Arch. Math. Logic (1990) 29:249–263

Archive for
Mathematical
Logic

Ordinal Notations Based on a Weakly Mahlo Cardinal

Michael Rathjen

Institut für Mathematische Logik und Grundlagenforschung,
Westfälische Wilhelms-Universität Münster, Einsteinstrasse 62,
D-4400 Münster, Federal Republic of Germany

Introduction

Up to now, the subsystem of second order arithmetic with Δ_2^1-comprehension and Bar-induction is the strongest for which we have a computation of its proof-theoretic ordinal. This was done in [9] by Jäger and Pohlers. The above mentioned system is neatly connected to an extension of Kripke-Platek set theory, called **KPi**. The standard model of **KPi** is the initial segment of the constructible hierarchy formed by the first recursively inaccessible ordinal. So a next step towards an ordinal analysis of the subsystem of second order arithmetic based on Π_2^1-comprehension is offered by a set theory which axiomatizes essential features of $\mathrm{L}(\mu_0)$, where μ_0 denotes the first recursively Mahlo ordinal. Let us call such a theory **KPM**. Now, μ_0 allows descriptions by concepts from higher recursion theory which do not prima facie rely on the realm of recursively large ordinals: μ_0 is the first ordinal which is not recursive in the superjump (see [7]), and μ_0 is the supremum of the closure ordinals of non monotonic Π_2^0-inductive definitions of a special kind (see [13]). So it is much to be hoped for that a proof theory of **KPM** would be a starting point for a proof theory of the superjump as well as a proof theory of non monotonic inductive definitions.

The purpose of this paper is to deliver a perspicuous ordinal notation system, which is sufficient for the proof-theoretic treatment of **KPM**.

There is previous work, which is important for this paper: In [3], Buchholz invented his ψ-functions. The paper [4] by Buchholz and Schütte presents the definition of the functions ψ_κ for κ ranging over regular cardinals $>\omega$ below I_0, where I_0 denotes the first weakly inaccessible cardinal. In [8], Jäger extended this hierarchy to obtain collapsing functions ψ_κ for every $\kappa<\Lambda_0$. Here, Λ_0 stands for the first cardinal μ which is a limit of ϱ-inaccessible cardinals for every $\varrho<\mu$ (see 1.1). The ψ-functions are closely related to the Θ-functions. Pohlers (see [11]) extended the Θ-hierarchy of functions by defining a doubly indexed hierarchy $\lambda\beta.\ \Theta\varrho\alpha\beta$, which is also based on ϱ-inaccessible cardinals. Eventually, Schütte (in [15]) extended Jäger's notation system. To get "names" for cardinals of a higher degree of inaccessibility, Schütte goes back to an old concept of his, the Klammersymbole.

1. Basic Notions and the Functions I_α

We work in Zermelo-Fraenkel set theory with the axiom of choice and, in addition, assume the existence of a weakly Mahlo cardinal (see [10, p. 61]). Throughout, let M denote the least weakly Mahlo cardinal.

Conventions. Small greek letters range over ordinals. (α,β), $(\alpha,\beta]$, $[\alpha,\beta)$, $[\alpha,\beta]$ denote intervals of ordinals in the usual sense.

ON stands for the class of ordinals and CARD for the class of *uncountable cardinals.* LIM denotes the class of *limit ordinals.*

The variables κ, π, τ, κ', π', τ' are reserved for regular uncountable cardinals $<\mathrm{M}$. If X is a set of ordinals, we write $X<\alpha$ as an abbreviation for $\forall\xi(\xi\in X\Rightarrow\xi<\alpha)$. $X\subset Y:\Leftrightarrow\forall\alpha(\alpha\in X\Rightarrow\alpha\in Y)$. $\alpha\notin Y:\Leftrightarrow\neg(\alpha\in X)$. $\alpha\leqq\beta:\Leftrightarrow\alpha<\beta\vee\alpha=\beta$.

An ordinal α is an *additive principal number* if $0<\alpha$ and $\xi+\eta<\alpha$ for all $\xi,\eta<\alpha$. AP denotes the class of additive principal numbers.

If X is a class of ordinals, there exists a unique increasing function F such that the domain of F, $\mathrm{dom}(F)$, is an ordinal α or ON and $X=\{F(\xi):\xi\in\mathrm{dom}(F)\}$. F is called the *enumeration function of* X and we use $\mathrm{Enum}(X)$ to denote F. We say that a limit ordinal λ is a *limit point* of X if $\lambda=\sup(X\cap\lambda)$.

$\mathrm{cl}(X):=X\cup\{\alpha:\alpha$ is limit point of $X\}$.

1.1. Definition. A cardinal α is *ϱ-inaccessible* if it is regular and limit of σ-inaccessible cardinals for all $\sigma<\varrho$. So the 0-inaccessible cardinals are exactly the regular cardinals, hence the 1-inaccessibles are weakly inaccessible cardinals. By induction on ϱ, we define

$$\mathrm{I}_\varrho:=\mathrm{Enum}(\mathrm{cl}\{\mu:\mu>\omega\wedge\mu\text{ is }\varrho\text{-inaccessible}\})$$

and write $\mathrm{I}\varrho\alpha$ in place of $\mathrm{I}_\varrho(\alpha)$.

κ is *weakly Mahlo* iff κ is a cardinal such that for every function $f:\kappa\to\kappa$ there exists a regular cardinal $\pi<\kappa$ such that $\forall\alpha<\pi(f(\alpha)<\pi)$.

Every weakly Mahlo cardinal μ is ϱ-inaccessible for all $\varrho\leqq\mu$ (see [6, CH.4, 3.4]). But μ is not the first cardinal ν such that ν is ν-inaccessible. There are μ cardinals ν below μ such that ν is ν-inaccessible. This is not the strongest statement one can prove in this connection. Indeed, Mahlo cardinals are larger than any cardinals which can be obtained by processes like these.

The functions I_α are used in [8] to develope a notation system which reflects some structural properties of the cardinal $\Lambda_0:=\min\{\xi>0:\forall\delta,\eta<\xi(\mathrm{I}\delta\eta<\xi)\}$.

But Λ_0 only shares a few properties with M, and likewise the corresponding notation system is not sufficient for the treatment of **KPM.**

2. The Veblen Hierarchy and the Functions Φ_α

The Veblen hierarchy is well known and studied in the literature. For references on these functions we refer the reader to [12] and [14, Sect. 13]. Especially [12] is recommended as a thorough and lucid introduction to this material. Therefore we just assemble basic properties of these functions.

2.1. Definition. By induction on α, we define

$$\varphi_\alpha := \mathrm{Enum}(\{\xi \in \mathrm{AP}: \forall \eta < \alpha(\varphi_\eta(\xi) = \xi)\})$$

and write $\varphi\alpha\beta$ for $\varphi_\alpha(\beta)$.

So φ_0 is just the enumeration function of the additive principal numbers, thus $\varphi 0\alpha = \omega^\alpha$.

We also consider a cardinal valued analogue of the Veblen hierarchy:

$$\Phi_\alpha := \mathrm{Enum}(\{\mu \in \mathrm{CARD}: \forall \eta < \alpha(\Phi_\eta(\mu) = \mu)\}.$$

By definition, we then have $\Phi_0(\xi) = \aleph_{1+\xi}$. We write $\Phi\alpha\beta$ for $\Phi_\alpha(\beta)$.

Properties of the function Φ stated in this paragraph without proof follow either immediately from the definition of Φ or may be proved like the corresponding properties of φ.

2.2. Lemma. (i) $\varphi\alpha\beta \in \mathrm{AP}$; $\Phi\alpha\beta \in \mathrm{CARD}$.

(ii) $\xi < \alpha \Rightarrow \varphi\xi(\varphi\alpha\beta) = \varphi\alpha\beta \wedge \Phi\xi(\Phi\alpha\beta) = \Phi\alpha\beta$.

(iii) $\beta < \gamma \Rightarrow \varphi\alpha\beta < \varphi\alpha\gamma \wedge \Phi\alpha\beta < \Phi\alpha\gamma$.

(iv) $\alpha < \beta \Rightarrow \varphi\alpha 0 < \varphi\beta 0 \wedge \Phi\alpha 0 < \Phi\beta 0$.

(v) $\alpha \leqq \varphi\alpha 0$; $\alpha \leqq \Phi\alpha 0$.

(vi) $\mu \in \mathrm{CARD} \wedge \alpha, \beta < \mu \Rightarrow \varphi\alpha\beta < \mu$.

(vii) *If μ is weakly inaccessible or a limit of weakly inaccessibles, then* $\forall \alpha < \mu \forall \beta < \mu(\Phi\alpha\beta < \mu)$.

Proof of (vii). Let κ be weakly inaccessible. Then $\forall \beta < \kappa(\Phi 0\beta = \aleph_{1+\beta} < \kappa)$ since $\aleph_\kappa = \kappa$. Now assume $0 < \alpha < \kappa$ and $\forall \eta < \alpha \forall \beta < \kappa(\Phi\eta\beta < \kappa)$. By the regularity of κ, to show $\forall \beta < \kappa(\Phi\alpha\beta < \kappa)$ it suffices to prove that $X := \{\zeta < \kappa: \forall \eta < \alpha(\Phi\eta\zeta = \zeta)\}$ is unbounded in κ. For $\delta < \kappa$ we define: $\zeta_0 = \delta$, $\zeta_{n+1} = \sup\{\Phi\eta\zeta_n : \eta < \alpha\}$, $\zeta = \sup\{\zeta_n : n < \omega\}$. Then $\forall \eta < \alpha(\Phi\eta\zeta = \zeta)$ and $\delta \leqq \zeta$. Moreover, $\zeta_n < \kappa$ by regularity of κ and the assumption above, hence $\zeta < \kappa$, again by regularity of κ. This establishes unboundedness of X. Finally, $\forall \alpha < \kappa \forall \beta < \kappa(\Phi\alpha\beta < \kappa)$ follows by induction on α.

If μ is a limit of weakly inaccessibles and $\alpha, \beta < \mu$, then there exists a weakly inaccessible $\kappa < \mu$ such that $\alpha, \beta < \mu$, so $\Phi\alpha\beta < \kappa < \mu$. $\square$

2.3. Proposition. *Let f be one of the functions φ or Φ. Then $f\alpha\beta = f\gamma\delta$ iff we have one of the following cases:*

1. *$\alpha < \gamma$ and $\beta = f\gamma\delta$,*
2. *$\alpha = \gamma$ and $\beta = \delta$,*
3. *$\gamma < \alpha$ and $f\alpha\beta = \delta$.*

2.4. Proposition. *Let f be one of the functions φ or Φ. Then $f\alpha\beta < f\gamma\delta$ iff we have one of the following cases:*

1. *$\alpha < \gamma$ and $\beta < f\gamma\delta$,*
2. *$\alpha = \gamma$ and $\beta < \delta$,*
3. *$\gamma < \alpha$ and $f\alpha\beta < \delta$.*

2.5. Proposition

(i) *For every $\gamma \in \mathrm{AP}$, there exist unique α and $\beta < \gamma$ such that $\gamma = \varphi\alpha\beta$.*

(ii) *For every $\mu \in \mathrm{CARD}$, there exist unique α and $\beta < \mu$ such that $\mu = \Phi\alpha\beta$.*

2.6. Definition. An ordinal α is said to be *strongly critical* if $\varphi\alpha 0=\alpha$. We let SC denote the class of strongly critical ordinals.

2.7. Lemma. *For every $\gamma\in \mathrm{AP}\backslash\mathrm{SC}$, there exist unique $\alpha,\beta<\gamma$ such that $\gamma=\varphi\alpha\beta$.*

2.8. Lemma. *For every $\alpha\notin \mathrm{AP}\cup\{0\}$, there exist unique $\alpha_1,\ldots,\alpha_n\in\mathrm{AP}$ such that $\alpha=\alpha_1+\ldots+\alpha_n$ and $\alpha>\alpha_1\geqq\ldots\geqq\alpha_n$.*

2.9. Definition. (i) $\gamma=_{\mathrm{NF}}\varphi\alpha\beta:\Leftrightarrow\gamma=\varphi\alpha\beta\wedge\alpha,\beta<\gamma$.
(ii) $\mu=_{\mathrm{NF}}\Phi\alpha\beta:\Leftrightarrow\mu=\Phi\alpha\beta\wedge\alpha,\ \beta<\gamma$.
(iii) $\alpha=_{\mathrm{NF}}\alpha_1+\ldots+\alpha_n:\Leftrightarrow\alpha=\alpha_1+\ldots+\alpha_n\wedge\alpha_1,\ldots,\alpha_n\in\mathrm{AP}\wedge\alpha>\alpha_1\geqq\ldots\geqq\alpha_n$.

3. The Functions χ_α

Henceforth we restrict ourselves to ordinals below $\mathrm{M}^\Gamma:=\min\{\alpha>\mathrm{M}:\alpha\in\mathrm{SC}\}$. In this paragraph, we define functions $\chi_\alpha:\mathrm{M}\to\mathrm{M}$ for $\alpha<\mathrm{M}^\Gamma$ by transfinite recursion on α. This hierarchy of functions enables us to define a "collapsing function" $\mathrm{D}:\mathrm{M}^\Gamma\to\mathrm{M}$ via $\mathrm{D}(\alpha):=\chi_\alpha(0)$, which sends ordinals $\alpha\in[\mathrm{M},\mathrm{M}^\Gamma)$ below M. The need to use functions indexed by ordinals from above M to reflect essential features of a Mahlo cardinal corresponds to the fact that M is not to be obtained by iteration combined with diagonalization of inaccessibility from below.

For $X\subset\mathrm{M}$, we set $\mathrm{cl}_{\mathrm{M}}(X):=X\cup\{\lambda<\mathrm{M}:\lambda\text{ is a limit point of }X\}$.

By κ, π, τ, κ', π', τ' we always denote regular uncountable cardinals $<\mathrm{M}$.

3.1. Definition. Every $\alpha<\mathrm{M}^\Gamma$ has a unique expression by means of strongly critical ordinals $<\mathrm{M}$, the constants 0, M and the functions $+,\varphi$. We denote this set of strongly critical ordinals $<\mathrm{M}$ by $\mathrm{SC}_{\mathrm{M}}(\alpha)$. The precise definition of $\mathrm{SC}_{\mathrm{M}}(\alpha)$ is given by the following induction on α.

1. $\mathrm{SC}_{\mathrm{M}}(0):=\mathrm{SC}_{\mathrm{M}}(\mathrm{M})=\emptyset$.
2. $\mathrm{SC}_{\mathrm{M}}(\alpha):=\{\alpha\}$ if $\alpha<\mathrm{M}$ and $\alpha\in\mathrm{SC}$.
3. $\mathrm{SC}_{\mathrm{M}}(\alpha):=\mathrm{SC}_{\mathrm{M}}(\alpha_1)\cup\ldots\cup\mathrm{SC}_{\mathrm{M}}(\alpha_n)$ if $\alpha=_{\mathrm{NF}}\alpha_1+\ldots+\alpha_n$.
4. $\mathrm{SC}_{\mathrm{M}}(\alpha):=\mathrm{SC}_{\mathrm{M}}(\gamma)\cup\mathrm{SC}_{\mathrm{M}}(\delta)$ if $\alpha=_{\mathrm{NF}}\varphi\gamma\delta$.

We set $\alpha^*:=\sup(\mathrm{SC}_{\mathrm{M}}(\alpha)\cup\{0\})$.

3.2. Lemma. $\alpha^*<\mathrm{M}$.

Proof. Obvious. □

3.3. Inductive Definition of $\mathrm{B}(\alpha,\beta)$ and χ_α for $\beta<\mathrm{M}$ and $\alpha<\mathrm{M}^\Gamma$.

(B1) $\beta\cup\{0,\mathrm{M}\}\subset\mathrm{B}^n(\alpha,\beta)$.
(B2) $\gamma=_{\mathrm{NF}}\gamma_1+\ldots+\gamma_k\wedge\gamma_1,\ldots,\gamma_k\in\mathrm{B}^n(\alpha,\beta)\Rightarrow\gamma\in\mathrm{B}^{n+1}(\alpha,\beta)$.
(B3) $\gamma=_{\mathrm{NF}}\varphi\delta\eta\wedge\delta,\eta\in\mathrm{B}^n(\alpha,\beta)\Rightarrow\gamma\in\mathrm{B}^{n+1}(\alpha,\beta)$.
(B4) $\pi\in\mathrm{B}^n(\alpha,\beta)\wedge\gamma<\pi\Rightarrow\gamma\in\mathrm{B}^{n+1}(\alpha,\beta)$.
(B5) $\delta,\eta\in\mathrm{B}^n(\alpha,\beta)\wedge\delta<\alpha\wedge\eta\in\mathrm{dom}(\chi_\delta)\Rightarrow\chi_\delta(\eta)\in\mathrm{B}^{n+1}(\alpha,\beta)$.
(B6) $\mathrm{B}(\alpha,\beta)=\bigcup\{\mathrm{B}^n(\alpha,\beta):n<\omega\}$.
(B7) $\chi_\alpha=\mathrm{Enum}(\mathrm{cl}_{\mathrm{M}}(\{\kappa:\kappa\notin\mathrm{B}(\alpha,\kappa)\wedge\alpha\in\mathrm{B}(\alpha,\kappa)\}))$.

Note that in the definition above (and always) π and κ range over regular cardinals, which are uncountable and $<\mathrm{M}$.

We write $\chi\alpha\beta$ for $\chi_\alpha(\beta)$ and set $\mathrm{IN}(\alpha):=\{\kappa:\kappa\notin\mathrm{B}(\alpha,\kappa)\wedge\alpha\in\mathrm{B}(\alpha,\kappa)\}$.

3.4. Remark. The definition of the functions χ_α resembles that of the Feferman-Aczel functions Θ_α (see [1] and [14]). The main difference is that for $\alpha>0$ every element of $\mathrm{IN}(\alpha)$ must be a weakly inaccessible cardinal. Hence, $\chi_1(0)$, already, is a cardinal which can't be proved to exist in ZFC. The first conjunct in the definition of $\mathrm{IN}(\alpha)$ is the essential part, the second requirement in the definition of $\mathrm{IN}(\alpha)$, that $\alpha\in \mathrm{B}(\alpha,\kappa)$, is raised for technical reasons. Such a "normality" condition has already been used by Buchholz (see [2]) to recursively define the functions $\bar{\Theta}_\alpha$. If χ'_α denotes the function which is defined without the constraint $\alpha\in \mathrm{B}(\alpha,\kappa)$, then the hierarchy $(\chi'_\alpha)_{\alpha<\mathrm{M}^\Gamma}$ grows more slowly than the χ_α-hierarchy, but it eventually gets the same job done. Mainly, the condition $\alpha\in \mathrm{B}(\alpha,\kappa)$ is raised for bypassing lots of difficulties, and thus accounts for the tractible nature of the χ_α.

3.5. Lemma. (i) $\alpha\leqq\gamma\wedge\beta\leqq\delta<\mathrm{M}\Rightarrow \mathrm{B}(\alpha,\beta)\subset \mathrm{B}(\gamma,\delta)$.
(ii) $\lambda\in \mathrm{LIM}\wedge\lambda<\mathrm{M}\Rightarrow \mathrm{B}(\alpha,\lambda)=\bigcup\{\mathrm{B}(\alpha,\xi):\xi<\lambda\}$.
(iii) $\chi 0\alpha=\aleph_{1+\alpha}=\Phi 0\alpha$ for $\alpha<\mathrm{M}$.
(iv) $\mu\in \mathrm{CARD}\wedge\mu<\mathrm{M}\wedge\mu\notin \mathrm{B}(\alpha,\mu)\Rightarrow \mathrm{B}(\alpha,\mu)\cap \mathrm{M}=\mu$.

Proof. (i): By induction on n, one easily sees that $\mathrm{B}^n(\alpha,\beta)\subset \mathrm{B}(\gamma,\delta)$.

(ii): By (i) $\bigcup\{\mathrm{B}(\alpha,\xi):\xi<\lambda\}\subset \mathrm{B}(\alpha,\lambda)$. $\mathrm{B}^n(\alpha,\lambda)\subset\bigcup\{\mathrm{B}(\alpha,\xi):\xi<\lambda\}$ follows by induction on n.

(iii): In the inductive definition of $\mathrm{B}(0,\kappa)$, the clause (B5) does not apply. Since $\delta,\eta<\kappa$ implies $\varphi\delta\eta<\kappa$ by 2.2 (vi), we have $\kappa\notin \mathrm{B}(0,\kappa)$ for every regular uncountable cardinal $\kappa<\mathrm{M}$. This proves (iii).

(iv): Suppose $\gamma\in \mathrm{B}^n(\alpha,\mu)\cap \mathrm{M}$. We prove by induction on n that $\gamma<\mu$. If $\gamma\in \mathrm{B}^n(\alpha,\mu)\cap \mathrm{M}$ holds by (B1)–(B4), then $\gamma<\mu$ follows either trivially or by induction hypothesis using 2.2 (vi) in the case $\gamma=_{\mathrm{NF}}\varphi\delta\eta$. Now suppose $\gamma\in \mathrm{B}^n(\alpha,\mu)$ holds by (B5). Then $\gamma=\chi\delta\eta$ where $\delta<\alpha$. This implies $\chi 0(\gamma+1)=\aleph_{\gamma+1}\in \mathrm{B}(\alpha,\mu)$ by (B5) and (iii). Application of (B4) yields $\aleph_{\gamma+1}\subset \mathrm{B}(\alpha,\mu)$, hence $\gamma<\mu$ since $\mu\notin \mathrm{B}(\alpha,\mu)$. □

3.6. Lemma. *For every* $\alpha<\mathrm{M}^\Gamma$, $\chi_\alpha:\mathrm{M}\to\mathrm{M}$ *is a normal function with* $\mathrm{dom}(\chi_\alpha)=\mathrm{M}$.

Proof. It suffices to show that $\mathrm{IN}(\alpha)$ is unbounded in M, i.e. $\forall\gamma<\mathrm{M}\,\exists\delta\in \mathrm{IN}(\alpha)$ $(\gamma<\delta)$. Moreover, thanks to the demand that M be a weakly Mahlo cardinal, we only have to show that

$$Z_\alpha:=\{\nu<\mathrm{M}:\nu\in \mathrm{CARD}\wedge\nu\notin \mathrm{B}(\alpha,\nu)\wedge\alpha\in \mathrm{B}(\alpha,\nu)\}$$

is closed and unbounded in M because this implies that Z_α contains M regular cardinals. Suppose $\langle\nu_\eta:\eta<\lambda\rangle$ is an increasing sequence of elements of Z_α, where $\lambda<\mathrm{M}$ is a limit. Let $\nu=\sup\{\nu_\eta:\eta<\lambda\}$. By 3.5 (ii), $\nu\in \mathrm{B}(\alpha,\nu)$ would imply $\nu\in \mathrm{B}(\alpha,\nu_\eta)$ for some $\eta<\lambda$ thus the contradiction $\nu_\eta\in \mathrm{B}(\alpha,\nu_\eta)$ by 3.5 (iv). This establishes $\nu\in Z_\alpha$, hence closure of Z_α.

Now assume $\alpha^*<\beta<\mathrm{M}$. This implies $\mathrm{SC}_\mathrm{M}(\alpha)\subset \mathrm{B}(\alpha,\beta)$. Consequently, $\alpha\in \mathrm{B}(\alpha,\beta)$ by (B2), (B3). By induction on n, we show that $|\mathrm{B}^n(\alpha,\beta)|<\mathrm{M}$, where $|\mathrm{B}^n(\alpha,\beta)|$ denotes the cardinality of $\mathrm{B}^n(\alpha,\beta)$. This is clearly true for $n=0$. In case $n=m+1$, by induction hypothesis and the regularity of M, there exists a $\nu<\mathrm{M}$ such that $\mathrm{B}^m(\alpha,\beta)\cap \mathrm{M}\subset\nu$. Therefore, $\mathrm{B}^n(\alpha,\beta)$ is contained in the closure of the set $X:=\mathrm{B}^m(\alpha,\beta)\cup\nu$ with respect to the mappings $+,\varphi_\eta$ and χ_ξ for $\xi,\eta\in X$ and $\xi<\alpha$. Since $|X|<\mathrm{M}$, we obtain $|\mathrm{B}^n(\alpha,\beta)|<\mathrm{M}$. This completes the induction. As a consequence, we get $|\mathrm{B}(\alpha,\beta)|<\mathrm{M}$ and for $\zeta:=\min\{\eta:\eta\notin \mathrm{B}(\alpha,\beta)\}$ that $\zeta<\mathrm{M}$. By the definition of ζ and $B(\alpha,\zeta)$, we obtain $B(\alpha,\beta)=B(\alpha,\zeta)$, hence $\zeta\notin B(\alpha,\zeta)$, thus $\zeta\in Z_\alpha$. Therefore, since we started with an arbitrary $\beta<M$ such that $\alpha^*<\beta$ and established the existence of a $\zeta\in Z_\alpha$ such that $\beta\leqq\zeta$, Z_α must be unbounded in M.

Since Z_α is closed and unbounded in M, we obtain that $\mathrm{IN}(\alpha)$ is unbounded in M, hence, by definition, $\chi_\alpha: M\to M$ is a normal function. □

3.7. Lemma. (i) $\alpha\in B(\alpha,\chi\alpha\beta)$.
(ii) $\chi\alpha 0$, $\chi\alpha(\eta+1)\in \mathrm{IN}(\alpha)$.
(iii) $\chi\alpha\beta\notin B(\alpha,\chi\alpha\beta)$.
(iv) $B(\alpha,\chi\alpha\beta)\cap M=\chi\alpha\beta$.
(v) $\beta<\delta<M\Rightarrow\chi\alpha\beta<\chi\alpha\delta$.

Proof. (i), (ii), and (v) are obvious by the definition of $\chi\alpha\beta$.

(iii): This is immediate if β is a successor or $\beta=0$. Now suppose that β is a limit. $\chi\alpha\beta\in B(\alpha,\chi\alpha\beta)$ implies the existence of a $\eta<\beta$ such that $\chi\alpha\beta\in B(\alpha,\chi\alpha(\eta+1))$, and thus $\chi\alpha(\eta+1)\in B(\alpha,\chi\alpha(\eta+1))$ by 3.5 (iv) since $\chi\alpha(\eta+1)<\chi\alpha\beta$ and $\chi\alpha(\eta+1)\in\mathrm{IN}(\alpha)$. Contradiction.

(iv) holds by (iii) and 3.5 (iv). □

3.8. Lemma. (i) $\gamma=_{\mathrm{NF}}\gamma_1+\ldots+\gamma_n\in B(\alpha,\beta)\Rightarrow\gamma_1,\ldots,\gamma_n\in B(\alpha,\beta)$.
(ii) $\gamma=_{\mathrm{NF}}\varphi\delta\eta\in B(\alpha,\beta)\Rightarrow\delta,\eta\in B(\alpha,\beta)$.
(iii) $\gamma\in B(\alpha,\beta)\Leftrightarrow\gamma^*\in B(\alpha,\beta)$.
(iv) $\alpha^*<\chi\alpha\beta$.
(v) $\chi\gamma\delta\in B(\alpha,\chi\alpha\beta)\Rightarrow\gamma,\delta\in B(\alpha,\chi\alpha\beta)\wedge\chi\gamma\delta<\chi\alpha\beta$.

Proof. (i): It is easily seen, by induction on m, that $\gamma\in B^m(\alpha,\beta)$ implies $\gamma_1,\ldots,\gamma_n\in B^m(\alpha,\beta)$. The proof of (ii) proceeds by the same induction.

(iii) is an immediate consequence of (i) and (ii) if one inducts on γ.

(iv): It follows from (iii) and 3.7(i) that $\alpha^*\in B(\alpha,\chi\alpha\beta)$. Hence $\alpha^*<\chi\alpha\beta$ by 3.2 and 3.7(iv).

(v): Suppose $\chi\gamma\delta\in B(\alpha,\chi\alpha\beta)$, then $\chi\gamma\delta<\chi\alpha\beta$ by 3.7(iv). Since $\gamma^*,\delta\leqq\chi\gamma\delta$, another application of 3.7(iv) yields $\gamma^*,\delta\in B(\alpha,\chi\alpha\beta)$, so $\gamma\in B(\alpha,\chi\alpha\beta)$ according to (iii). □

Using 3.7(iv) and 3.8(iii), we obtain

3.9. Lemma. $\gamma\in B(\alpha,\chi\alpha\beta)\Leftrightarrow\gamma^*<\chi\alpha\beta$.

3.10. Definition. $\mu=_{\mathrm{NF}}\chi\alpha\beta:\Leftrightarrow\mu=\chi\alpha\beta\wedge\beta<\mu$.

3.11. Lemma. $\chi\alpha\beta=\chi\gamma\delta\wedge\beta,\delta<\chi\alpha\beta\Rightarrow\alpha=\gamma\wedge\beta=\delta$.

Proof. By 3.8(iv), $\chi\alpha\beta=\chi\gamma\delta$ implies $\gamma^*<\chi\alpha\beta$, thus $\gamma\in B(\alpha,\chi\alpha\beta)$. Suppose $\gamma<\alpha$. Then $\chi\gamma\delta\in B(\alpha,\chi\alpha\beta)$ follows from $\gamma,\delta\in B(\alpha,\chi\alpha\beta)$ by (B5). But this contradicts $\chi\alpha\beta\notin B(\alpha,\chi\alpha\beta)$. By symmetry, we also exclude $\alpha<\gamma$, so $\alpha=\gamma$. Since χ_α is an increasing function, this proves the assertion. □

3.12. Lemma. $\gamma<\alpha\wedge\gamma\in B(\alpha,\chi\alpha\beta)\wedge\delta<\chi\alpha\beta\Rightarrow\chi\gamma\delta<\chi\alpha\beta$.

Proof. From the hypothesis we obtain $\chi\gamma\delta\in B(\alpha,\chi\alpha\beta)$, thus $\chi\gamma\delta<\chi\alpha\beta$ by 3.7(iv). □

3.13. Lemma. *Assume $\mu=\chi\alpha\beta$ and $\nu=_{\mathrm{NF}}\chi\gamma\delta$. Then $\mu<\nu$ holds iff one of the following cases holds:*

1. $\alpha<\gamma$, $\beta<\nu$ *and* $\alpha\in\mathrm{B}(\gamma,\nu)$,
2. $\alpha=\gamma$ *and* $\beta<\gamma$,
3. $\gamma<\alpha$ *and* $(\mu\leqq\delta\vee\gamma\notin\mathrm{B}(\alpha,\mu))$.

Proof. If $\alpha<\gamma$, then $\mu<\nu\Leftrightarrow\beta<\nu\wedge\alpha\in\mathrm{B}(\gamma,\nu)$ holds by 3.8(v) and 3.12. In case $\alpha=\gamma$, we have $\mu<\nu\Leftrightarrow\beta<\gamma$ because χ_α is an increasing function. Now suppose $\gamma<\alpha$. From $\mu\leqq\delta\vee\gamma\notin\mathrm{B}(\alpha,\mu)$ we obtain $\mu<\nu$, this is because $\delta<\nu$ and $\gamma\in\mathrm{B}(\gamma,\nu)\subset\mathrm{B}(\alpha,\nu)$. Since $\delta<\mu\wedge\gamma\in\mathrm{B}(\alpha,\mu)$ implies $\nu\in\mathrm{B}(\alpha,\mu)$ and moreover $\nu<\mu$, we also have $\mu<\nu\Rightarrow\mu\leqq\delta\vee\gamma\notin\mathrm{B}(\alpha,\mu)$ in this case. This proves our assertion. □

From 3.9 and 3.13, we obtain the following characterization of $\mu<\nu$:

3.14. Corollary. *Assume $\mu=\chi\alpha\beta$ and $\nu=_{\mathrm{NF}}\chi\gamma\delta$. Then $\mu<\nu$ iff we have one of the following cases:*

1. $\alpha<\gamma$, $\beta<\nu$ *and* $\alpha^*<\nu$,
2. $\alpha=\gamma$ *and* $\beta<\gamma$,
3. $\gamma<\alpha$ *and* $(\mu\leqq\delta\vee\mu\leqq\gamma^*)$.

3.15. Lemma. *Let $\mu=_{\mathrm{NF}}\Phi\alpha\beta<\mathrm{M}$, $0<\alpha$ and let $\nu=_{\mathrm{NF}}\chi\gamma\delta$. Then we have:*

(i) $\mu\neq\nu$.

(ii) $\mu<\nu\Leftrightarrow(\gamma=0\wedge\mu<\delta)\vee(0<\gamma\wedge\alpha,\beta<\nu)$.

Proof. If $0<\gamma$, then ν is weakly inaccessible or a limit of weakly inaccessibles, thus (i) and (ii) follow from 2.2(vii). If $\gamma=0$, then $\nu=\Phi0\delta$, so (i) and (ii) hold by 2.3 and 2.4. □

4. Relations between I_α and χ_α

In this paragraph, we show that the functions I_α and χ_α for $\alpha<\Lambda_0$ coincide on M, where Λ_0 denotes the least ordinal $\zeta>0$ such that $\forall\delta,\eta<\zeta(\mathrm{I}\delta\eta<\zeta)$.

4.1. Lemma. *By recursion on n, we define $\delta_0=0$ and $\delta_{n+1}=\chi\delta_n0$. We set $\lambda=\sup_{n<\omega}\delta_n$.*

(i) $\delta_n<\chi\mathrm{M}0\wedge\delta_n<\delta_{n+1}$.

(ii) $\forall\alpha<\lambda(\alpha\in\mathrm{B}(\alpha,0))$.

(iii) $\forall\alpha<\lambda\forall\beta<\mathrm{M}[\mathrm{I}\alpha\beta=\chi\alpha\beta]$.

(iv) $\Lambda_0=\sup\{\delta_n: n<\omega\}$.

Proof. (i): This is easily verified by induction on n using 3.13.

(ii): Suppose $\alpha<\lambda$. Then there exists n such that $\alpha<\delta_n$. Hence we have $\alpha\in\mathrm{B}(\delta_{n+1},0)$ since $\delta_n\subset\mathrm{B}(\delta_{n+1},0)$. Let η be minimal with the property $\alpha\in\mathrm{B}(\eta,0)$. The above discussion shows $\eta<\lambda$. If $\eta=0$, then $\alpha\in\mathrm{B}(\alpha,0)$ holds. In case $0<\eta$, we have $\eta=\gamma+1$ for some γ, so $\alpha\notin\mathrm{B}(\gamma,0)$ and $\alpha\in\mathrm{B}(\gamma+1,0)$. If $\gamma\notin\mathrm{B}(\gamma,0)$, then one sees easily by induction on n that $\mathrm{B}^n(\gamma+1,0)\subset\mathrm{B}(\gamma,0)$, which leads to the contradiction $\mathrm{B}(\gamma+1,0)=\mathrm{B}(\gamma,0)$. Hence $\gamma\in\mathrm{B}(\gamma,0)$. If $\gamma=0$, then $\gamma<\alpha$. If $0<\gamma$, then $\chi0(\gamma+1)\subset\mathrm{B}(\gamma,0)$, thus $\gamma<\alpha$ since $\alpha\notin\mathrm{B}(\gamma,0)$. This verifies $\gamma+1\leqq\alpha$. Hence, from $\alpha\in\mathrm{B}(\gamma+1,0)$, we obtain $\alpha\in\mathrm{B}(\alpha,0)$.

To show (iii), we proceed by induction on $\alpha<\lambda$. One has $\chi 0\beta = \mathrm{I}0\beta$. This is because $\kappa \notin \mathrm{B}(0,\kappa)$ for every regular $\kappa<\mathrm{M}$. Now suppose that the assertion is true for $\xi<\alpha$. If κ is α-inaccessible, then $\forall\xi<\alpha\forall\eta<\kappa(\mathrm{I}\xi\eta=\chi\xi\eta<\kappa)$, hence $\kappa\notin\mathrm{B}(\alpha,\kappa)$. According to 3.5(iv), from $\kappa\notin\mathrm{B}(\alpha,\kappa)$, we get $\chi\xi\eta<\kappa$ if $\xi<\alpha$, $\xi\in\mathrm{B}(\alpha,\kappa)$ and $\eta<\kappa$. By (ii), we have $\xi\in\mathrm{B}(\alpha,\kappa)$ for every $\xi<\alpha$, thus $\forall\xi<\alpha\forall\eta<\kappa(\chi\xi\eta=\mathrm{I}\xi\eta<\kappa)$, which proves α-inaccessibility of κ. So we have established that $\mathrm{IN}(\alpha)=\{\kappa<\mathrm{M}:\omega<\kappa\wedge\kappa$ is α-inaccessible$\}$, which becomes $\forall\beta<\mathrm{M}\ (\chi\alpha\beta=\mathrm{I}\alpha\beta)$. The proof is complete.

(iv): If $\xi,\eta<\lambda$, then there exists n such that $\xi,\eta<\delta_n$, so $\mathrm{I}\xi\eta=\chi\xi\eta<\delta_{n+1}$ by (iii) and 3.13, hence $\Lambda_0\leqq\lambda$. Now let us show that $\delta_n<\Lambda_0$. The case $n=0$ is trivial. If $n=m+1$, then $\delta_m<\Lambda_0$ by induction hypothesis, so $\delta_n=\mathrm{I}\delta_m 0<\Lambda_0$ by (iii) and the definition of Λ_0. Hence $\lambda\leqq\Lambda_0$, and we are done. □

4.2. Corollary. (i) $\Lambda_0<\chi\mathrm{M}0$.
(ii) $\forall\alpha<\Lambda_0\forall\beta<\mathrm{M}(\mathrm{I}\alpha\beta=\chi\alpha\beta)$.

5. The Functions ψ_κ

In this paragraph we define ordinal functions $\psi_\kappa:\mathrm{M}^\Gamma\to\kappa$ for all regular cardinals κ of the shape $\chi\alpha\beta$. ψ_κ "collapses" elements of M^Γ below κ. The invention of these functions is due to Buchholz. The article [5] presents the definition of the functions ψ_κ for κ ranging over regular cardinals $>\omega$ below the first weakly inaccessible (hence for successor cardinals). In [8], Jäger extended the hierarchy to obtain collapsing functions ψ_κ for every $\kappa<\Lambda_0$. Our exposition here is inspired by [8].

5.1. Definition. (i) $\mathrm{R}:=\{\chi\alpha 0:\alpha<\mathrm{M}^\Gamma\}\cup\{\chi\alpha(\beta+1):\alpha<\mathrm{M}^\Gamma\wedge\beta<\mathrm{M}\}$.
(ii) For $\kappa\in\mathrm{R}$ we set

$$\kappa^- = \begin{cases}\chi\alpha\beta & \text{if} \quad \kappa=\chi\alpha(\beta+1)\\ \alpha^* & \text{if} \quad \kappa=\chi\alpha 0\end{cases}.$$

Convention: In this and the following paragraphs, $\kappa,\pi,\tau,\kappa',\pi',\tau'$ are supposed to range over elements of R.

5.2. Lemma. (i) $\kappa^-<\kappa$.
(ii) *If* $\kappa=_{\mathrm{NF}}\chi\alpha\beta$, *then* $\alpha^*\leqq\kappa^-$.

Proof. Immediate from 3.8(iv). □

5.3. Inductive Definition of $\mathrm{C}_\kappa(\alpha)$ and $\psi_\kappa(\alpha)$ for $\alpha<\mathrm{M}^\Gamma$ by recursion on α.
(C1) $\kappa^-\cup\{\kappa^-,\mathrm{M}\}\subset\mathrm{C}^n_\kappa(\alpha)$.
(C2) $\gamma=_{\mathrm{NF}}\gamma_1+\ldots+\gamma_k\wedge\gamma_1,\ldots,\gamma_k\in\mathrm{C}^n_\kappa(\alpha)\Rightarrow\gamma\in\mathrm{C}^{n+1}_\kappa(\alpha)$.
(C3) $\gamma=_{\mathrm{NF}}\varphi\delta\eta\wedge\delta,\eta\in\mathrm{C}^n_\kappa\Rightarrow\gamma\in\mathrm{C}^{n+1}_\kappa(\alpha)$.
(C4) $\pi\in\mathrm{C}^n_\kappa(\alpha)\cap\kappa\wedge\gamma<\pi\wedge\pi\in\mathrm{R}\Rightarrow\gamma\in\mathrm{C}^{n+1}_\kappa(\alpha)$.
(C5) $\gamma=_{\mathrm{NF}}\chi\delta\eta\wedge\delta,\eta\in\mathrm{C}^n_\kappa(\alpha)\Rightarrow\gamma\in\mathrm{C}^{n+1}_\kappa(\alpha)$.
(C6) $\gamma=_{\mathrm{NF}}\Phi\delta\eta\wedge\delta,\eta\in\mathrm{C}^n_\kappa(\alpha)\wedge 0<\delta\wedge\delta,\eta<\mathrm{M}\Rightarrow\gamma\in\mathrm{C}^{n+1}_\kappa(\alpha)$.
(C7) $\beta<\alpha\wedge\pi,\ \beta\in\mathrm{C}^n_\kappa(\alpha)\wedge\beta\in\mathrm{C}_\pi(\beta)\Rightarrow\psi_\pi(\beta)\in\mathrm{C}^{n+1}_\kappa(\alpha)$.
(C8) $\mathrm{C}_\kappa(\alpha)=\bigcup\{\mathrm{C}^n_\kappa(\alpha):n\in\omega\}$.
(C9) $\psi_\kappa(\alpha)=\min\{\xi:\xi\notin\mathrm{C}_\kappa(\alpha)\}$.
We write $\psi\kappa\alpha$ for $\psi_\kappa(\alpha)$.

5.4. Lemma. (i) $|C_\kappa(\alpha)|<\kappa$ *($|C_\kappa(\alpha)|:=$ cardinality of $C_\kappa(\alpha)$).*
(ii) $\psi\kappa\alpha\in(\kappa^-,\kappa)$.

Proof. (i): It is enough to show that $|C^n_\kappa(\alpha)|<\kappa$. This is easily accomplished by an induction on n using the regularity of κ.
(ii) follows from (i). $\square$

5.5. Lemma. (i) $\alpha<\beta\Rightarrow C_\kappa(\alpha)\subset C_\kappa(\beta)\wedge\psi\kappa\alpha\leqq\psi\kappa\beta$.
(ii) $\alpha<\beta\wedge\alpha\in C_\kappa(\alpha)\Rightarrow\psi\kappa\alpha<\psi\kappa\beta$.
(iii) $\kappa\in C_\kappa(\alpha)$.
(iv) $\psi\kappa\alpha\in SC$.
(v) $\mu=_{NF}\chi\xi\eta\Rightarrow\mu\neq\psi\kappa\alpha$.
(vi) $\mu=_{NF}\Phi\xi\eta\Rightarrow\mu\neq\psi\kappa\alpha$.

Proof. (i) is obvious if one passes through the inductive definitions of the sets.
(ii): By (i), from $\alpha<\beta$ and $\alpha\in C_\kappa(\alpha)$, we obtain $\alpha\in C_\kappa(\beta)\cap\beta$, thus $\psi\kappa\alpha\in C_\kappa(\beta)$ holds by (ii) and (C7), so $\psi\kappa\alpha<\psi\kappa\beta$ since $\psi\kappa\beta\notin C_\kappa(\beta)$ and $\psi\kappa\alpha\leqq\psi\kappa\beta$.
(iii): Suppose $\kappa=_{NF}\chi\delta\eta$. If $\eta=0$, then $\delta^*=\kappa^-$, hence $SC_M(\delta)\subset C_\kappa(\alpha)$, which implies $\delta\in C_\kappa(\alpha)$ by (C1)–(C3), thus $\kappa\in C_\kappa(\alpha)$ by (C5). If $\eta=\zeta+1$, the discussion above shows that $\delta\in C_\kappa(\alpha)$, as well. This is because $\delta^*\leqq\kappa^-$ holds by 5.2(ii). Moreover, we have $\zeta<\kappa^-+1\subset C_\kappa(\alpha)$ in this case, so $\kappa=\chi\delta(\zeta+1)\in C_\kappa(\alpha)$ thanks to (C1)–(C3) and (C5).
(iv) follows from closure of $C_\kappa(\alpha)$ under $+$ and φ.
(v): Suppose $\mu=_{NF}\chi\xi\eta$. Then $\xi^*,\eta<\mu$. So, if $\mu=\psi\kappa\alpha$, then $\xi^*,\eta\in C_\kappa(\alpha)$, thus $\mu\in C_\kappa(\alpha)$, which contradicts $\psi\kappa\alpha\notin C_\kappa(\alpha)$.
(vi): Suppose, for a contradiction, $\mu=_{NF}\Phi\xi\eta=\psi\kappa\alpha$. Then $\xi,\eta<\psi\kappa\alpha$, so $\xi,\eta\in C_\kappa(\alpha)$, which yields $\psi\kappa\alpha=\mu\in C_\kappa(\alpha)$. Contradiction. $\square$

5.6. Lemma. (i) $\psi\pi\gamma<\kappa<\pi\Rightarrow\psi\pi\gamma\leqq\kappa^-$.
(ii) $\kappa<\pi\Rightarrow\psi\pi\gamma\notin(\kappa^-,\kappa)$.

Proof. (i): Suppose $\kappa=_{NF}\chi\beta\xi$. Then we have $\beta^*,\xi\leqq\kappa^-$. This shows that $\kappa^-<\psi\pi\gamma$ implies $\beta^*,\xi\in C_\pi(\gamma)$, whence $\kappa\in C_\pi(\gamma)$. So if $\kappa^-<\psi\pi\gamma$ and $\kappa<\pi$ hold, then $\kappa\subset C_\pi(\gamma)$ by (C4), which ensures that $\kappa\leqq\psi\pi\gamma$. This proves our claim.
(ii) follows immediately from (i). $\square$

5.7. Proposition. $C_\kappa(\alpha)\cap\kappa=\psi\kappa\alpha$.

Proof. Let $\kappa=\chi\beta\xi$. If $\beta>0$ and $\eta\in C_\kappa(\alpha)\cap\kappa$, then we get $\eta<\chi0(\eta+1)\subset C_\kappa(\alpha)\cap\kappa$ by (C5) and (C4), and thus the assertion. So we only need to consider the case $\beta=0$. We induct on n to prove $C^n_\kappa(\alpha)\cap\kappa\subset\psi\kappa\alpha$.

If $\delta\in C^n_\kappa(\alpha)\cap\kappa$ holds by (C1)–(C4), then $\delta<\psi\kappa\alpha$ follows either trivially or by I.H. ($=$induction hypothesis) using 5.5(iv) in the case of (C3).

Assume that $\delta\in C^n_\kappa(\alpha)\cap\kappa$ holds by (C5). Then $\delta=_{NF}\chi\gamma\zeta$ for some $\gamma,\zeta\in C^{n-1}_\kappa(\alpha)$. By I.H., we know that $\zeta<\psi\kappa\alpha$. Since $\delta<\kappa=\chi0\xi$, either $\gamma=0\wedge\zeta<\xi$ or $0<\gamma\wedge\delta\leqq\xi$ holds by 3.13. In the first case, we get $\delta\leqq\kappa^-$, hence $\delta<\psi\kappa\alpha$. $\xi\leqq\kappa^-$ implies $\delta<\psi\kappa\alpha$ in the second case.

If $\delta=_{NF}\Phi\gamma\zeta\in C^n_\kappa(\alpha)\cap\kappa$ holds by (C6), then $0<\gamma$. Therefore, we obtain $\chi0\delta=\delta<\chi0\xi=\kappa$, hence $\delta<\xi\leqq\kappa^-<\psi\kappa\alpha$.

Finally, let us assume that $\delta=\psi\pi\gamma\in C^n_\kappa(\alpha)\cap\kappa$, where $\gamma\in C_\pi(\gamma)$, $\gamma<\alpha$ and $\pi,\gamma\in C^{n-1}_\kappa(\alpha)$. If $\pi<\kappa$, then we have $\pi<\psi\kappa\alpha$ by I.H., whence $\delta<\psi\kappa\alpha$. If $\pi=\kappa$, then

$\delta<\psi\kappa\alpha$ follows from 5.5(ii). Now suppose $\kappa<\pi$. By 5.6(ii), since $\psi\pi\gamma<\kappa$, this enforces $\psi\pi\gamma\leqq\kappa^-$, thus $\delta<\psi\kappa\alpha$. $\square$

5.8. Lemma. *Let* $\pi<\kappa$.

(i) $\pi\leqq\psi\kappa\alpha\Rightarrow\psi\pi\beta<\psi\kappa\alpha$.
(ii) $\psi\kappa\alpha<\pi\Rightarrow\psi\kappa\alpha<\psi\pi\beta$.
(iii) $\psi\kappa\alpha\neq\psi\pi\beta$.

Proof. (i) holds since $\psi\pi\beta<\pi$.

(ii): Suppose $\psi\kappa\alpha<\pi$. Then $\psi\kappa\alpha\leqq\pi^-$ holds by 5.6(i), hence $\psi\kappa\alpha<\psi\pi\beta$.

(iii) is a consequence of (i) and (ii). $\square$

5.9. Corollary. $\psi\kappa\alpha=\psi\pi\beta\wedge\alpha\in C_\kappa(\alpha)\wedge\beta\in C_\pi(\beta)\Rightarrow\kappa=\pi\wedge\alpha=\beta$.

Proof. 5.8(iii) and 5.5(ii). $\square$

5.10. Proposition. *Let* $\alpha\in C_\kappa(\alpha)$ *and* $\beta\in C_\pi(\beta)$. *Then* $\psi\pi\beta<\psi\kappa\alpha$ *iff we have one of the following cases:*

(1) $\pi<\kappa\wedge\pi<\psi\kappa\alpha$,
(2) $\pi=\kappa\wedge\beta<\alpha$,
(3) $\kappa<\pi\wedge\psi\pi\beta<\kappa$.

Proof. Suppose $\pi<\kappa$. Then clearly (1) $\Rightarrow\psi\pi\beta<\psi\kappa\alpha$. If $\psi\pi\beta<\psi\kappa\alpha$, then $\pi\leqq\psi\kappa\alpha$ by 5.6(ii), so $\pi<\psi\kappa\alpha$ holds by 5.5(v). This establishes $\psi\pi\beta<\psi\kappa\alpha\Rightarrow$ (1). In case $\pi=\kappa$, the assertion follows from 5.5(ii).

Now suppose $\kappa<\pi$. In this case, we have $\psi\pi\beta\notin(\kappa^-,\kappa)$ by 5.6(ii), hence (3) holds iff $\psi\pi\beta<\psi\kappa\alpha$. $\square$

5.11. Definition. $\gamma=_{\mathrm{NF}}\psi\kappa\alpha:\Leftrightarrow\gamma=\psi\kappa\alpha\wedge\alpha\in C_\kappa(\alpha)$.

5.12. Lemma. (i) $\gamma=_{\mathrm{NF}}\gamma_1+\ldots+\gamma_n\in C_\kappa^m(\alpha)\Rightarrow\gamma_1,\ldots,\gamma_n\in C_\kappa^m(\alpha)$.

(ii) $\gamma=_{\mathrm{NF}}\varphi\xi\eta\in C_\kappa^m(\alpha)\vee\gamma=_{\mathrm{NF}}\Phi\xi\eta\in C_\kappa^m(\alpha)\Rightarrow\xi,\eta\in C_\kappa^m(\alpha)$.
(iii) $\gamma=_{\mathrm{NF}}\chi\xi\eta\in C_\kappa^m(\alpha)\Rightarrow\xi,\eta\in C_\kappa(\alpha)$.
(iv) $\gamma=\chi\xi\eta\in C_\kappa(\alpha)\cap\kappa\Rightarrow\xi,\eta\in C_\kappa(\alpha)$.
(v) $\pi<\kappa\wedge\gamma=\psi\pi\beta\in C_\kappa(\alpha)\Rightarrow\pi\in C_\kappa(\alpha)$.
(vi) $\kappa\leqq\pi\wedge\kappa^-<\psi\pi\beta\wedge\gamma=_{\mathrm{NF}}\psi\pi\beta\in C_\kappa(\alpha)\Rightarrow\beta<\alpha\wedge\pi,\beta\in C_\kappa(\alpha)$.
(vii) $\pi\in C_\kappa(\alpha)\Rightarrow\pi^-\in C_\kappa(\alpha)$.

Proof. One easily proves (i) and (ii) by induction on m.

(iii): Suppose $\gamma=_{\mathrm{NF}}\chi\xi\eta\in C_\kappa^m(\alpha)$. We proceed by induction on m to show $\xi,\eta\in C_\kappa(\alpha)$. According to 3.15(i) and 5.5(v), $\gamma\in C_\kappa^m(\alpha)$ must hold by (C1), (C4) or (C5). If this is the case via (C1) or (C4), then $\xi^*,\eta\in C_\kappa(\alpha)$, thus $\xi,\eta\in C_\kappa(\alpha)$. In the remaining case, the assertion follows from 3.11.

(iv): By 5.7, the hypothesis yields $\xi^*,\eta\leqq\gamma<\psi\kappa\alpha$, so $\xi^*,\eta\in C_\kappa(\alpha)$, thus $\xi,\eta\in C_\kappa(\alpha)$.

(v): By 5.6(ii), $\pi<\kappa\Rightarrow\psi\kappa\alpha\notin(\pi^-,\pi)$. Hence our hypothesis implies $\pi\leqq\psi\kappa\alpha$, so $\pi<\psi\kappa\alpha$ by 5.5(v), and we are done.

(vi): If $\kappa<\pi$ and $\kappa^-<\psi\pi\beta$, then $\kappa\leqq\psi\pi\beta$ holds by 5.6(ii), hence $\psi\pi\beta\in C_\kappa(\alpha)$ must hold by (C7), thus $\pi,\beta\in C_\kappa(\alpha)$ and $\beta<\alpha$. If $\kappa=\pi$, then $\beta<\alpha$ and, since $\beta\in C_\kappa(\beta)$, we also have $\beta\in C_\kappa(\alpha)$. Moreover, $\kappa\in C_\kappa(\alpha)$ by 5.5(iii). This proves the claim.

(vii) follows from (i)–(iii). $\square$

5.13. Lemma. *If* $\beta<\chi\alpha\beta$, *then:*

(i) $\chi\alpha\beta<\psi\kappa\xi \Leftrightarrow \chi\alpha\beta<\kappa \wedge \chi\alpha\beta\in \mathrm{C}_\kappa(\xi)$.

If $\gamma=_{\mathrm{NF}}\Phi\alpha\beta$, *then:*

(ii) $\gamma<\psi\kappa\xi \Leftrightarrow \alpha,\beta<\psi\kappa\alpha \wedge \gamma<\kappa$.

Proof. (i) and (ii) follow from 5.7. □

The next lemma provides a characterization of the relation $\beta<\chi\alpha\beta$ by means of subterms of α and β, solely.

5.14. Lemma. *Let* $\beta<\mathrm{M}$.

(i) $\beta\notin \mathrm{SC} \Rightarrow \beta<\chi\alpha\beta$.

(ii) *If* $\beta=_{\mathrm{NF}}\chi\gamma\delta$, *then:* $\beta<\chi\alpha\beta \Leftrightarrow \gamma\leqq\alpha \vee \beta\leqq\alpha^*$.

(iii) *If* $\beta=_{\mathrm{NF}}\psi\kappa\xi$ *and* $\kappa=_{\mathrm{NF}}\chi\gamma\delta$, *then:* $\beta<\chi\alpha\beta \Leftrightarrow \gamma\leqq\alpha \vee \kappa\leqq\alpha^* \vee \alpha\notin \mathrm{C}_\kappa(\xi)$.

(iv) *If* $\beta=_{\mathrm{NF}}\Phi\gamma\delta$ *and* $0<\gamma$, *then:* $\beta<\chi\alpha\beta \Leftrightarrow 0<\alpha$.

Proof. (i) is obvious by 2.2(vi).

(ii): "$\Rightarrow$": $\alpha<\gamma \wedge \alpha^*<\beta$ yields that $\alpha\in \mathrm{B}(\gamma,\beta)$, and, moreover, $\forall\xi<\beta(\chi\alpha\xi<\beta)$ by (B5) and 3.7(iv), which becomes $\chi\alpha\beta=\beta$.

"$\Leftarrow$": Since $\alpha^*<\chi\alpha\beta$, we clearly have $\beta\leqq\alpha^* \Rightarrow \beta<\chi\alpha\beta$. If $\gamma=\alpha$, then $\beta=\chi\alpha\delta<\chi\alpha\beta$ since $\delta<\beta$. Now suppose that $\gamma<\alpha$. Note that $\gamma^*,\delta<\beta$. Hence $\gamma,\delta\in \mathrm{B}(\alpha,\chi\alpha\beta)$. This implies $\beta=\chi\gamma\delta<\chi\alpha\beta$ by (B5) and 3.7(iv).

(iii): "$\Rightarrow$": From $\alpha<\gamma \wedge \alpha^*<\kappa \wedge \alpha\in \mathrm{C}_\kappa(\xi)$, it follows that $\forall\eta<\beta(\chi\alpha\eta\in \mathrm{C}_\kappa(\xi)\cap\kappa)$, hence $\chi\alpha\beta=\beta$. "$\Leftarrow$": Note that $\alpha\notin \mathrm{C}_\kappa(\xi)$ implies $\alpha^*\notin \mathrm{C}_\kappa(\xi)$. So, since $\alpha^*<\chi\alpha\beta$, we get $\kappa\leqq\alpha^* \vee \alpha\notin \mathrm{C}_\kappa(\xi) \Rightarrow \beta<\chi\alpha\beta$. We know that $\delta\leqq\kappa^-<\psi\kappa\xi$, hence $\kappa=\chi\gamma\delta<\chi\gamma\beta$. This verifies $\beta<\chi\alpha\beta$ in case that $\alpha=\gamma$. Finally, suppose that $\gamma<\alpha$. If $\chi\alpha\beta<\kappa$, then we get $\chi\alpha\beta\leqq\gamma^*$ by 3.14, since $\delta<\beta\leqq\chi\alpha\beta$. But this is impossible because $\gamma^*\leqq\kappa^-<\beta$. Hence we have $\kappa\leqq\chi\alpha\beta$, which implies $\beta<\chi\alpha\beta$.

(iv) is a consequence of 2.2(ii) and 2.2(vii). □

6. The Notation System T(M)

We isolate a countable set of ordinals, T(M), such that each element of T(M) can be denoted uniquely using only the symbols $0, \mathrm{M}, +, \varphi, \chi, \Phi, \psi$.

6.1. Inductive Definition of T(M) and $\mathrm{G}\gamma<\omega$ for $\gamma\in \mathrm{T(M)}$.

(T1) $0, \mathrm{M}\in \mathrm{T(M)}$ and $\mathrm{G}0=\mathrm{GM}=0$.

(T2) $\gamma=_{\mathrm{NF}}\gamma_1+\ldots+\gamma_n \wedge \gamma_1,\ldots,\gamma_n\in \mathrm{T(M)}$
$\Rightarrow \gamma\in \mathrm{T(M)} \wedge \mathrm{G}\gamma=\max\{\mathrm{G}\gamma_1,\ldots,\mathrm{G}\gamma_n\}+1$.

(T3) $\gamma=_{\mathrm{NF}}\varphi\delta\eta<\mathrm{M} \wedge \delta,\eta\in \mathrm{T(M)} \Rightarrow \gamma\in \mathrm{T(M)} \wedge \mathrm{G}\gamma=\max\{\mathrm{G}\delta,\mathrm{G}\eta\}+1$.

(T4) $\gamma=_{\mathrm{NF}}\varphi 0\eta \wedge \mathrm{M}<\eta \wedge \eta\in \mathrm{T(M)} \Rightarrow \gamma\in \mathrm{T(M)} \wedge \mathrm{G}\gamma=(\mathrm{G}\eta)+1$.

(T5) $\gamma=_{\mathrm{NF}}\chi\delta\eta \wedge \delta,\eta\in \mathrm{T(M)} \Rightarrow \gamma\in \mathrm{T(M)} \wedge \mathrm{G}\gamma=\max\{\mathrm{G}\delta,\mathrm{G}\eta\}+1$.

(T6) $\gamma=_{\mathrm{NF}}\Phi\delta\eta<\mathrm{M} \wedge 0<\delta \wedge \delta,\eta\in \mathrm{T(M)} \Rightarrow \gamma\in \mathrm{T(M)} \wedge \mathrm{G}\gamma=\max\{\mathrm{G}\delta,\mathrm{G}\eta\}+1$.

(T7) $\gamma=_{\mathrm{NF}}\psi\kappa\alpha \wedge \kappa,\alpha\in \mathrm{T(M)} \wedge \alpha<\mathrm{M} \Rightarrow \gamma\in \mathrm{T(M)} \wedge \mathrm{G}\gamma=\max\{\mathrm{G}\kappa,\mathrm{G}\alpha\}+1$.

6.2. Remark. It follows from 2.2(vi), 2.5, 2.8, 3.11, 3.15, 5.5(iv), (v), (vi), and 5.9 that every ordinal $\gamma\in \mathrm{T(M)}$ is an element of T(M) due to exactly one of the rules (T1)–(T7) and that its degree $\mathrm{G}\gamma<\omega$ is uniquely determined.

6.3. Theorem. $\mathrm{T(M)} \cap \mathrm{M} = \mathrm{C}_{\chi 00}(\psi\Xi 0) \cap \psi\Xi 0$, *where* $\Xi := \chi\varepsilon_{\mathrm{M}+1}0$ *and* $\varepsilon_{\mathrm{M}+1} := \varphi 1(\mathrm{M}+1)$.

Proof. We set $X := \mathrm{C}_{\chi 00}(\psi\Xi 0) \cap \psi\Xi 0$ and $\mathrm{T} := \mathrm{T(M)} \cap \mathrm{M}$. Note that $\mathrm{T(M)} \subset \varepsilon_{\mathrm{M}+1}$.

"$\subset$": By induction on $\mathrm{G}\gamma$, we show that $\gamma \in \mathrm{T} \Rightarrow \gamma \in \mathrm{X} \cap \mathrm{C}_\Xi(0)$. If $\gamma =_{\mathrm{NF}} \chi\delta\eta \in \mathrm{T}$, then $\mathrm{SC_M}(\delta) \cup \{\eta\} \subset \mathrm{T}$, so $\mathrm{SC_M}(\delta) \cup \{\eta\} \subset \mathrm{X} \cap \mathrm{C}_\Xi(0)$ by I.H. Since $\delta < \varepsilon_{\mathrm{M}+1}$, we obtain $\chi\delta\eta \in \mathrm{C}_\Xi(0) \cap \Xi$, thus $\chi\delta\eta < \psi\Xi 0$ holds by 5.7, hence $\gamma \in \mathrm{X} \cap \mathrm{C}_\Xi(0)$. If $\gamma =_{\mathrm{NF}} \psi\kappa\alpha \in \mathrm{T}$, then $\kappa, \alpha \in \mathrm{T}$, so $\kappa, \alpha \in \mathrm{X} \cap \mathrm{C}_\Xi(0)$ by I.H. This implies $\gamma \in \mathrm{X} \cap \mathrm{C}_\Xi(0)$. The remaining cases are easily verified.

"$\supset$": Let $\mathrm{X}^n := \mathrm{C}^n_{\chi 00}(\psi\Xi 0) \cap \psi\Xi 0$. We prove $\mathrm{X}^n \subset \mathrm{T}$ by induction on n. Note that (C4) does not apply since there is no π satisfying $\pi < \chi 00$. Suppose $\gamma =_{\mathrm{NF}} \psi\kappa\alpha \in \mathrm{X}^n$. Then $\kappa, \alpha \in \mathrm{C}^{n-1}_{\chi 00}(\psi\Xi 0)$ and $\alpha < \psi\Xi 0$. $\Xi < \kappa$ would imply $\Xi \in \mathrm{C}_\kappa(\alpha) \cap \kappa$, and thus lead to the contradiction $\Xi < \psi\kappa\alpha$. Hence we have $\kappa < \psi\Xi 0$ by 5.10. Therefore, by I.H., we obtain $\kappa, \alpha \in \mathrm{T}$, hence $\gamma \in \mathrm{T}$. Next suppose $\gamma =_{\mathrm{NF}} \chi\delta\eta$. Then $\mathrm{SC_M}(\delta) \cup \{\eta\} \subset \mathrm{X}^{n-1}$, hence $\mathrm{SC_M}(\delta) \cup \{\eta\} \subset \mathrm{T}$ by I.H., which implies $\gamma \in \mathrm{T}$. Similarly, one proves the other cases. □

6.4. Lemma. *Let* α, β *be elements of* T(M), *and let* $\gamma, \delta, \kappa \in \mathrm{T(M)} \cap \mathrm{M}$. *Then:*

(i) $\alpha + \beta$, $\omega^\alpha \in \mathrm{T(M)}$.

(ii) $\varphi\gamma\delta$, $\chi\alpha\gamma$, $\Phi\gamma\delta \in \mathrm{T(M)}$.

(iii) $\alpha^* \in \mathrm{T(M)} \wedge \mathrm{G}\alpha^* \leqq \mathrm{G}\alpha$.

(iv) $\kappa^- \in \mathrm{T(M)} \wedge \mathrm{G}\kappa^- \leqq \mathrm{G}\kappa$.

Proof. Obvious. □

6.5. Lemma. $\Lambda_0 = \psi(\chi\mathrm{M}0)0$.

Proof. $\delta_n \in \mathrm{C}_{\chi\mathrm{M}0}(0)$ is easily shown by induction on n. Now $\delta_n < \chi\mathrm{M}0$. Therefore, we get the inequality "$\leqq$" by 4.1(iv) and 5.7. To prove "$\geqq$", it suffices to show $\mathrm{C}^n_{\chi\mathrm{M}0}(0) \cap \chi\mathrm{M}0 \subset \Lambda_0$. To this end, we induct on n. Note that the clause (C7) does not apply. Now suppose $\gamma \in \mathrm{C}^n_{\chi\mathrm{M}0}(0) \cap \chi\mathrm{M}0$. If this is the case by (C1)–(C4) or (C6), then the assertion follows immediately by I.H. If $\gamma =_{\mathrm{NF}} \chi\alpha\beta$, then $\alpha, \beta < \chi\mathrm{M}0$, thus $\alpha, \beta < \Lambda_0$ holds by I.H. This implies $\alpha, \beta < \delta_n$ for some $n < \omega$, hence $\gamma < \delta_{n+1} < \Lambda_0$. The same argument yields $\gamma < \Lambda_0$ in case that $\gamma =_{\mathrm{NF}} \Phi\alpha\beta$ with $0 < \alpha$. □

7. A Primitive Recursive Notation System

We assume that $\langle \ldots \rangle$ is a primitive recursive coding function on finite sequences of natural numbers. The function $\mathrm{V}: \mathrm{T(M)} \to \mathbb{N}$ is given by the following recursion with respect to $\mathrm{G}\alpha$:

$$\mathrm{V}(\alpha) = \begin{cases} \langle 0, 0 \rangle & \text{if } \alpha = 0 \\ \langle 1, 0 \rangle & \text{if } \alpha = \mathrm{M} \\ \langle 2, \mathrm{V}(\alpha_1), \ldots, \mathrm{V}(\alpha_n) \rangle & \text{if } \alpha =_{\mathrm{NF}} \alpha_1 + \ldots + \alpha_n \\ \langle 3, \mathrm{V}(\beta), \mathrm{V}(\gamma) \rangle & \text{if } \alpha =_{\mathrm{NF}} \varphi\beta\gamma \\ \langle 4, \mathrm{V}(\beta), \mathrm{V}(\gamma) \rangle & \text{if } \alpha =_{\mathrm{NF}} \chi\beta\gamma \\ \langle 5, \mathrm{V}(\beta), \mathrm{V}(\gamma) \rangle & \text{if } \alpha =_{\mathrm{NF}} \Phi\beta\gamma \quad \text{and} \quad 0 < \beta \\ \langle 6, \mathrm{V}(\kappa), \mathrm{V}(\beta) \rangle & \text{if } \alpha =_{\mathrm{NF}} \psi\kappa\beta \end{cases}$$

Let $\mathfrak{T} \subset \mathbb{N}$ be the image of V. The mapping V induces a wellordering $\lhd$ on $\mathfrak{T}$ via $V(\beta) \lhd V(\gamma) :\Leftrightarrow \beta < \gamma$. The rest of the paper is devoted to proving that $\mathfrak{T}$ is a primitive recursive set of natural numbers and that $\lhd$ allows a primitive recursive definition. Indeed, this task can be executed by simply exhibiting such primitive recursive definitions of $\mathfrak{T}$ and $\lhd$, which on its part is independent of the existence of weakly Mahlo cardinals. Thus, only the wellfoundedness of $\lhd$ relies on a large cardinal hypothesis. But it is also possible to rid of this assumption as will be shown in a forthcoming paper. Indeed, it is possible to develop the notation system on the basis of recursively large ordinals by replacing each occurrence of a cardinal notion by its "recursive analogue". But then proofs will become more difficult since the proofs of 3.6 and 5.4(ii) are based on "cardinality" considerations which will no longer be available, then. So our assumption that there exists a Mahlo cardinal turns out to be an exaggeration, but it helps simplifying proofs.

Now, in the previous paragraphs, we have already done a lot of work to deliver recursive characterizations of equality and inequality between ordinals. The main obstacle which prevents us from converting those results into a recursive definition of $\lhd$ and $\mathfrak{T}$ is that they comprise conditions like $\gamma \in C_\kappa(\xi)$. This problem gives rise to the following definition.

7.1. Inductive Definition of the coefficient sets $K_\kappa \gamma$ for $\gamma \in T(M)$.

(H1) $K_\kappa 0 = K_\kappa M = \emptyset$.

(H2) $K_\kappa \gamma = K_\kappa \gamma_1 \cup \ldots \cup K_\kappa \gamma_n$ if $\gamma =_{NF} \gamma_1 + \ldots + \gamma_n$.

(H3) $K_\kappa \gamma = K_\kappa \delta \cup K_\kappa \eta$ if $\gamma =_{NF} \varphi\delta\eta$ or $\gamma =_{NF} \chi\delta\eta$ or $\gamma =_{NF} \Phi\delta\eta$.

(H4) If $\gamma =_{NF} \psi\pi\beta$, then

$$K_\kappa \gamma = \begin{cases} \emptyset, & \text{if } \gamma \leqq \kappa^- \\ K_\kappa \pi, & \text{if } \kappa^- < \gamma \wedge \pi < \kappa. \\ \{\beta\} \cup K_\kappa \pi \cup K_\kappa \beta, & \text{if } \kappa^- < \gamma \wedge \kappa \leqq \pi \end{cases}$$

7.2. Lemma. *Let* $\gamma \in T(M)$. *Then:* $\gamma \in C_\kappa(\xi) \Leftrightarrow K_\kappa \gamma < \xi$.

Proof by induction on $G\gamma$. We set $C := C_\kappa(\xi)$, $K\gamma := K_\kappa \gamma$.

1. If γ is not of the shape $\psi\pi\beta$, then the assertion follows from the I.H. using 5.12(i), (ii), (iii).

2. Suppose $\gamma =_{NF} \psi\pi\beta$.

2.1. $\gamma \leqq \kappa^-$. Then $\gamma \in C$ and $K\gamma = \emptyset < \xi$.

2.2. $\kappa^- < \gamma \wedge \pi < \kappa$. From $\gamma \in C$ we get $\pi \in C$ by 5.12(v), thus $K\gamma = K\pi < \xi$ by I.H. If $K\gamma < \xi$, then $\pi \in C$ by I.H., so $\pi < \psi\kappa\xi$, which implies $\gamma \in C$.

2.3. $\kappa^- < \gamma \wedge \kappa \leqq \pi$. Suppose $\gamma \in C$. By 5.12(vi), we then have $\pi, \beta \in C$ and $\beta < \xi$. So the I.H. yields $K\pi \cup K\beta < \xi$, hence $K\gamma < \xi$. Vice versa, if $K\gamma < \xi$, then $\beta < \xi$ and, by I.H., $\pi, \beta \in C$, thus $\gamma \in C$ by (C7).

We now turn to a definition which will allow us to decide for $\alpha, \beta \in T(M)$ whether $\alpha, \beta < \varphi\alpha\beta$ or $\alpha, \beta < \Phi\alpha\beta$.

7.3. Inductive Definition of $e(\gamma)$ for $\gamma \in T(M)$.

1. $e(\gamma) = 0$ if $\gamma \notin AP$.
2. $e(\gamma) = \alpha$ if $\gamma =_{NF} \varphi\alpha\beta$.
3. $e(\gamma) = \gamma$ otherwise.

7.4. Inductive Definition of $E(\gamma)$ for $\gamma \in T(M) \cap M$.

1. $E(\gamma)=0$ if $\gamma \notin SC$ or $\gamma =_{NF} \chi 0\beta$ or $\gamma =_{NF} \psi(\chi 0\alpha)\beta$, where $\alpha < \chi 0\alpha$.
2. $E(\gamma)=\alpha$ if $\gamma =_{NF} \Phi\alpha\beta$ and $0<\alpha$.
3. $E(\gamma)=\gamma$ otherwise.

7.5. Lemma. *Let* α, β *be elements of* $T(M)$. *Let* $\delta, \eta \in T(M) \cap M$ *and* $0<\delta$. *Then:*

(i) $\alpha, \beta < \varphi\alpha\beta \Leftrightarrow e(\beta) \leqq \alpha \wedge [0<\beta \vee e(\alpha)<\alpha \vee \alpha=0]$.

(ii) $\delta, \eta < \Phi\delta\eta \Leftrightarrow E(\eta) \leqq \delta \wedge [0<\eta \vee E(\delta)<\delta]$.

Proof. The proof consists of a straightforward but cumbersome distinction by cases using various results from earlier paragraphs. But this checking is best done on scratch paper. □

Now, thanks to 7.2, it is fairly clear by 2.2(vi), 2.4, 3.14, 5.5(v), (vi), 5.10, 5.13 (for $<$), and 5.14, 7.5 (for $=_{NF}$) how to give a simultaneous inductive definition of the set T(M) and the relations $<$ and $=_{NF}$ on T(M), which can be converted into a simultaneous primitive recursive definition of $\mathfrak{T}$ and $\lhd$. We omit the details, since this would amount to a mere repetition of the content of the above mentioned results.

7.6. Remark. Let $\tilde{C}_\kappa(\alpha)$ denote the set defined in the same way as $C_\kappa(\alpha)$ except for the omission of clause (C6) (closure under Φ). Let $\tilde{\psi}\kappa\alpha := \min\{\xi : \xi \notin \tilde{C}_\kappa(\alpha)\}$. Then it can be shown that $\tilde{\psi}(\chi 10)0 = \Phi 10$, $\tilde{\psi}\varepsilon_{(\chi 10)+1} = \chi\varepsilon_{(\chi 10)+1}0$ and $\tilde{\psi}(\chi\alpha\beta)0 = \psi(\chi\alpha\beta)0$ for $1<\alpha$ and $\chi\alpha\beta \in R$. So we could as well have chosen to define our notation system on the basis of the function $\tilde{\psi}$ without requiring closure under Φ. The main reason why we have included the function Φ in the build-up of T(M) is our desire to express the proof-theoretic ordinals of several subsystems of **KPi** by means of ordinals of T(M). By using Φ, those notations become more transparent.

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Received April 20, 1989/in revised form August 17, 1989