# On the regular extension axiom and its variants

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**Abstract**

The regular extension axiom, **REA**, was first considered by Peter Aczel in the context of Constructive Zermelo-Fraenkel Set Theory as an axiom that ensures the existence of many inductively defined sets. **REA** has several natural variants. In this note we gather together metamathematical results about these variants from the point of view of both classical and constructive set theory.

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## 1 Introduction

The first large set axiom proposed in the context of constructive set theory was the *Regular Extension Axiom*, **REA**, which Aczel introduced to accommodate inductive definitions in Constructive Zermelo-Fraenkel Set Theory, **CZF** (cf. [Acz78], [Acz86]).

**Definition: 1.1** For sets $A, B$ let ${}^{A}B$ be the class of all functions with domain $A$ and with range contained in $B$. Let $\mathbf{mv}({}^{A}B)$ be the class of all sets $R \subseteq A \times B$ satisfying $\forall u{\in}A\, \exists v{\in}B\, \langle u, v\rangle \in R$.

The expression $\mathbf{mv}({}^{A}B)$ should be read as the collection of *multi-valued functions* from the set $A$ to the set $B$.

On the basis of **CZF**, ${}^{A}B$ is a set for all sets $A, B$, while, in general, it cannot be shown that $\mathbf{mv}({}^{A}B)$ is a set.

**Definition: 1.2** A set $C$ is said to be *regular* if it is transitive, inhabited (i.e. $\exists u\; u \in C$) and for any $u{\in}C$ and $R \in \mathbf{mv}({}^{u}C)$ there exists a set $v \in C$ such that

$$\forall x{\in}u\, \exists y{\in}v\, \langle x, y\rangle \in R \;\wedge\; \forall y{\in}v\, \exists x{\in}u\, \langle x, y\rangle \in R.$$

We write $\mathbf{Reg}(C)$ to express that $C$ is regular.

**REA** is the principle

$$\forall x\, \exists y\, (x \subseteq y \;\wedge\; \mathbf{Reg}(y)).$$

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**Definition: 1.3** There are interesting weakened notions of regularity. A transitive inhabited set $C$ is *weakly regular* if for any $u \in C$ and $R \in \mathbf{mv}({}^{u}C)$ there exists a set $v \in C$ such that

$$\forall x \in u \, \exists y \in v \, \langle x, y \rangle \in R.$$

We write $\mathbf{wReg}(C)$ to express that $C$ is weakly regular. The *Weak Regular Extension Axiom* (**wREA**) is as follows:

*Every set is a subset of a weakly regular set.*

A transitive inhabited set $C$ is *functionally regular* if for any $u \in C$ and function $f : u \to C$, $\mathbf{ran}(f) \in C$. We write $\mathbf{fReg}(C)$ to express that $C$ is functionally regular. The *Functional Regular Extension Axiom* (**fREA**) is as follows:

*Every set is a subset of a functionally regular set.*

**Definition: 1.4** A class $A$ is said to be $\bigcup$*-closed* if for all $x \in A$, $\bigcup x \in A$.

A class $A$ is said to be *closed under Exponentiation (Exp-closed)* if for all $x, y \in A$, ${}^{x}y \in A$.

One is naturally led to consider strengthenings of the notion of a regular set, for instance that the set should also be $\bigcup$-closed and Exp-closed. A transitive inhabited set $C$ is said to be *strongly regular* if $C$ is regular, $\bigcup$-closed and Exp-closed. The *Strong Regular Extension Axiom* (**sREA**) is as follows:

*Every set is a subset of a strongly regular set.*

**Lemma: 1.5** (**CZF**) *If $A$ is regular then $A$ is weakly regular and functionally regular.*

**Proof:** Obvious. □

**Lemma: 1.6** (**CZF**) *If $A$ is functionally regular and $\mathbf{2} \in A$, then $A$ is closed under Pairing, that is $\forall x, y \in A \; \{x, y\} \in A$.*

**Proof:** Given $x, y \in A$ define a function $g : \mathbf{2} \to A$ by $g(\mathbf{0}) = x$ and $g(\mathbf{1}) = y$. Then $\{x, y\} = \mathbf{ran}(g) \in A$. □

**Definition: 1.7** If $A$ is a set, $H(A)$ denotes the smallest class $Y$ such that whenever $b \in A$ and $f : b \to Y$, then the range of $f$ is in $Y$. $H(A)$ is the class of sets hereditarily an image of a set in $A$, where $b$ is an image of $a$ if there is a function from $a$ onto $b$. For example $H(\omega)$ is the class of hereditarily finite sets and $H(\omega \cup \{\omega\})$ is the class of hereditarily countable sets.

**Proposition: 1.8** (**CZF** + **wREA**) *For every set $A$, $H(A)$ is a set.*

**Proof:** Inspection of the proof of [Acz86], Theorem 5.2 shows that one needs only **CZF** + **wREA** (rather than **CZF** + **REA**) to show that every bounded inductive definition inductively defines a set. As $H(A)$ is defined by a bounded inductive definition, it follows from [Acz86], Corollary 5.3 that $H(A)$ is a set. □

## 2 Proof-theoretic strengths

With respect to proof-theoretic strength, the axioms **wREA**, **REA**, and **sREA** have a markedly different effect when added to **CZF** than when added to **ZF**.

**Theorem: 2.1** *(i)* **CZF** *has the same proof-theoretic strength as classical Kripke-Platek set theory (with the Infinity axiom).*

*(ii)* **CZF+wREA** *has a (much) greater proof-theoretic strength than* **CZF**. *The theories* **CZF + wREA**, **CZF + REA**, *and* **CZF + sREA** *have the same proof-theoretic strength as the subsystem of second order arithmetic with $\mathbf{\Delta}^1_2$-Comprehension and Bar Induction.*

**Proof:** See [R01], Theorem 4.7. □

**Open Problem.** It is not known whether **CZF + fREA** is stronger than **CZF**.

One can easily show that **ZF + REA** is not stronger than **ZF**. But it turns out that **ZF + sREA** is stronger than **ZF**.

**Lemma: 2.2** *On the basis of* **ZFC**, *a set $B$ is regular if and only if $B$ is functionally regular.*

**Proof:** Obvious. □

**Proposition: 2.3** **ZFC** *proves* **REA**, *and hence* **ZF** *and* **ZF + REA** *have the same proof-theoretic strength.*

**Proof:** The axiom of choice implies that arbitrarily large regular cardinals exists and that for each regular cardinal $\kappa$, $H(\kappa)$ is a regular set. Given any set $b$ let $\mu$ be the cardinality of $\mathbf{TC}(b) \cup \{b\}$. Then the next cardinal after $\mu$, denoted $\mu^+$, is regular and $b \in H(\mu^+)$.□

**Lemma: 2.4** (**ZF**) *If $A$ is a functionally regular $\bigcup$-closed set with $\mathbf{2} \in A$, then the least ordinal not in $A$, $o(A)$, is a regular ordinal.*

**Proof:** If $f : \alpha \to o(A)$, where $\alpha < o(A)$, then $\alpha \in A$ and thus $\mathbf{ran}(f) \in A$, and hence $\bigcup \mathbf{ran}(f) \in A$. Since $\mathbf{ran}(f)$ is a set of ordinals, $\bigcup \mathbf{ran}(f)$ is an ordinal, too. Let $\beta = \bigcup \mathbf{ran}(f)$. Then $\beta \in A$. Note that $\beta + 1 \in A$ as well since $\mathbf{2} \in A$ entails that $A$ is closed under Pairing and $\beta + 1 = \bigcup\{\beta, \{\beta\}\}$. Since $f : \alpha \to \beta + 1$ this shows that $o(A)$ is a regular ordinal. □

**Proposition: 2.5** **ZFC** *does not prove that there exists a strongly regular set containing $\omega$. Moreover, in* **ZFC** *the existence of a strongly regular set containing $\omega$ implies that there exists a weakly inaccessible cardinal.*

**Proof:** For a contradiction assume

$$\mathbf{ZFC} \vdash \exists A\,[\mathbf{Reg}(A) \ \wedge\ \omega{\in}A \ \wedge\ A \texttt{ is Exp-closed and } \bigcup\texttt{-closed}].$$

In the following we work in **ZFC**. By Lemma 2.4 $\kappa = o(A)$ is a regular uncountable cardinal. We claim that $\kappa$ is a limit cardinal, too. Let $\rho < \kappa$ and $F : {}^{\rho}2 \to \mu$ be a surjective function. Suppose $\kappa \leq \mu$. Then let $X = \{g \in {}^{\rho}2 |\, F(g) < \kappa\}$. Note that

$$\{F(g) |\, g{\in}X\} = \kappa$$

since $F$ is surjective. Since $A$ is Exp-closed we have ${}^{({}^{\rho}2)}2 \in A$. Define a function $G : {}^{\rho}2 \to 2$ by $G(h) = 1$ if $h{\in}X$, and $G(h) = 0$ otherwise. Then $G \in A$. Further, define $j : G \to A$ by $j(\langle h, i\rangle) = F(h)$ if $i = 1$, and $j(\langle h, i\rangle) = 0$ otherwise. Then $\mathbf{ran}(j) \in A$. However, $\mathbf{ran}(j) = \{F(g) |\, g{\in}X\} \cup \{0\} = \kappa$, yielding the contradiction $\kappa{\in}\kappa$.

As a result, $\mu < \kappa$ and therefore $\kappa$ cannot be a successor cardinal. Consequently we have shown the existence of a weakly inaccessible cardinal. But that cannot be done in **ZFC** (providing **ZF** is consistent). □

## 3 On the total failure of wREA in CZF

The first interesting consequence of **wREA** is that the class of hereditarily countable sets, $HC = H(\omega \cup \{\omega\})$, constitutes a set. In the Leeds-Manchester Proof Theory Seminar, Peter Aczel asked whether **CZF** is at least strong enough to show that $HC$ is a set. This section is devoted to showing that this is not the case.

**Definition: 3.1** The *Axiom of Countable Choice*, $\mathbf{AC}_\omega$, states that whenever $F$ is a function with with domain $\omega$ such that $\forall i{\in}\omega\, \exists y{\in}F(i)$, then there exists a function $f$ with domain $\omega$ such that $\forall i{\in}\omega\, f(i){\in}F(i)$.

**Proposition: 3.2** $\mathbf{CZF} + \mathbf{AC}_\omega$ *does not prove that HC is a set.*

**Proof:** It has been shown by Rathjen (cf. [R94]) that $\mathbf{CZF} + \mathbf{AC}_\omega$ has the same proof-theoretic strength as Kripke-Platek set theory, **KP**. The proof-theoretic ordinal of $\mathbf{CZF} + \mathbf{AC}_\omega$ is the so-called Bachmann-Howard ordinal $\psi_{\Omega_1}\varepsilon_{\Omega_1+1}$. Let

$$T := \mathbf{CZF} + \mathbf{AC}_\omega + H(\omega \cup \{\omega\}) \texttt{ is a set.}$$

Another theory which has proof-theoretic ordinal $\psi_{\Omega_1}\varepsilon_{\Omega_1+1}$ is the intuitionistic theory of arithmetic inductive definitions $\mathbf{ID}^i_1$ (cf. [P89, BFPS]). We aim at showing that $T$ proves the consistency of $\mathbf{ID}^i_1$. The latter implies that $T$ proves the consistency of $\mathbf{CZF} + \mathbf{AC}_\omega$ as well, yielding the desired Proposition, owing to Gödel's Incompleteness Theorem.

Let $L_{HA}(P)$ be the language of Heyting arithmetic augmented by a new unary predicate symbol $P$. The language of $\mathbf{ID}^i_1$ comprises $L_{HA}$ and in addition contains a unary predicate symbol $I_\phi$ for each formula $\phi(u, P)$ of $L_{HA}(P)$ in which $P$ occurs only positively.

The axioms of $\mathbf{ID}_1^i$ comprise those of Heyting arithmetic with the induction scheme for natural numbers extended to the language of $\mathbf{ID}_1^i$ plus the following axiom schemes relating to the predicates $I_\phi$:

$$(ID_\phi^1) \qquad \forall x\,[\phi(x, I_\phi) \to I_\phi(x)]$$
$$(ID_\phi^2) \qquad \forall x\,[\phi(x, \psi) \to \psi(x)] \;\to\; \forall x\,[I_\phi(x) \to \psi(x)]$$

for every formula $\psi$, where $\phi(x, \psi)$ arises from $\phi(x, P)$ by replacing every occurrence of a formula $P(t)$ in $\phi(x, P)$ by $\psi(t)$.

Arguing in $T$ we want to show that $\mathbf{ID}_1^i$ has a model. The domain of the model will be $\omega$. The interpretation of $\mathbf{ID}_1^i$ in $T$ is given as follows. The quantifiers of $\mathbf{ID}_1^i$ are interpreted as ranging over $\omega$. The arithmetic constant 0 and the functions $+1, +, \cdot$ are interpreted by their counterparts on $\omega$. It remains to provide an interpretation for the predicates $I_\phi$, where $\phi(u, P)$ is a $P$ positive formula of $L_{HA}(P)$. Let $\phi(u, v)^*$ be the set-theoretic formula which arises from $\phi(u, P)$ by, firstly, restricting all quantifiers to $\omega$, secondly, replacing all subformulas of the form $P(t)$ by $t \in v$, and thirdly, replacing the arithmetic constant and function symbols by their set-theoretic counterparts. Let

$$\Gamma_\phi(A) = \{x \in \omega \mid \phi(x, A)^*\}$$

for every subset $A$ of $\omega$, and define a mapping $x \mapsto \Gamma_\phi^x$ by recursion on $H(\omega \cup \{\omega\})$ via

$$\Gamma_\phi^x = \Gamma_\phi(\bigcup_{u \in x} \Gamma_\Phi^u).$$

Finally put

$$I_\phi^* = \bigcup_{x \in H(\omega \cup \{\omega\})} \Gamma_\phi^x.$$

It is obvious that the above interpretation validates the arithmetic axioms of $\mathbf{ID}_1^i$. The validity of the interpretation of $(ID_\phi^1)$ follows from

$$\Gamma_\phi(I_\phi^*) \subseteq I_\phi^*. \tag{1}$$

Let $HC = H(\omega \cup \{\omega\})$. Before we prove (1) we show

$$\Gamma_\phi^{\in a} \subseteq \Gamma_\phi^a \tag{2}$$

for $a \in HC$, where $\Gamma_\phi^{\in a} = \bigcup_{x \in a} \Gamma_\phi^x$. (2) is shown by Set Induction on $a$. The induction hypothesis then yields, for $x \in a$,

$$\Gamma_\phi^{\in x} \subseteq \Gamma_\phi^x \subseteq \Gamma_\phi^{\in a}.$$

Thus, by monotonicity of the operator $\Gamma_\phi$,

$$\Gamma_\phi(\Gamma_\phi^{\in x}) = \Gamma_\phi^x \subseteq \Gamma_\phi(\Gamma_\phi^{\in a}) = \Gamma_\phi^a,$$

and hence $\Gamma_\phi^{\in a} \subseteq \Gamma_\phi^a$, confirming (2).

To prove (1) assume $n \in \Gamma_\phi(I_\phi^*)$. Then $\phi(n, \bigcup_{x\in HC}\Gamma_\phi^x)^*$ by definition of $\Gamma_\Phi$. Now, since $\bigcup_{x\in HC}\Gamma_\phi^x$ occurs positively in the latter formula one can show, by induction on the construction of $\phi$, that

$$\phi(n, \Gamma_\phi^a)^* \tag{3}$$

for some $a \in HC$. The atomic cases are obvious. The crucial case is when $\phi(n, v)^*$ is of the form $\forall k \in \omega \psi(k, n, v)$. Inductively one then has

$$\forall k \in \omega\, \exists y \in HC\, \psi(k, n, \Gamma_\phi^y).$$

Employing Strong Collection, there exists $R \in \mathbf{mv}({}^\omega HC)$ such that

$$\forall k \in \omega\, \exists y\, [\langle k, y\rangle \in R \ \wedge\ \psi(k, n, \Gamma_\phi^y).$$

Using $\mathbf{AC}_\omega$ there exists a function $f : \omega \to HC$ such that $\forall k \in \omega\, \langle k, f(k)\rangle \in R$ and hence

$$\forall k \in \omega\, \psi(k, n, \Gamma_\phi^{f(k)}).$$

Let $b = \mathbf{ran}(f)$. It follows from (2) that $\Gamma_\phi^{f(k)}) \subseteq \Gamma_\phi^b$, and thus, by positivity of the occurrence of $P$ in $\phi$ we get,

$$\forall k \in \omega\, \psi(k, n, \Gamma_\phi^b))^*.$$

The validity of the interpretation of $(ID_\phi^2)$ can be seen as follows. Assume

$$\forall i{\in}\omega\, [\phi(i, X) \to i{\in}X], \tag{4}$$

where $X$ is a definable class. We want to show $I_\phi^* \subseteq X$. It suffices to show $\Gamma_\phi^a \subseteq X$ for all $a \in HC$. We proceed by induction on $a{\in}HC$. The induction hypothesis provides $\Gamma_\phi^{\in a} \subseteq X$. Monotonicity of $\Gamma_\phi$ yields $\Gamma_\phi(\Gamma_\phi^{\in a}) = \Gamma_\phi^a \subseteq \Gamma_\phi(X)$. By (2) it holds $\Gamma_\phi(X) \subseteq X$. Hence $\Gamma_\phi^a \subseteq X$.

We have now shown within $T$ that $\mathbf{ID}_1^i$ has a model. Note also that, arguing in $T$, this model is a set as the mapping $\phi(u, P) \mapsto I_\phi^*$ is a function when we assume a coding of the syntax of $\mathbf{ID}_1^i$. As a result, by formalizing the notion of truth for this model, $T$ proves the consistency of $\mathbf{ID}_1^i$, establishing the Proposition. □

## 4 ZF, fREA, and $HC$

One version of the regular extension axiom is provable in **ZF**. **ZF** proves **fREA**, though this is not a triviality. Here we shall slightly generalize [Jech82], Theorem 1, where it was shown that **ZF** proves that $HC$ is a set.

**Proposition: 4.1** $\mathbf{ZF} \vdash \forall x\ H(x)$ `is a set`; *in particular* $\mathbf{ZF} \vdash \mathbf{fREA}$.

**Proof:** Every set $x$ is contained in a transitive set $A$ with $\omega \subseteq A$. Thus if we can show that $H(A)$ is a set we have found a set comprising $x$ which is functionally regular. The main task of the proof is therefore to show that $H(A)$ is a set. Let $\rho$ be the supremum of all ordinals which are order types of well-orderings of subsets of $A$. (A well-ordering of a set $B$ is a relation $R \subseteq B \times B$ such $R$ linearly orders the elements of $B$ and for every non-empty $X \subseteq B$ there exists an $R$-least element in $X$, i.e. $\exists u \in X \, \forall v \in X \, \neg vRu$.) Note that $\rho$ exists owing to Power Set, Separation, Replacement, and Union. Also note that $\rho$ is a cardinal $\geq \omega$ and for every well-ordering $R$ of a subsetet of $A$, the order-type of $R$ is less than $\rho$.

Let $\kappa = \rho^+$ (where $\rho^+$ denotes the least cardinal bigger than $\rho$). We shall show that $\text{rank}(s) < \kappa$ for every $s \in H(A)$, and thus

$$H(A) \subseteq V_\kappa. \tag{5}$$

For a set $X$ let $\bigcup^n X$ be the $n$-fold union of $X$, i.e., $\bigcup^0 X = X$, and $\bigcup^{n+1} X = \bigcup(\bigcup^n X)$. Note that

$$\text{rank}(X) = \{\text{rank}(u) | \; u \in \mathbf{TC}(X)\} = \bigcup_{n \in \omega} \{\text{rank}(u) | \; u \in \bigcup^n X\}.$$

Let $\Theta$ be the set of all non-empty finite sequences of ordinals $< \rho$. We shall define a function $F$ on $H(A) \times \omega \times \Theta$ such that for each $s \in H(A)$, if $F_s$ denotes the function $F_s(n,t) = F(s,n,t)$, then $F_s$ maps $\omega \times \Theta$ onto $\text{rank}(s)$. Since there is a bijection between $\Theta$ and $\rho$ (cf. [Ku], 10.13), we then have $\text{rank}(s) < \kappa$, and thus $s \in V_\kappa$. We define the function $F$ by recursion on $n$. For each $n$, we denote by $F^n_s$ the function $F^n_s(t) = F(s,n,t)$. For $n = 0$ we let for each $s \in H(A)$ and each $\beta < \rho$,

$$F^0_s(\langle\beta\rangle) \quad = \quad \text{the } \beta\text{th element of } \{\text{rank}(u) | \; u{\in}s\}$$

if the set $\{\text{rank}(u) | \; u{\in}s\}$ has order-type $> \beta$, and $F^0_s(\langle\beta\rangle) = 0$ otherwise. If $t \in \Theta$ is not of the form $\langle\beta\rangle$, we put $F^0_s(t) = 0$.

Since there exists $b{\in}A$ and $g : b \to H(A)$ such that $s = \mathbf{ran}(g)$, the order type of $\{\text{rank}(x) | \; x{\in}s\}$ is an ordinal $< \rho$, owing to $b \subseteq A$. And hence $F^0_s$ maps $\Theta$ onto the set $\{\text{rank}(x) | \; x{\in}s\}$.

For $n = 1$, $s \in H(A)$, and $\beta_0, \beta_1 < \rho$ we let

$$F^1_s(\langle\beta_0, \beta_1\rangle) = \text{the } \beta_1\text{th element of } \{F^0_u(\langle\beta_0\rangle) | \; u{\in}s\},$$

if it exists, and $F^1_s(\langle\beta_0, \beta_1\rangle) = 0$ otherwise. If $t \in \Theta$ is not of the form $\langle\beta_0, \beta_1\rangle$, let $F^1_s(t) = 0$. In general, let

$$F^{n+1}_s(\langle\beta_0, \ldots, \beta_{n+1}\rangle) \quad = \quad \text{the } \beta_{n+1}\text{th element of } \{F^n_u(\langle\beta_0, \ldots, \beta_n\rangle) | \; u{\in}s\},$$

if it exists, and $F^{n+1}_s(\langle\beta_0, \ldots, \beta_{n+1}\rangle) = 0$ otherwise. If $t \in \Theta$ is not of the form $\langle\beta_0, \ldots, \beta_{n+1}\rangle$, let $F^{n+1}_s(t) = 0$.

For each $s \in H(A)$ and each $\langle\beta_0, \ldots, \beta_n\rangle \in \Theta$, the order-type of the set

$$\{F^n_u(\langle\beta_0, \ldots, \beta_n\rangle) | \; u{\in}s\}$$

is an ordinal $< \rho$. Hence $F_s^{n+1}$ maps $\Theta$ onto the set

$$\{F_u^n(\langle \beta_0, \ldots, \beta_n \rangle)|\ u{\in}s \wedge \langle \beta_0, \ldots, \beta_n \rangle \in \rho \times \cdots \times \rho\}.$$

It follows by induction that for each $n$ and for each $s \in H(A)$, the function $F_s^n$ maps $\Theta$ onto the set $\{\text{rank}(u)|\ u \in \bigcup^n s\}$. For each $s \in H(A)$, $F_s$ therefore maps $\omega \times \Theta$ onto the set $\{\text{rank}(u)|\ u \in \mathbf{TC}(s)\} = \text{rank}(s)$.

This concludes the proof of (5). Finally, by Separation, it follows that $H(A)$ is a set. □

By definition, $HC$ is functionally regular, but **ZF** is not capable of showing that $HC$ enjoys any of the other regularity properties.

**Proposition: 4.2** *If* **ZF** *is consistent, then* **ZF** *does not prove that $HC$ is weakly regular.*

**Proof:** Assume that **ZF** is consistent. Let $T$ be the theory **ZF** plus the assertion that the real numbers are a union of countably many countable sets. By results of Feferman and Levy it follows that $T$ is consistent as well (see [FL] or [Jech73], Theorem 10.6). In the following we argue in $T$ and identify the set of reals, $\mathbb{R}$, with the set of functions from $\omega$ to $\omega$. Working towards a contradiction, assume that $HC$ is weakly regular. Let $\mathbb{R} = \bigcup_{n \in \omega} X_n$, where each $X_n$ is countable and infinite. By induction on $n{\in}\omega$ one verifies that $n{\in}HC$ for every $n{\in}\omega$, and thus $\omega \in HC$. If $f : \omega \to \omega$ define $f^*$ by $f^*(n) = \langle n, f(n) \rangle$. Then $f^* : \omega \to HC$ as $HC$ is closed under Pairing, and hence $f = \mathbf{ran}(f^*) \in HC$. As a result, $\mathbb{R} \subseteq HC$ and, moreover, $X_n \in HC$ since each $X_n$ is countable. Furthermore, $\{X_n|\ n{\in}\omega\} \in HC$.

For each $X_n$ let

$$\mathcal{G}_n = \{f : \omega \to X_n|\ f \text{ is 1-1 and onto}\}.$$

Note that $\mathcal{G}_n \subseteq HC$. Define $R \in \mathbf{mv}({}^{\{X_n|\ n\in\omega\}}HC)$ by

$$\langle X_n, f \rangle \in R \text{ iff } f \in \mathcal{G}_n.$$

By weak regularity there exists $B \in HC$ such that

$$\forall n{\in}\omega\, \exists f{\in}B\ \langle X_n, f \rangle \in R.$$

Now pick $g : \omega \to B$ such that $B = \mathbf{ran}(g)$. For every $x \in \mathbb{R}$ define $J(x)$ as follows. Select the least $n$ such that $x{\in}X_n$ and then pick the least $m$ such that $\langle X_n, g(m) \rangle \in R$, and let

$$J(x) = \langle n, (g(m))^{-1}(x) \rangle,$$

where $(g(m))^{-1}$ denotes the inverse function of $g(m)$. It follows that

$$J : \mathbb{R} \to \omega \times \omega$$

is a 1-1 function, implying the contradiction that $\mathbb{R}$ is countable. □

**Corollary: 4.3** *If* **ZF** *is consistent, then so is the theory*

$$\mathbf{ZF} + HC \textit{ is not } \bigcup\textit{-closed.}$$

**Proof:** This follows from Proposition 2.4 and Proposition 4.2. □

## 5 Some AC-like consequences of wREA in ZF

Though **wREA** does not imply any of the familiar forms of **AC** (cf. Corollary 6.4) we will show one result to follow from **wREA** that is usually considered a consequence of the axiom of choice.

**Proposition: 5.1** (**ZF**) *If $A$ is a weakly regular set with $\omega \in A$, then rank$(A)$ is an uncountable ordinal of cofinality $> \omega$.*

**Proof:** Set $\kappa = \text{rank}(A)$. Obviously $\omega < \kappa$. Suppose $f : \omega \to \kappa$. Define $R \subseteq \omega \times A$ by $nRa$ iff $f(n) < \text{rank}(a)$. Since for every ordinal $f(n)$ there exists a set $a{\in}A$ with rank $> f(n)$, $R$ is a total relation. Employing the weak regularity of $A$, there exists a set $b{\in}A$ such that $\forall n{\in}\omega\, \exists x{\in}b\; f(n) < \text{rank}(x)$. As a result, $f : \omega \to \text{rank}(b)$ and $\text{rank}(b) < \kappa$. This shows that the cofinality of $\kappa$ is bigger than $\omega$. □

**Corollary: 5.2** (**ZF**) **wREA** *implies that, for any set $X$, there is a cardinal $\kappa$ such that $X$ cannot be mapped onto a cofinal subset of $\kappa$.*

**Proof:** Let $A$ be a weakly regular set such that $X \in A$. Set $\kappa = \text{rank}(A)$. Aiming at a contradiction, suppose there exists $f : X \to \kappa$ such that $\mathbf{ran}(f)$ is a cofinal subset of $\kappa$. Define $R \subseteq X \times A$ by $uRa$ iff $f(u) < \text{rank}(a)$. Since for every ordinal $f(u)$ there exists a set $a{\in}A$ with $\text{rank}(a) > f(u)$, $R$ is a total relation. Employing the weak regularity of $A$, there exists a set $b{\in}A$ such that $\forall u{\in}X\, \exists y{\in}b\; f(u) < \text{rank}(y)$. As a result, $f : X \to \text{rank}(b)$ and $\text{rank}(b) < \kappa$. But the latter contradicts the assumption that $\mathbf{ran}(f)$ is a cofinal subset of $\kappa$. □

**Corollary: 5.3** (**ZF**) **wREA** *implies that there are class many regular cardinals.*

**Proof:** If $\alpha$ is an ordinal then by the previous result there exists a cardinal $\kappa$ such that $\alpha$ cannot be mapped onto a cofinal subset of $\kappa$. Let $\pi$ be the cofinality of $\kappa$. Then $\pi$ is a regular cardinal $> \alpha$. □

## 6 ZF models of REA

In this section it is shown that **REA** is hard to avoid in that "most" of the models of **ZF** satisfy **REA**.

**Definition: 6.1** There is a weak form of the axiom of choice, which holds in a plethora of **ZF** universes. The *axiom of small violations of choice*, **SVC**, has been studied by A. Blass [Bl79]. It says in some sense, that all failure of choice occurs within a single set. **SVC** is the assertion that there is a set $S$ such that, for every set $a$, there exists an ordinal $\alpha$ and a function from $S \times \alpha$ onto $a$.

**Theorem: 6.2** (**ZF**) **SVC** *implies* **REA**.

**Proof:** This follows from [R02], Theorem 4.8. □

From the previous theorem and results in [Bl79] it follows that **REA** holds in all permutation models and symmetric models. A *permutation model* (cf. [Jech73], chapter 4) is specified by giving a model $V$ of **ZFC** with atoms in which the atoms form a set $A$, a group $\mathcal{G}$ of permutations of $A$, and a normal filter $\mathcal{F}$ of subgroups of $\mathcal{G}$. The permutation model then consists of the hereditarily symmetric elements of $V$.

A *symmetric model* (cf. [Jech73], chapter 5), is specified by giving a ground model $M$ of **ZFC**, a complete Boolean algebra $B$ in $M$, an $M$-generic filter $G$ in $B$, a group $\mathcal{G}$ of automorphisms of $B$, and a normal filter of subgroups of $\mathcal{G}$. The symmetric model consists of the elements of $M[G]$ that hereditarily have symmetric names.

If $B$ is a set then HOD($B$) denotes the class of sets hereditarily ordinal definable over $B$.

**Corollary: 6.3** *The usual models of classical set theory without choice satisfy* **REA**. *More precisely, every permutation model and symmetric model satisfies* **REA**. *Furthermore, if* $V$ *is a universe that satisfies* **ZF**, *then for every transitive set* $A \in V$ *and any set* $B \in V$ *the submodels* $L(A)$ *and* $HOD(B)$ *satisfy* **REA**.

**Proof:** This follows from Theorem 6.2 in conjunction with [Bl79], Theorems 4.2, 4.3, 4.4, 4.5. □

**Corollary: 6.4** (**ZF**) **REA** *does not imply the countable axiom of choice,* $\mathbf{AC}_\omega$, *and* **DC**. *Moreover,* **REA** *does not imply any of the mathematical consequences of* **AC** *of [Jech73], chapter 10. Among those consequences are the existence of a basis for any vector space and the existence of the algebraic closure of any field.*

**Proof:** This follows from Corollary 6.2 and [Jech73], chapter 10. □

# 7 ZF models for ¬wREA

The only models of **ZF** + ¬**REA** and **ZF** + ¬**wREA** known to us are the models of

$$\mathbf{ZF} + \texttt{All uncountable cardinals are singular}$$

given by Gitik [Gi80] who showed the consistency of the latter theory from the assumption that

$$\mathbf{ZFC} + \forall\alpha\,\exists\kappa > \alpha\,(\kappa\ \texttt{is a strongly compact cardinal})$$

is consistent. This large cardinal assumption might seem exaggerated, but it is known that the consistency of all uncountable cardinals being singular cannot be proved without assuming the consistency of the existence of some large cardinals. For instance, it was shown in [DJ81] that if $\aleph_1$ and $\aleph_2$ are both singular one can obtain an inner model with a measurable cardinal.

**Corollary: 7.1** *If*

$$\mathbf{ZF} + \texttt{All uncountable cardinals are singular}$$

*is consistent, then so are the following theories:*

1. $\mathbf{ZF} + \texttt{There are no weakly regular sets containing } \omega$.
2. $\mathbf{ZF} + \texttt{There are no } \bigcup\texttt{-closed functionally regular sets containing } \omega$.

**Proof:** (i) follows from Proposition 5.1.

(ii): By Proposition 2.4, the existence of a functionally regular $\bigcup$-closed set $A$ with $\omega \in A$ would yield the existence of an uncountable regular ordinal. □

It would be very interesting to construct models of $\mathbf{ZF} + \neg\mathbf{REA}$ and $\mathbf{ZF} + \neg\mathbf{wREA}$ that do not hinge on large cardinal assumptions.

## References


[Acz78] P. Aczel: *The type theoretic interpretation of constructive set theory.* In: MacIntyre, A. and Pacholski, L. and Paris, J, editor, *Logic Colloquium '77* (North Holland, Amsterdam 1978) 55–66.

[Acz82] P. Aczel: *The type theoretic interpretation of constructive set theory: Choice principles.* In: A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium* (North Holland, Amsterdam 1982) 1–40.

[Acz86] P. Aczel: *The type theoretic interpretation of constructive set theory: Inductive definitions.* In: R.B. et al. Marcus, editor, *Logic, Methodology and Philosophy of Science VII* (North Holland, Amsterdam 1986) 17–49.

[AR02] P. Aczel, M. Rathjen: *Notes on constructive set theory*, Technical Report 40, Institut Mittag-Leffler (The Royal Swedish Academy of Sciences, 2001). *http://www.ml.kva.se/preprints/archive2000-2001.php*

[Ba75] J. Barwise: *Admissible Sets and Structures* (Springer-Verlag, Berlin, Heidelberg, New York, 1975).

[Bl79] A. Blass: *Injectivity, projectivity, and the axiom of choice.* Transactions of the AMS 255 (1979) 31–59.

[BFPS] W. Buchholz, S. Feferman, W. Pohlers, W. Sieg: *Iterated inductive definitions and subsystems of analysis*, Lecture Notes in Math. 897 (Springer, Berlin, 1981) 78–142.

[DJ81] T. Dodd and R. Jensen: *The core model.* Annals of Mathematical Logic 20 (1981) 43-75.

[FL] S. Feferman and A. Levy: *Independence results in set theory by Cohen's method. II.* (abstract) Notices of the american Mathematical Society 10 (1963) 593.

[Gi80] M. Gitik: *All uncountable cardinals can be singular.* Israel Journal of Mathematics 35 (1980) 61–88.

[Jech73] T. Jech: *The axiom of choice.* (North-Holland, Amsterdam, 1973).

[Jech82] T. Jech: *On hereditarily countable sets.* Journal of symbolic Logic 47 (1982) 43–47.

[Ku] K. Kunen: *Set Theory: An introduction to independence proofs.* (North-Holland, Amsterdam, 1980).

[P89] W. Pohlers: *Proof theory.* Lecture Notes in Mathematics 1407 (Springer, Berlin, 1989).

[R94] M. Rathjen: *The strength of some Martin-Löf type theories.* Archive for Mathematical Logic, 33:347–385, 1994.

[R01] M. Rathjen: *The anti-foundation axiom in constructive set theories.* In: G. Mints, R. Muskens (eds.): *Games, Logic, and Constructive Sets.* (CSLI Publications, Stanford, 2003).

[R02] M. Rathjen: *Choice principles in constructive and classical set theories.* To appear in: Z. Chatzidakis, P. Koepke, W. Pohlers (eds.): *Proceedings of the Logic Colloquium 2002.*