2$ are symbols for primitive recursive relations. $L_2$ contains predicate symbols for the primitive recursive relations of equality and inequality and possibly more (or all) primitive recursive relations. If $R$ is a predicate symbol in $L_2$ we denote by $R^N$ the primitive recursive relation it stands for. If $t_1, \ldots, t_n$ are closed terms the formula $R(t_1, \ldots, t_n)$ ($\neg R(t_1, \ldots, t_n)$) is said to be true if $R^N(t_1^N, \ldots, t_n^N)$ is true (is false).

(ii) Henceforth a sequent will be a finite set of $L_2$-formulas without free number variables.

(iii) A sequent $\Gamma$ is axiomatic if it satisfies at least one of the following conditions:

1. $\Gamma$ contains a true literal, i.e. a true formula of either form $R(t_1, \ldots, t_n)$ or $\neg R(t_1, \ldots, t_n)$, where $R$ is a predicate symbol in $L_2$ for a primitive recursive relation and $t_1, \ldots, t_n$ are closed terms.

2. $\Gamma$ contains formulas $s \in U$ and $t \notin U$ for some set variable $U$ and terms $s, t$ with $s^N = t^N$.

(iv) A sequent is reducible or a redex if it is not axiomatic and contains a formula which is not a literal.

Definition 4.3 For $Q \subseteq \mathbb{N}$ define

$$D_Q(n) = \begin{cases} \bar{n} \in U_0 & \text{if } n \in Q \\ \bar{n} \notin U_0 & \text{otherwise} \end{cases}$$
For the proof of Theorem 4.1 it is convenient to have a finite axiomatization of arithmetic comprehension.

**Lemma 4.4** ACA\(_0\) can be axiomatized via a single \(\Pi^1_2\) sentence \(\forall X C(X)\).

**Proof:** [17], Lemma VIII.1.5.

**Definition 4.5** Let \(<Q\) be a well-ordering. Let \(B(U_i)\) be the formula \(\exists Y B_0(U_0, Y, U_i)\) of Lemma 3.2.

A \(Q\)-deduction chain is a finite string
\[ \Gamma_0, \Gamma_1, \ldots, \Gamma_k \]
of sequents \(\Gamma_i\) constructed according to the following rules:

(i) \(\Gamma_0 = \neg D_Q(0), \neg B(U_0), \neg C(U_0)\).

(ii) \(\Gamma_i\) is not axiomatic for \(i < k\).

(iii) If \(i < k\) and \(\Gamma_i\) is not reducible then
\[ \Gamma_{i+1} = \Gamma_i, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}) \]

(iv) Every reducible \(\Gamma_i\) with \(i < k\) is of the form
\[ \Gamma_i', E, \Gamma''_i \]
where \(E\) is not a literal and \(\Gamma_i'\) contains only literals.

\(E\) is said to be the **redex** of \(\Gamma_i\).

Let \(i < k\) and \(\Gamma_i\) be reducible. \(\Gamma_{i+1}\) is obtained from \(\Gamma_i = \Gamma_i', E, \Gamma''_i\) as follows:

1. If \(E \equiv E_0 \lor E_1\) then
\[ \Gamma_{i+1} = \Gamma_i', E_0, E_1, \Gamma''_i, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}) \]

2. If \(E \equiv E_0 \land E_1\) then
\[ \Gamma_{i+1} = \Gamma_i', E_j, \Gamma''_i, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}) \]
where \(j = 0\) or \(j = 1\).

3. If \(E \equiv \exists x F(x)\) then
\[ \Gamma_{i+1} = \Gamma_i', F(\bar{m}), \Gamma''_i, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}), E \]
where \(m\) is the first number such that \(F(\bar{m})\) does not occur in \(\Gamma_0, \ldots, \Gamma_i\).

4. If \(E \equiv \forall x F(x)\) then
\[ \Gamma_{i+1} = \Gamma_i', F(\bar{m}), \Gamma''_i, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}) \]
for some \(m\).
5. If \( E \equiv \exists X F(X) \) then
\[
\Gamma_{i+1} = \Gamma'_i, F(U_m), \Gamma''_i, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1}), E
\]
where \( m \) is the first number such that \( F(U_m) \) does not occur in \( \Gamma_0, \ldots, \Gamma_i \).

6. If \( E \equiv \forall X F(X) \) then
\[
\Gamma_{i+1} = \Gamma'_i, F(U_m), \Gamma''_i, \neg D_Q(i+1), \neg B(U_{i+1}), \neg C(U_{i+1})
\]
where \( m \) is the first number such that \( m \neq i + 1 \) and \( U_m \) does not occur in \( \Gamma_i \).

The set of \( Q \)-deduction chains forms a tree \( D_Q \) labeled with strings of sequents. We will first consider the case that \( D_Q \) is not well-founded. Then \( D_Q \) contains an infinite path \( P \). Now define a set \( M \) via
\[
(M)_i = \{ t^n | t \notin U_i \text{ occurs in } P \}.
\]
Set \( M = (\mathbb{N}; ((M)_i | i \in \mathbb{N}), +, \cdot, 0, 1, <) \).

For a formula \( F \), let \( F \in P \) mean that \( F \) occurs in \( P \), i.e. \( F \in \Gamma \) for some \( \Gamma \in P \).

**Claim:** Under the assignment \( U_i \mapsto (M)_i \) we have:
\[
F \in P \quad \Rightarrow \quad M \models \neg F. \tag{3}
\]

The Claim will imply that \( M \) is an \( \omega \)-model of \( \text{ACA} \). Also note that \( (M)_0 = Q \), thus \( Q \) is in \( M \).

The proof of (3) follows by induction on \( F \) using Lemma 4.6 below. The upshot of the foregoing is that we can prove Theorem 4.1 under the assumption that \( D_Q \) is ill-founded for all sets \( Q \subseteq \mathbb{N} \).

**Lemma 4.6** Let \( Q \) be an arbitrary subset of \( \mathbb{N} \) and \( D_Q \) be the corresponding deduction tree. Moreover, suppose \( D_Q \) is not well-founded. Then \( D_Q \) has an infinite path \( P \). \( P \) has the following properties:

1. \( P \) does not contain literals which are true in \( \mathbb{N} \).
2. \( P \) does not contain formulas \( s \in U_i \) and \( t \notin U_i \) for constant terms \( s \) and \( t \) such that \( s^N = t^N \).
3. If \( P \) contains \( E_0 \lor E_1 \) then \( P \) contains \( E_0 \) and \( E_1 \).
4. If \( P \) contains \( E_0 \land E_1 \) then \( P \) contains \( E_0 \) or \( E_1 \).
5. If \( P \) contains \( \exists x F(x) \) then \( P \) contains \( F(\bar{n}) \) for all \( n \).
6. If \( P \) contains \( \forall x F(x) \) then \( P \) contains \( F(\bar{n}) \) for some \( n \).
7. If \( P \) contains \( \exists X F(X) \) then \( P \) contains \( F(U_m) \) for all \( m \).
8. If \( P \) contains \( \forall X F(X) \) then \( P \) contains \( F(U_m) \) for some \( m \).
9. \( P \) contains \( \neg B(U_m) \) for all \( m \).
10. \( P \) contains \( \neg C(U_m) \) for all \( m \).
11. \( P \) contains \( \neg D_Q(m) \) for all \( m \).
Proof: Standard.

Corollary 4.7 If $D_Q$ is ill-founded then there exists a countable coded $\omega$-model of $\text{ACA}_0$ containing $Q$ which satisfies $\forall Z \exists Y B_0(Q, Y, Z)$.

The remainder of the paper will be devoted to ruling out the possibility that, whenever $Q$ is a well-ordering, $D_Q$ can be a well-founded tree. This is the place where cut elimination for the infinitary proof system of ramified analysis, $\text{RA}^*$ (see [15], part C), enters the stage. In a nutshell, the idea is that a well-founded $D_Q$ gives rise to a derivation of the empty sequent (contradiction) in $\text{RA}^*$ which can be ruled out by showing cut elimination for $\text{RA}^*$ using transfinite induction up to $\varphi\exists X$, where $X$ is a well ordering not much longer than $Q$. However, to simplify the technical treatment we first introduce an intermediate system $\Delta^1_{\omega}-\text{CR}_Q\infty$ based on the $\Delta^1_{\omega}$-comprehension rule and the $\omega$-rule. This theory basically coincides with Schütte’s system $\text{DA}^*$ (see [15], part C). It is not difficult to see that a well-founded $D_Q$ can be viewed as a derivation of the empty sequent in $\Delta^1_{\omega}-\text{CR}_Q\infty$. The last step towards reaching a contradiction consists in embedding $\Delta^1_{\omega}-\text{CR}_Q\infty$ into $\text{RA}^*$. Here we can basically follow [15] Theorem 22.14.

4.2 The infinitary calculus $\Delta^1_{\omega}-\text{CR}_Q\infty$

In what follows we fix $Q \subseteq \mathbb{N}$ such that $<_Q$ is a well-ordering. In the main, the system $\Delta^1_{\omega}-\text{CR}_Q\infty$ is obtained from $\text{ACA}_0$ by adding the $\Delta^1_{\omega}$-comprehension rule, the $\omega$-rule and the basic diagram of $Q$. The language of $\Delta^1_{\omega}-\text{CR}_Q\infty$ is the same as that of $\text{ACA}_0$ but the notion of formula comes enriched with set terms. Formulas and set terms are defined simultaneously. Literals are formulas. Every set variable is a set term. If $A(x)$ is a formula without set quantifiers (i.e. arithmetical) then $\{x \mid A(x)\}$ is a set term. If $P$ is a set term and $t$ is a numerical term then $t \in P$ and $t /\notin P$ are formulas. The other formation rules pertaining to $\land, \lor, \forall x, \exists x, \forall X, \exists X$ are as per usual.

We will be working in a Tait-style formalization of the second order arithmetic with formulas in negation normal form, i.e. negations only in front of atomic formulas. Due to the $\omega$-rule there is no need for formulas with free numerical variables. Thus all sequents below are assumed to consist of formulas without free numerical variables.

**Axioms of $\Delta^1_{\omega}-\text{CR}_Q\infty$**

(i) $\Gamma, L$ where $L$ is a true literal.

(ii) $\Gamma, s \in U, t /\notin U$ where $s^N = t^N$.

(iii) $\Gamma, s \in U_0$ if $s^N \in Q$.

(iv) $\Gamma, s /\notin U_0$ if $s^N /\notin Q$.

**Rules of $\Delta^1_{\omega}-\text{CR}_Q\infty$**

$$\text{(\&)} \quad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \land B}$$
(∀) \[ \frac{\Gamma, A_i}{\Gamma, A_0 \lor A_1} \] where \( i \in \{0, 1\} \).

(Cut) \[ \frac{\Gamma, A}{\Gamma, \neg A} \]

(ω) \[ \frac{\Gamma, F(\bar{n})}{\Gamma, \forall x F(x)} \] for all \( n \)

(∃) \[ \frac{\Gamma, F(t)}{\Gamma, \exists x F(x)} \]

(∀) \[ \frac{\Gamma, F(P)}{\Gamma, \forall X F(X)} \] for all set terms \( P \)

(∃) \[ \frac{\Gamma, F(P)}{\Gamma, \exists X F(X)} \] where \( P \) is a set term.

(Δ₁⁻CR) \[ \frac{\forall x \forall Y A_0(x,Y) \leftrightarrow \exists Y A_1(x,Y)}{\Gamma, \exists X \forall x [x \in X \leftrightarrow \forall Y A(x,Y)]} \] with \( A_0, A_1 \) arithmetical.

(ST₁) \[ \frac{\Gamma, A(t)}{\Gamma, t \in P} \] where \( P \) is the set term \( \{ x \mid A(x) \} \).

(ST₂) \[ \frac{\Gamma, \neg A(t)}{\Gamma, t \notin P} \] where \( P \) is the set term \( \{ x \mid A(x) \} \).

\( \Delta^1_{\text{CR}} \) is a sequent calculus version of the system \( \text{DA}^* \) of [15, §20]. The language of \( \text{DA}^* \), though, is based on the connectives \( \bot, \forall, \rightarrow \) while \( \Delta^1_{\text{CR}} \) has the connectives \( \land, \lor, \forall, \exists, \neg \) and formulas are in negation normal form, i.e. the negation sign appears only in front of atomic formulas. The other main difference is that the deduction system of \( \text{DA}^* \) is the Schütte calculus of positive and negative forms whereas \( \Delta^1_{\text{CR}} \)’s is the Gentzen sequent calculus.

**Lemma 4.8** We shall use \( \Delta^1_{\text{CR}} \) to convey that the sequent \( \Gamma \) is derivable in \( \Delta^1_{\text{CR}} \).

Pivotal properties of \( \Delta^1_{\text{CR}} \) we shall exploit are the following:

(a) \( n \in Q \Rightarrow \Delta^1_{\text{CR}} \vdash \bar{n} \in U_0 \).

(b) \( n \notin Q \Rightarrow \Delta^1_{\text{CR}} \vdash \bar{n} \notin U_0 \).

(c) \( \Delta^1_{\text{CR}} \vdash \text{WO}(U_0) \).

(d) \( \Delta^1_{\text{CR}} \vdash \exists Y H_A(U_0,Y) \) for all arithmetical formulas \( A(u,Y) \) having no other free numerical variables than \( u \).

**Proof:** (a) and (b) are immediate by the axioms (iii) and (iv) of \( \Delta^1_{\text{CR}} \).

(c) follows by (outer) transfinite induction on \( \prec_Q \), crucially using the \( \omega \)-rule. This is standard but it seems to be a challenge to find a reference. Via the axioms (iii) and (iv), the role of \( Q \) is
Lemma 4.10. If those derivations will be "longer" than the lengths of the foregoing derivations, then the ordinal \( \omega \) to the ordinal \( 1 \), which is the same as \( \Delta \). The rationale for the choice of \( \omega \) is that it gives us enough elbow room for calibrating the lengths of the foregoing derivations. Hence via (\( \exists_i \)) we arrive at

\[
\Delta \vdash \neg \text{Prog}_Q(U), \exists \forall(y < Q \ n \rightarrow y \in U) \land \neg \in Q, \in Q,
\]

which is the same as \( \Delta \vdash \neg \text{Prog}_Q(U), \in Q \). Thus, by induction on \( <_Q \), (4) follows.

(d) also follows by transfinite induction on \( <_Q \) using \( \Delta \). A reference will be provided in Lemma 4.10.

We shall need to measure the length of the previous derivations. For (c) and (d) the lengths of those derivations will be "longer" than \( Q \), though not "much longer". Let \( \tau \) be the ordinal giving the order-type of \( Q \). It is easy to cook up a new ordering \( Q^\star \) in an elementary way from \( Q \) corresponding to the ordinal \( \omega^2 + \omega \cdot \tau + \omega \) in such a way that \( \text{RCA}_0 \) suffices to prove \( \text{WO}(Q) \rightarrow \text{WO}(Q^\star) \) (see [9]). The rationale for the choice of \( \omega^2 + \omega \cdot \tau + \omega \) is that it gives us enough elbow room for calibrating the lengths of the foregoing derivations.

From the standing assumption that \( Q \) is a well-ordering we get that \( Q^\star \) is a well-ordering, too.

**Definition 4.9** If \( \alpha \) is an element of the field of \( < Q^\star \), we use the notation \( \Delta \vdash \text{CR}_Q \vdash \alpha \Gamma \) to convey that the sequent \( \Gamma \) is deducible in \( \Delta \vdash \text{CR}_Q \), via a derivation of length \( \leq \alpha \). More formally, this relation is defined by recursion on \( \alpha \) as follows: \( \Delta \vdash \text{CR}_Q \vdash \alpha \Gamma \) holds if if either \( \Gamma \) is an axiom of \( \Delta \vdash \text{CR}_Q \) or \( \Gamma \) is the conclusion of a \( \Delta \vdash \text{CR}_Q \)-inference with premises \( (\Gamma_i)_{i \in I} \) such that for every \( i \in I \) there exists \( \beta_i < Q^\star \) with \( \Delta \vdash \text{CR}_Q \vdash \beta \Gamma_i \).

**Lemma 4.10** 1. \( \Delta \vdash \text{CR}_Q \vdash D_Q(n) \) for all \( n \) with \( 0 \) being the least element of \( Q \).

2. \( \Delta \vdash \text{CR}_Q \vdash C(U) \) for some \( \alpha \in \text{field}(Q^\star) \) and all free set variables \( U \).

3. \( \Delta \vdash \text{WO}(U_0) \) for some \( \beta \in \text{field}(Q^\star) \).

4. \( \Delta \vdash \text{CR}_Q \vdash \exists Y H_A(U_0, Y) \) for some \( \gamma \in \text{field}(Q^\star) \) for all arithmetical formulas \( A(u, Y) \) having no other free numerical variables than \( u \).
5. \( \Delta_1^{1}-\text{CR}_\infty^Q \vdash^\delta B(U) \) for some \( \delta \in \text{field}(Q^*) \) and all free set variables \( U \).

**Proof:** (1) is an immediate consequence of Lemma 4.8 (a) and (b). (2) follows since the rule (\( \exists_2 \)) gives arithmetical comprehension. (3) and (4) correspond to Lemma 4.8 (c) and (d), respectively. A detailed proof of (4) amounts to basically the same as that of [15, \$21 Lemma 14]. (5) is an immediate consequence of (4). \( \square \)

Recall that, by Corollary 4.7, there exists a countable coded \( \omega \)-model of \( \text{ACA}_0 \) containing \( Q \) and satisfying \( \forall Z \exists Y B_0(Q,Y,Z) \) providing \( D_Q \) is ill-founded. Now let us assume that \( Q \) is a well-ordering and that \( D_Q \) is well-founded. Then \( D_Q \) can be viewed as a deduction with hidden cuts involving formulas of the shape \( \neg B(U_{i+1}), \neg C(U_{i+1}) \) and \( \neg D_Q(i+1) \). Note that by Lemma 4.10, \( \Delta_1^{1}-\text{CR}_\infty^Q \vdash^0 D_Q(n), \Delta_1^{1}-\text{CR}_\infty^Q \vdash C(U), \) and \( \Delta_1^{1}-\text{CR}_\infty^Q \vdash B(U) \) for some \( \alpha, \gamma \in \text{field}(Q^*) \). Thus if \( \Gamma \) is the sequent attached to a node \( \tau \) of \( D_Q \) and \( (\Gamma_i)_{i \in I} \) is an enumeration of the sequents attached to the immediate successor nodes of \( \tau \) in \( D_Q \) then the transition

\[
(\Gamma_i)_{i \in I} \quad \Gamma
\]

can be viewed as a combination of four inferences in \( \Delta_1^{1}-\text{CR}_\infty^Q \), the first one being a logical inferences and the other three being cuts. By interspersing \( D_Q \) with cuts and adding three cuts with cut formulas \( \neg C(U_0), \neg B(U_0) \) and \( \neg D_Q(0) \) at the bottom we obtain a derivation \( \tilde{D}_Q \) in \( \Delta_1^{1}-\text{CR}_\infty^Q \) of the empty sequent. Since the preceding line of arguments can be done in \( \text{ACA}_0 \) we arrive at the following:

**Corollary 4.11 (ACA\(_0\))** If \( Q \) is a well-ordering and \( D_Q \) is well-founded then there is a derivation \( \tilde{D}_Q \) in \( \Delta_1^{1}-\text{CR}_\infty^Q \) of the empty sequent.

To finish the paper we thus have to show that the latter is impossible. This we shall do by embedding \( \Delta_1^{1}-\text{CR}_\infty^Q \) into a system \( \text{RA}^\infty \) defined below. Note that an upper bound for the length of \( \tilde{D}_Q \) is provided by \( (\alpha + \gamma + \rho) \cdot 4 \), where \( \rho \) corresponds to the Kleene-Brouwer ordering on \( D_Q \).

5 Ramified Analysis \( \text{RA}^\infty \)

The theories \( \text{RA}_\rho \) are designed to capture Gödel’s notion of constructibility restricted to sets of natural numbers. They use ordinal indexed variables \( X^\alpha, Y^\alpha, Z^\alpha, \ldots \) for \( \alpha < \rho \), with the intended meaning that level 0 variables range over sets definable by numerical quantification, and level \( \alpha > 0 \) variables range over sets definable by numerical quantification and level \( \alpha \) set quantification. The proof-theoretic ordinal of \( \text{RA}_\alpha \) is \( \varphi \alpha \). We are interested in an infinitary version of ramified analysis.

**Definition 5.1** \( \text{RA}^\infty \) is basically the same system as \( \text{RA}^\ast \) in [15, \$22]. One difference is that the language of \( \text{RA}^\ast \) is based on the connectives \( \bot, \forall, \to \) while \( \text{RA}^\infty \) has \( \land, \lor, \forall, \exists, \neg \) and formulas are in negation normal form, i.e. the negation sign appears only in front of atomic formulas. The other difference is that the deduction system of \( \text{RA}^\ast \) is the Schütte calculus of positive and negative forms whereas \( \text{RA}^\infty \)’s is the Gentzen sequent calculus.

The formulas of \( \text{RA}^\infty \) do not have free numerical variables. Literals are formulas of the form \( R(t_1, \ldots, t_n) \) and \( \neg R(t_1, \ldots, t_n) \) with \( R \) being a symbol for a primitive recursive relation and \( t_1, \ldots, t_n \) being closed numerical terms.
\( \text{RA}^\infty \) uses ordinal indexed free set variables \( U^\alpha, V^\alpha, W^\alpha, \ldots \) and bound set variables \( X^\beta, Y^\beta, Z^\beta, \ldots \) with \( \beta > 0 \), where the ordinals are assumed to be elements of some countable well-ordering \( R \).

The set terms and formulas together with their levels are generated as follows (cf. [15, §22]):

1. Every literal is a formula of level 0.
2. Every free set variable \( U^\alpha \) is a set term of level \( \alpha \).
3. If \( P \) is a set term of level \( \alpha \) and \( t \) is a numerical term, then \( t \in P \) and \( t \notin P \) are formulas of level \( \alpha \).
4. If \( A \) and \( B \) are formulas of levels \( \alpha \) and \( \beta \), then \( A \lor B \) and \( A \land B \) are formulas of level \( \max(\alpha, \beta) \).
5. If \( F(0) \) is a formula of level \( \alpha \), then \( \forall x F(x) \) and \( \exists x F(x) \) are formulas of level \( \alpha \) and \( \{ x \mid F(x) \} \) is a set term of level \( \alpha \).
6. If \( F(U^\beta) \) is a formula of level \( \alpha \) and \( \beta > 0 \), then \( \forall X^\beta F(X^\beta) \) is a formula of level \( \max(\alpha, \beta) \).

**Definition 5.2** The calculus \( \text{RA}_Q^\infty \)

**Axioms**

\( \Gamma, L \) where \( L \) is a true literal.
\( \Gamma, s \in U^\alpha, t \notin U^\alpha \) where \( s^N = t^N \).
\( \Gamma, s \in U_0 \) if \( s^N \in Q \).
\( \Gamma, s \notin U_0 \) if \( s^N \notin Q \).

**Rules**

\((\land), (\lor), (\omega), \) numerical \((\exists)\) and \((\text{Cut})\) as per usual;

\[
(\exists^\alpha) \quad \frac{\Gamma, F(P)}{\Gamma, \exists X^\alpha F(X^\alpha)} \quad \text{P set term of level < } \alpha.
\]

\[
(\forall^\alpha) \quad \frac{\Gamma, F(P)}{\Gamma, \forall X^\alpha F(X^\alpha)} \quad \text{for all set terms } P \text{ of level < } \alpha
\]

\[
(ST_1) \quad \frac{\Gamma, F(t)}{\Gamma, t \in \{ x \mid F(x) \}}
\]

\[
(ST_2) \quad \frac{\Gamma, \neg F(t)}{\Gamma, t \notin \{ x \mid F(x) \}}
\]
Definition 5.3 The cut rank of a formula $A$ in $\mathbf{RA}_Q^\infty$, $|A|$, is defined as follows (cf. [15, §22]):

1. $|L| = 0$ for arithmetical literals $L$.

2. $|t \in U^\alpha| = |t \notin U^\alpha| = \omega \cdot \alpha$.

3. $|B_0 \land B_1| = |B_0 \lor B_1| = \max(|B_0|, |B_1|) + 1$.

4. $|\forall x B(x)| = |\exists x B(x)| = |t \in \{x \mid B(x)\}| = |t \notin \{x \mid B(x)\}| = |B(0)| + 1$.

5. $|\forall X^\alpha A(X^\alpha)| = |\exists X^\alpha A(X^\alpha)| = \max(\omega \cdot \gamma, |A(U^0)| + 1)$, where $\gamma$ is the level of $\forall X^\alpha A(X^\alpha)$.

By recursion on $\alpha$ we define the relation $\mathbf{RA}_Q^\infty \vdash_\rho^{\alpha} \Gamma$ as follows: $\mathbf{RA}_Q^\infty \vdash_\rho^{\alpha} \Gamma$ holds if either $\Gamma$ is an axiom of $\mathbf{RA}_Q^\infty$ or $\Gamma$ is the conclusion of an $\mathbf{RA}_Q^\infty$-inference with premisses $(\Gamma_i)_{i \in I}$ such that for every $i \in I$ there exists $\beta_i < \alpha$ with $\mathbf{RA}_Q^\infty \vdash_\beta^{\beta_i} \Gamma_i$ and, moreover, if this inference is a cut with cut formula $A$ then $|A| < \rho$.

The following three statements are proved in [15] for the system $\mathbf{RA}^*$, It is routine to transfer them to $\mathbf{RA}_Q^\infty$ since cut elimination in a Schütte calculus of positive and negative is closely related to cut elimination in sequent calculi. Moreover, the additional axioms pertaining to $Q$ do not impede the cut elimination process.

Theorem 5.4 (Cut-elimination I)

$\mathbf{RA}_Q^\infty \vdash_{\eta+1}^{\alpha} \Gamma \Rightarrow \mathbf{RA}_Q^\infty \vdash_\rho^{\alpha} \Gamma$.

Proof: Similar to [15, §22 Lemma 4].

Theorem 5.5 (Cut-elimination II)

$\mathbf{RA}_Q^\infty \vdash_{\omega^\beta}^{\alpha} \Gamma \Rightarrow \mathbf{RA}_Q^\infty \vdash_{0^\beta}^{\omega^\alpha} \Gamma$.

Proof: Similar to [15, Theorem 22.7].

For a formula $F$ of the language of $\Delta_1^1\text{-CR}_Q^\infty$ let $F^\sigma$ be the result of replacing every bound variable $X$ by $X^\sigma$ and every free set variable by a set term of a level $< \sigma$. For $\Gamma = \{F_1, \ldots, F_n\}$ let $\Gamma^\sigma = \{F_1^\sigma, \ldots, F_n^\sigma\}$.

Theorem 5.6 (Interpretation Theorem)

$\Delta_1^1\text{-CR}_Q^\infty \vdash_\alpha^{\sigma} \Gamma \Rightarrow \mathbf{RA}_Q^\infty \vdash_{\sigma^\omega+\omega^+\omega^\alpha}^{\omega^\sigma+\omega^+\omega^\alpha} \Gamma^\sigma$ for all $\sigma$ of the form $\omega^\alpha \cdot \beta$ with $\beta \neq 0$.

Proof: This is basically the same as [15, Theorem 22.14].

There are different ways of formalizing infinite deductions in theories like $\mathbf{PA}$. We just mention [16] and [7].

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5.1 Finishing the proof of the main Theorem

Recall that in order to finish the proof of Theorem 4.1 we want to show that \( D_Q \) is not well-founded whenever \( Q \) is a well-ordering. By Corollary 4.11, if \( Q \) is a well-ordering and \( D_Q \) is well-founded then there is a derivation \( D_Q \) in \( \Delta^1\text{-CR}_Q \) of the empty sequent. By the Interpretation Theorem 5.6 we would then get a derivation in \( \text{RA}_Q^\infty \) of the empty sequent. Using the principle \( \text{WO}(\mathcal{X}) \rightarrow \text{WO}(\varphi\mathcal{X}) \) we can then employ the cut elimination Theorem 5.5 to obtain a cut-free derivation of the empty sequent in \( \text{RA}_Q^\infty \). But this is impossible.

From Corollary 4.7 we can thus conclude that for every well-ordering \( \tilde{Q} \) there exists a countable coded \( \omega \)-model of \( \text{ACA}_0 \) containing \( \tilde{Q} \) and satisfying

\[
\forall Z \exists Y B_0(\tilde{Q}, Y, Z).
\]

From this we would like to infer that for every well-ordering \( Q \) and every set \( Z_0 \) there exists a set \( Y \) such that \( B_0(\tilde{Q}, Y, Z_0) \). We can do this by encoding \( Q \) and \( Z_0 \) in a well-ordering \( \tilde{Q} \) from which \( Q \) and \( Z_0 \) can be retrieved in any \( \omega \)-model of \( \text{ACA}_0 \) containing \( \tilde{Q} \). One way of doing this is to define the new ordering \( \tilde{Q} \) by letting

\[
\langle n, m \rangle <_{\tilde{Q}} \langle n', m' \rangle \quad \text{iff} \quad \begin{cases} n = n' = 0 & \land \ m <_{\tilde{Q}} m' \\ n = n' = 1 & \land \ m, m' \in Z_0 & \land \ m < m' \\ n = 0 & \land \ n' = 1 & \land \ m \in \text{field}(Q) & \land \ m' \in Z_0. \end{cases}
\]

Obviously \( \tilde{Q} \) is a well-ordering, too, and any \( \omega \)-model \( M \) of \( \text{ACA}_0 \) containing \( \tilde{Q} \) will contain \( Z_0 \) as well. Moreover, \( M \models \exists Y B_0(\tilde{Q}, Z_0) \) implies \( M \models \exists Y B_0(Q, Z_0) \). Hence, in view of Lemma 3.2, we get \( \text{ATR}_0 \), thereby finishing the proof of Theorem 4.1.

6 Finishing the proof of Theorem 1.3

One direction of Theorem 1.3 follows from Theorem 4.1. The other direction is implicit in the proof of [15] Theorem 21.6.

7 Prospectus

The methodology exemplified in the proof of Theorem 1.3 should have many more applications. Every cut elimination theorem in ordinal-theoretic proof theory potentially encapsulates a theorem of type 1.3. The first author has looked at two more examples and sketched proofs of the pertaining theorems. A familiar function from proof theory is the \( \Gamma \)-function where \( \alpha \mapsto \Gamma_\alpha \) enumerates the fixed points of the \( \varphi \)-function. Since the proof of the next result has only been sketched we classify it as a conjecture.\(^1\)

\textbf{Conjecture 7.1} Over \( \text{RCA}_0 \) the following are equivalent:

\begin{enumerate}
\item Every set \( X \) is contained in a countable coded \( \omega \)-model of \( \text{ATR}_0 \).
\item \( \forall \mathcal{X} [\text{WO}(\mathcal{X}) \rightarrow \text{WO}(\Gamma_\mathcal{X})] \).
\end{enumerate}

\(^1\)Recently this conjecture has been proved in [14].
The direction (i)⇒(ii) follows from [12, 4.13, 4.16].

For an example from impredicative proof theory one would perhaps first turn to the ordinal representation system used for the ordinal analysis of the theory $\text{ID}_1$ of non-iterated inductive definitions, which can be expressed in terms of the $\theta$-function (cf. [4]). $\text{ID}_1$ has the same strength as the subsystem of second order arithmetic based on bar induction, $\text{BI}$ (cf. [4, 5, 13]). In Simpson’s book the acronym used for $\text{BI}$ is $\Pi^1_\omega\text{-TI}_0$ (cf. [17, §VII.2]). In place of the function $\theta$ we prefer to work with simpler ordinal representations based on the $\psi$-function introduced in [3] or the $\vartheta$-function of [13]. For definiteness we refer to [13]. Given a well-ordering $\mathcal{X}$, the relativized versions $\vartheta^\mathcal{X}$ and $\psi^\mathcal{X}$ of the $\vartheta$-function and the $\psi$-function, respectively, are obtained by adding all the ordinals from $\mathcal{X}$ to the sets $C_n(\alpha, \beta)$ of [13, §1] and $C_n(\alpha)$ of [13, Definition 3.1] as initial segments, respectively. The resulting well-orderings $\vartheta^\mathcal{X}(\varepsilon_{\Omega+1})$ and $\psi^\mathcal{X}(\varepsilon_{\Omega+1})$ are equivalent owing to [13, Corollary 3.2].

Again, as the following statement has not been buttressed by a complete proof we formulate it as a conjecture.

**Conjecture 7.2** Over $\text{RCA}_0$ the following are equivalent:

(i) $\text{RCA}_0 +$ Every set $X$ is contained in a countable coded $\omega$-model of $\text{BI}$.

(ii) $\forall X \left[ \text{WO}(X) \longrightarrow \text{WO}(\psi^X(\varepsilon_{\Omega+1})) \right]$.

**References**


