Chapter 15
Constructive Zermelo-Fraenkel Set Theory, Power Set, and the Calculus of Constructions

Michael Rathjen

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15.1 Introduction

If the power set operation is considered as a definite operation, but the universe of all sets is regarded as an indefinite totality, we are led to systems of set theory having Power Set as an axiom but only Bounded Separation axioms and intuitionistic logic for reasoning about the universe at large. The study of subsystems of **ZF** formulated in intuitionistic logic with Bounded Separation but containing the Power Set axiom was apparently initiated by Pozsgay (1971, 1972) and then pursued more systematically by Tharp (1971), Friedman (1973a), and Wolf (1974). These systems are actually semi-intuitionistic as they contain the law of excluded middle for bounded formulae. Pozsgay had conjectured that his system is as strong as **ZF**, but Tharp and Friedman proved its consistency in **ZF** using a modification of Kleene's method of realizability. Wolf established the equivalence in strength of several related systems.

In the classical context, weak subsystems of **ZF** with Bounded Separation and Power Set have been studied by Thiele (1968), Friedman (1973b) and more recently

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M. Rathjen (✉)
Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, UK
e-mail: rathjen@maths.leeds.ac.uk

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at great length by Mathias (2001). Mac Lane has singled out and championed a particular fragment of **ZF**, especially in his book *Form and Function* Mac Lane (1992). *Mac Lane Set Theory*, christened **MAC** in Mathias (2001), comprises the axioms of Extensionality, Null Set, Pairing, Union, Infinity, Power Set, Bounded Separation, Foundation, and Choice. **MAC** is naturally related to systems derived from topos-theoretic notions and, moreover, to type theories.

Type theories à la Martin-Löf embodying weak forms of Power Set (such as the calculus of constructions with its impredicative type of propositions) have been studied by Aczel (1986, 2000) and Gambino (1999).

Intuitionistic Zermelo-Fraenkel set theory, **IZF**, is obtained from **CZF**, by adding the full separation axiom scheme and the power set axiom. The strength of **CZF** plus full separation, as has been shown by Lubarsky (2006), is the same as that of second order arithmetic, using a straightforward realizability interpretation in classical second order arithmetic and the fact that second order Heyting arithmetic is already embedded in **CZF** plus full separation. This paper is concerned with the strength of **CZF** augmented by the power set axiom, $\mathbf{CZF}_{\mathcal{P}}$. It will be shown that it is of the same strength as Power Kripke–Platek set theory, $\mathbf{KP}(\mathcal{P})$, as well as a certain system of type theory, $\mathbf{MLV_P}$, which is a version of the calculus of constructions with one universe. It is perhaps worth pointing out that $\mathbf{KP}(\mathcal{P})$ is not the theory **KP** plus power set, **Pow**. An upper bound for the proof-theoretic strength of $\mathbf{KP} + \mathbf{Pow}$ is Zermelo's set theory, **Z**, so that it doesn't even prove the existence of $V_{\omega+\omega}$ whereas $\mathbf{KP}(\mathcal{P})$ proves the existence of V_α for any ordinal α.

The reduction of $\mathbf{CZF}_{\mathcal{P}}$ to $\mathbf{KP}(\mathcal{P})$ uses a realizability interpretation wherein a realizer for an existential statement provides a set of witnesses for the existential quantifier rather than a single witness. Tharp (1971) also used realizability to give an interpretation of a semi-intuitionistic set theory closely related to Pozsgay's system. Tharp's realizers are codes for $\Sigma_1^{\mathcal{P}}$ definable partial functions, i.e., functions whose graphs are Σ_1 in the powerset operation $\mathcal{P}(x)$, which is taken as a primitive. For the realizability interpretation he needs a $\Sigma_1^{\mathcal{P}}$-definable search operation on the set-theoretic universe and in point of fact assumes $V = L$. As it turns out, this realizability interpretation could be formalized in $\mathbf{KP}(\mathcal{P}) + V = L$. However, the assumption $V = L$ is not harmless in this context since $\mathbf{KP}(\mathcal{P}) + V = L$ is a much stronger theory than $\mathbf{KP}(\mathcal{P})$ (cf. Mathias 2001; Rathjen 2012), and therefore one would like to remove this hypothesis. This paper shows that this can be achieved by using a notion of realizability with sets of witnesses in the existential quantifier case, and thereby yields a realizability interpretation of a theory in a theory of equal proof-theoretic strength.

The reduction of $\mathbf{KP}(\mathcal{P})$ to $\mathbf{CZF}_{\mathcal{P}}$ is based on results from Rathjen (2012) whose proofs are obtained via techniques from ordinal analysis. They can be used to show that $\mathbf{KP}(\mathcal{P})$ is reducible to **CZF** with the *Negative Power Set Axiom*. As **CZF** plus the negative powerset can be interpreted in $\mathbf{MLV_P}$, utilizing work from Aczel (2000) and Gambino (1999), and the latter type theory has a types-as-classes interpretation in $\mathbf{CZF}_{\mathcal{P}}$, the circle will be completed. We also get a characterization of a subtheory of Tharp's set theory Tharp (1971). The theory in Tharp (1971) has the following axioms (cf. Sect. 15.2.1): Extensionality, Empty Set, Pairing, Union, Powerset, Infinity, Set Induction, Strong Collection,[1] Excluded Middle for power bounded formulae[2] and an axiom **Ord-Im** which asserts that every set is the image of an ordinal, i.e., for every set x there exists an ordinal α and a surjective function $f : \alpha \to x$.

In the Theorem below we use several acronyms. **RDC** stands for the relativized dependent choices axiom. Given a family of sets $(B_a)_{a\in A}$ over a set A we define the dependent product $\prod_{a\in A} B_a$ and the dependent sum $\sum_{a\in A} B_a$ as follows:

$$\prod_{a\in A} B_a := \{f \mid \mathrm{Fun}(f) \wedge \mathrm{dom}(f) = A \wedge \forall z \in A\ f(z) \in B_a\}$$

$$\sum_{a\in A} B_a := \{\langle a, u\rangle \mid a \in A \wedge u \in B_a\}$$

where $\mathrm{Fun}(f)$ signifies that f is a function and $\mathrm{dom}(f)$ stands for its domain.

Let X be the smallest class of sets containing ω and all elements of ω which is closed under dependent products and sums. $\mathbf{\Pi\Sigma-AC}$ asserts that every set A in X is a base, i.e., if $(B_a)_{a\in A}$ is family of sets over A such that B_a is inhabited for every $a \in A$ then there exists a function f with domain A such that $\forall a \in A\ f(a) \in A_a$ (for more information on this axiom see Aczel 1982, Rathjen 2006a, Rathjen and Tupailo 2006).

The negative power set axiom, $\mathbf{Pow}^{\neg\neg}$ for short, asserts that for every set a there exists a set c containing all the subsets x of a for which $\forall u \in a(\neg\neg u \in x \to u \in x)$ holds.

The intuitionistic version of $\mathbf{KP}(\mathcal{P})$ will be denoted by $\mathbf{IKP}(\mathcal{P})$. Both $\mathbf{KP}(\mathcal{P})$ and $\mathbf{IKP}(\mathcal{P})$ can be subjected to ordinal analysis which reduces them to theories $\mathbf{Z} + \{\text{`}V_\tau \text{ exists'}\}_{\tau\in\mathrm{BH}}$ and $\mathbf{IZ} + \{\text{`}V_\tau \text{ exists'}\}_{\tau\in\mathrm{BH}}$, respectively. Here $\mathbf{Z}$ stands for classical Zermelo set theory and $\mathbf{IZ}$ for its intuitionistic version. BH refers to an ordinal representation system for the Bachmann-Howard ordinal (cf. Rathjen and Weiermann 1993). For $\tau \in \mathrm{BH}$ the statement 'V_τ *exists*' expresses that the powerset operation can be iterated τ times.

Theorem 15.1. *The following theories are of the same proof-theoretic strength.*

(i) $\mathbf{CZF}_{\mathcal{P}}$
(ii) $\mathbf{CZF}_{\mathcal{P}} + \mathbf{RDC} + \mathbf{\Pi\Sigma-AC}$
(iii) $\mathbf{KP}(\mathcal{P})$
(iv) $\mathbf{IKP}(\mathcal{P})$
(v) *Tharp's (1971) quasi-intuitionistic set theory but without* **Ord-Im**.
(vi) $\mathbf{MLV_P}$
(vii) $\mathbf{CZF} + \mathbf{Pow}^{\neg\neg}$
(viii) $\mathbf{Z} + \{\text{`}V_\tau \textit{ exists'}\}_{\tau\in\mathrm{BH}}$
(ix) $\mathbf{IZ} + \{\text{`}V_\tau \textit{ exists'}\}_{\tau\in\mathrm{BH}}$

Presenting a proof of Theorem 15.1 is the main goal of this article.

[1] Curiously, Tharp calls this scheme Replacement.

[2] The $\Delta_0^{\mathcal{P}}$-formulae of Definition 15.1.

15.2 The Theories CZF and KP($\mathcal{P}$)

15.2.1 CZF

We briefly summarize the language and axioms of **CZF**, a variant of Myhill's CST (see Myhill 1975). The language of **CZF** is based on the same first order language as that of classical Zermelo-Fraenkel Set Theory, whose only non-logical symbol is $\in$. The logic of **CZF** is intuitionistic first order logic with equality. Among its non-logical axioms are *Extensionality*, *Pairing* and *Union* in their usual forms. **CZF** has additionally axiom schemata which we will now proceed to summarize.

Infinity: $\exists x \forall u \big[u \in x \leftrightarrow \big(\emptyset = u \vee, \exists \exists v \in x\ u = v + 1\big)\big]$ where $v + 1 = v \cup \{v\}$.

Set Induction: $\forall x [\forall y \in x A(y) \rightarrow A(x)] \rightarrow \forall x A(x)$

Bounded Separation: $\forall a \exists b \forall x [x \in b \leftrightarrow x \in a \wedge A(x)]$

for all *bounded* formulae A. A set-theoretic formula is *bounded* or *restricted* if it is constructed from prime formulae using $\neg, \wedge, \vee, \exists, \rightarrow, \forall x \in y$ and $\exists x \in y$ only.

Strong Collection: For all formulae A,

$$\forall a \big[\forall x \in a \exists y A(x, y) \ \rightarrow\ \exists b\, [\forall x \in a\, \exists y \in b\, A(x, y) \wedge \forall y \in b\, \exists x \in a\, A(x, y)]\big].$$

Subset Collection: For all formulae B,

$$\forall a \forall b \exists c \forall u \big[\forall x \in a\, \exists y \in b\ B(x, y, u) \ \rightarrow$$
$$\exists d \in c\, [\forall x \in a\, \exists y \in d\ B(x, y, u) \wedge \forall y \in d\, \exists x \in a\ B(x, y, u)]\big].$$

The Powerset Axiom, **Pow**, is the following:

$$\forall x\, \exists y\, \forall z\, (z \subseteq x \rightarrow z \in y).$$

Remark 15.1. Subset Collection plays no role when we study $\mathbf{CZF}_{\mathcal{P}}$ since it is a consequence of **Pow** and the other axioms of **CZF**.

To save us work when proving realizability of the axioms of **CZF** it is useful to know that the axiom scheme of bounded separation can be deduced from a single instance (in the presence of strong collection).

Lemma 15.1. *Let* **Binary Intersection** *be the statement* $\forall x \forall y \exists z\, x \cap y = z$. *If* $\mathbf{CZF}_0$ *denotes* **CZF** *without bounded separation and subset collection, then every instance of bounded separation is provable in* $\mathbf{CZF}_0$ + Binary Intersection.

Proof. Aczel and Rathjen (2001, Proposition 4.8) is a forerunner of this result. It is proved in the above form in Aczel and Rathjen (2010, Corollary 9.5.7). □

15.2.2 Kripke–Platek Set Theory

A particularly interesting (classical) subtheory of **ZF** is Kripke–Platek set theory, **KP**. Its standard models are called *admissible sets*. One of the reasons that this is an important theory is that a great deal of set theory requires only the axioms of **KP**. An even more important reason is that admissible sets have been a major source of interaction between model theory, recursion theory and set theory (cf. Barwise 1975). **KP** arises from **ZF** by completely omitting the power set axiom and restricting separation and collection to bounded formulae. These alterations are suggested by the informal notion of 'predicative'. To be more precise, the axioms of **KP** consist of *Extensionality, Pair, Union, Infinity, Bounded Separation*

$$\exists x\, \forall u\, [u \in x \leftrightarrow (u \in a \,\wedge\, A(u))]$$

for all bounded formulae $A(u)$, *Bounded Collection*

$$\forall x \in a\, \exists y\; B(x, y) \;\rightarrow\; \exists z\, \forall x \in a\, \exists y \in z\, B(x, y)$$

for all bounded formulae $B(x, y)$, and *Set Induction*

$$\forall x\; [(\forall y \in x\, C(y)) \rightarrow C(x)] \;\rightarrow\; \forall x\, C(x)$$

for all formulae $C(x)$.

A transitive set A such that $(A, \in)$ is a model of **KP** is called an *admissible set*. Of particular interest are the models of **KP** formed by segments of Gödel's *constructible hierarchy* L. The constructible hierarchy is obtained by iterating the definable powerset operation through the ordinals

$$\begin{aligned} \mathrm{L}_0 &= \emptyset, \\ \mathrm{L}_\lambda &= \bigcup\{\mathrm{L}_\beta : \beta < \lambda\}\; \lambda \text{ limit} \\ \mathrm{L}_{\beta+1} &= \big\{X :\; X \subseteq \mathrm{L}_\beta;\; X \text{ definable over } \langle \mathrm{L}_\beta, \in\rangle\big\}. \end{aligned}$$

So any element of L of level α is definable from elements of L with levels $< \alpha$ and the parameter L_α. An ordinal α is *admissible* if the structure $(\mathrm{L}_\alpha, \in)$ is a model of **KP**.

Remark 15.2. Our system **KP** is not quite the same as the theory **KP** in Mathias' paper (Mathias 2001, p. 111). There **KP** does not have an axiom of Infinity and set induction only holds for Σ_1 formulae, or what amounts to the same, Π_1 foundation ($A \neq \emptyset \rightarrow \exists x \in A\; x \cap A = \emptyset$ for Π_1 classes A).

15.2.3 Power Kripke–Platek Set Theory

We use subset bounded quantifiers $\exists x \subseteq y \ \ldots$ and $\forall x \subseteq y \ \ldots$ as abbreviations for $\exists x(x \subseteq y \wedge \ldots)$ and $\forall x(x \subseteq y \rightarrow \ldots)$, respectively.

We call a formula of $\mathcal{L}_{\in}$ $\Delta_0^{\mathcal{P}}$ if all its quantifiers are of the form $Q\, x \subseteq y$ or $Q\, x{\in}y$ where Q is $\forall$ or $\exists$ and x and y are distinct variables.

Definition 15.1. The $\Delta_0^{\mathcal{P}}$ formulae are the smallest class of formulae containing the atomic formulae closed under $\wedge, \vee, \exists, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall x \subseteq a,\ \exists x \subseteq a.$$

Definition 15.2. $\mathbf{KP}(\mathcal{P})$ has the same language as **ZF**. Its axioms are the following: Extensionality, Pairing, Union, Infinity, Powerset, $\Delta_0^{\mathcal{P}}$-Separation and $\Delta_0^{\mathcal{P}}$-Collection.

The transitive models of $\mathbf{KP}(\mathcal{P})$ have been termed **power admissible** sets in Friedman (1973b).

Remark 15.3. Alternatively, $\mathbf{KP}(\mathcal{P})$ can be obtained from **KP** by adding a function symbol $\mathcal{P}$ for the powerset function as a primitive symbol to the language and the axiom

$$\forall y\,[y \in \mathcal{P}(x) \leftrightarrow y \subseteq x]$$

and extending the schemes of Δ_0 Separation and Collection to the Δ_0 formulae of this new language.

Lemma 15.2. $\mathbf{KP}(\mathcal{P})$ *is* **not** *the same theory as* $\mathbf{KP} + \mathbf{Pow}$. *Indeed,* $\mathbf{KP} + \mathbf{Pow}$ *is a much weaker theory than* $\mathbf{KP}(\mathcal{P})$ *in which one cannot prove the existence of* $V_{\omega+\omega}$.

Proof. Note that in the presence of full Separation and Infinity there is no difference between our system **KP** and Mathias's (2001) **KP**. It follows from Mathias (2001, Theorem 14) that $\mathbf{Z}{+}\mathbf{KP}{+}\mathbf{AC}$ is conservative over $\mathbf{Z}{+}\mathbf{AC}$ for stratifiable sentences. **Z** and $\mathbf{Z}+\mathbf{AC}$ are of the same proof-theoretic strength as the constructible hierarchy can be simulated in **Z**; a stronger statement is given in (Mathias, 2001, Theorem 16). As a result, **Z** and $\mathbf{Z} + \mathbf{KP}$ are of the same strength. As $\mathbf{KP} + \mathbf{Pow}$ is a subtheory of $\mathbf{Z} + \mathbf{KP}$, we have that $\mathbf{KP} + \mathbf{Pow}$ is not stronger than **Z**. If $\mathbf{KP} + \mathbf{Pow}$ could prove the existence of $V_{\omega+\omega}$ it would prove the consistency of **Z**. On the other hand $\mathbf{KP}(\mathcal{P})$ prove the existence of V_α for every ordinal α and hence proves the existence of arbitrarily large transitive models of Z. □

Remark 15.4. Our system $\mathbf{KP}(\mathcal{P})$ is not quite the same as the theory $\mathbf{KP}^{\mathcal{P}}$ in Mathias' paper (Mathias 2001, 6.10). The difference between $\mathbf{KP}(\mathcal{P})$ and $\mathbf{KP}^{\mathcal{P}}$ is that in the latter system set induction only holds for $\Sigma_1^{\mathcal{P}}$ formulae, or what amounts to the same, $\Pi_1^{\mathcal{P}}$ foundation ($A \neq \emptyset \rightarrow \exists x \in A\ x \cap A = \emptyset$ for $\Pi_1^{\mathcal{P}}$ classes A).

15.2.4 Extended E-Recursive Functions

We would like to have unlimited application of sets to sets, i.e. we would like to assign a meaning to the symbol $[a](x)$ where a and x are sets. In generalized recursion theory this is known as *E-recursion* or *set recursion* (see, e.g., Normann 1978 or Sacks 1990, Chap. X). However, we shall introduce an extended notion of E-computability, christened $E_{\wp}$-computability, rendering the functions $exp(a,b) = {}^{a}b$ and $\mathcal{P}(x) = \{u \mid u \subseteq x\}$ computable as well, (where ${}^{a}b$ denotes the set of all functions from a to b). Moreover, the constant function with value ω is taken as an initial function in $E_{\wp}$-computability. $E_{\wp}$-computability is closely related to power recursion, where the power set operation is regarded to be an initial function. The latter notion has been studied by Moschovakis (1976) and Moss (1995).

There is a lot of leeway in setting up $E_{\wp}$-recursion. The particular schemes we use are especially germane to our situation. Our construction will provide a specific set-theoretic model for the elementary theory of operations and numbers **EON** (see, e.g., Beeson 1985, VI.2, or the theory **APP** as described in Troelstra and van Dalen 1988, Chap. 9, Sect. 3). We utilize encoding of finite sequences of sets by the usual pairing function $\langle\,,\rangle$ with $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$, letting $\langle x\rangle = x$ and $\langle x_1, \ldots, x_n, x_{n+1}\rangle = \langle\langle x_1, \ldots, x_n\rangle, x_{n+1}\rangle$. We use functions $()_0$ and $()_1$ to retrieve the left and right components, respectively, of an ordered pair $a = \langle x, y\rangle$, i.e., $(a)_0 = x$ and $(a)_1 = y$.

Below we use the notation $[x](y)$ rather than the more traditional $\{x\}(y)$ to avoid any ambiguity with the singleton set $\{x\}$.

Definition 15.3. (**CZF**$_{\mathcal{P}}$, **KP**($\mathcal{P}$)) First, we select distinct non-zero natural numbers $\mathbf{k}$, $\mathbf{s}$, $\mathbf{p}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{s_N}$, $\mathbf{p_N}$, $\mathbf{d_N}$, $\bar{\mathbf{0}}$, $\bar{\boldsymbol{\omega}}$, $\boldsymbol{\gamma}$, $\boldsymbol{\rho}$, $\boldsymbol{\upsilon}$, $\boldsymbol{\pi}$, $\boldsymbol{i_1}$, $\boldsymbol{i_2}$, $\boldsymbol{i_3}$, and $\bar{\wp}$ which will provide indices for special $E_{\wp}$-recursive partial (class) functions. Inductively we shall define a class $\mathbb{E}$ of triples $\langle e, x, y\rangle$. Rather than "$\langle e, x, y\rangle \in \mathbb{E}$", we shall write "$[e](x) \simeq y$", and moreover, if $n > 0$, we shall use $[e](x_1, \ldots, x_n) \simeq y$ to convey that

$$[e](x_1) \simeq \langle e, x_1\rangle \wedge [\langle e, x_1\rangle](x_2) \simeq \langle e, x_1, x_2\rangle \wedge \ldots \wedge [\langle e, x_1, \ldots, x_{n-1}\rangle](x_n) \simeq y.$$

We shall say that $[e](x)$ is defined, written $[e](x)\downarrow$, if $[e](x) \simeq y$ for some y. Let $\mathbb{N} := \omega$. $\mathbb{E}$ is defined by the following clauses:

$$[\mathbf{k}](x, y) \simeq x$$
$$[\mathbf{s}](x, y, z) \simeq [[x](z)]([y](z))$$
$$[\mathbf{p}](x, y) \simeq \langle x, y\rangle$$
$$[\mathbf{p_0}](x) \simeq (x)_0$$
$$[\mathbf{p_1}](x) \simeq (x)_1$$
$$[\mathbf{s_N}](n) \simeq n + 1 \text{ if } n \in \mathbb{N}$$

$$
\begin{aligned}
[\mathbf{p_N}](0) &\simeq 0 \\
[\mathbf{p_N}](n+1) &\simeq n \text{ if } n \in \mathbb{N} \\
[\mathbf{d_N}](n,m,x,y) &\simeq x \text{ if } n,m \in \mathbb{N} \text{ and } n = m \\
[\mathbf{d_N}](n,m,x,y) &\simeq y \text{ if } n,m \in \mathbb{N} \text{ and } n \neq m \\
[\bar{\mathbf{0}}](x) &\simeq 0 \\
[\bar{\boldsymbol{\omega}}](x) &\simeq \omega \\
[\boldsymbol{\pi}](x,y) &\simeq \{x,y\} \\
[\boldsymbol{\upsilon}](x) &\simeq \bigcup x \\
[\boldsymbol{\gamma}](x,y) &\simeq x \cap (\bigcap y) \\
[\boldsymbol{\rho}](x,y) &\simeq \{[x](u) \mid u \in y\} \text{ if } [x](u) \text{ is defined for all } u \in y \\
[\boldsymbol{i}_1](x,y,z) &\simeq \{u \in x \mid y \in z\} \\
[\boldsymbol{i}_2](x,y,z) &\simeq \{u \in x \mid u \in y \to u \in z\} \\
[\boldsymbol{i}_3](x,y,z) &\simeq \{u \in x \mid u \in y \to z \in u\} \\
[\bar{\wp}](x) &\simeq \mathcal{P}(x).
\end{aligned}
$$

Note that $[\mathbf{s}](x,y,z)$ is not defined unless $[x](z)$, $[y](z)$ and $[[x](z)]([y](z))$ are already defined. The clause for **s** is thus to be read as a conjunction of the following clauses: $[\mathbf{s}](x) \simeq \langle \mathbf{s}, x\rangle$, $[\langle \mathbf{s}, x\rangle](y) \simeq \langle \mathbf{s}, x, y\rangle$ and, if there exist a, b, c such that $[x](z) \simeq a$, $[y](z) \simeq b$, $[a](b) \simeq c$, then $[\langle \mathbf{s}, x, y\rangle](z) \simeq c$. Similar restrictions apply to $\boldsymbol{\rho}$.

Lemma 15.3. ($\mathbf{CZF}_{\mathcal{P}}$, $\mathbf{IKP}(\mathcal{P})$) *$\mathbb{E}$ is an inductively defined class and $\mathbb{E}$ is functional in that for all e, x, y, y',*

$$\langle e, x, y\rangle \in \mathbb{E} \wedge \langle e, x, y'\rangle \in \mathbb{E} \Rightarrow y = y'.$$

Proof. The inductive definition of $\mathbb{E}$ falls under the heading of Aczel and Rathjen (2001, Theorem 11.4). If $[e](x) \simeq y$ the uniqueness of y follows by induction on the stages (see Aczel and Rathjen 2001, Lemma 5.2) of that inductive definition. □

Definition 15.4. *Application terms* are defined inductively as follows:

(i) The constants The constants $\mathbf{k}$, $\mathbf{s}$, $\mathbf{p}$, $\mathbf{p_0}$, $\mathbf{p_1}$, $\mathbf{s_N}$, $\mathbf{p_N}$, $\mathbf{d_N}$, $\bar{\mathbf{0}}$, $\bar{\boldsymbol{\omega}}$, $\boldsymbol{\gamma}$, $\boldsymbol{\rho}$, $\boldsymbol{\upsilon}$, $\boldsymbol{\pi}$, $\boldsymbol{i}_1$, $\boldsymbol{i}_2$ $\boldsymbol{i}_3$, and $\bar{\wp}$ singled out in Definition 15.3 are *application terms*;
(ii) Variables are *application terms*;
(iii) If s and t are *application terms* then (st) is an *application term*.

Definition 15.5. Application terms are easily formalized in $\mathbf{CZF}_{\mathcal{P}}$. However, rather than translating application terms into the set – theoretic language of $\mathbf{CZF}_{\mathcal{P}}$, we define the translation of expressions of the form $t \simeq u$, where t is an application term and u is a variable. The translation proceeds along the way that t was built up:

$$[c \simeq u]^\wedge \quad \text{is} \quad c = u \text{ if } c \text{ is a constant or a variable;}$$
$$[(st) \simeq u]^\wedge \quad \text{is} \quad \exists x \exists y([s \simeq x]^\wedge \wedge [t \simeq y]^\wedge \wedge \langle x, y, u \rangle \in \mathbb{E}).$$

Abbreviations. For application terms $s, t, t_1, \ldots, t_n$ we will use:

$s(t_1, \ldots, t_n)$ as a shortcut for $((\ldots(st_1)\ldots)t_n)$; *(parentheses associated to the left)*;
$st_1 \ldots t_n$ as a shortcut for $s(t_1, \ldots, t_n)$;
$t{\downarrow}$ as a shortcut for $\exists x(t \simeq x)^\wedge$; ($t$ *is defined*)
$(s \simeq t)^\wedge$ as a shortcut for $(s{\downarrow} \vee t{\downarrow}) \rightarrow \exists x((s \simeq x)^\wedge \wedge (t \simeq x)^\wedge)$.

A *closed* application term is an application term that does not contain variables. If t is a closed application term and $a_1, \ldots, a_n, b$ are sets we use the abbreviation

$$t(a_1, \ldots, a_n) \simeq b \quad \text{for} \ \exists x_1 \ldots x_n \exists y \big(x_1 = a_1 \wedge \ldots \wedge x_n = a_n \wedge y = b \wedge [t(x_1, \ldots, x_n) \simeq y]^\wedge\big).$$

Definition 15.6. Every closed application term gives rise to a partial class function. A partial n-place (class) function Υ is said to be an $E_\wp$*-recursive partial function* if there exists a closed application term t_Υ such that

$$\text{dom}(\Upsilon) = \{(a_1, \ldots, a_n) \mid t_\Upsilon(a_1, \ldots, a_n) \downarrow\}$$

and for all for all sets $(a_1, \ldots, a_n) \in \text{dom}(\Upsilon)$,

$$t_\Upsilon(a_1, \ldots, a_n) \simeq \Upsilon(a_1, \ldots, a_n).$$

In the latter case, t_Υ is said to be an *index* for Υ.

If Υ_1, Υ_2 are $E_\wp$-recursive partial functions, then $\Upsilon_1(\mathbf{a}) \simeq \Upsilon_2(\mathbf{a})$ iff neither $\Upsilon_1(\mathbf{a})$ nor $\Upsilon_2(\mathbf{a})$ are defined, or $\Upsilon_1(\mathbf{a})$ and $\Upsilon_2(\mathbf{a})$ are defined and equal.

The next two results can be proved in the theory **APP** and thus hold true in any applicative structure. Thence the particular applicative structure considered here satisfies the Abstraction Lemma and Recursion Theorem (see e.g. Feferman 1979 or Beeson 1985).

Lemma 15.4 (Abstraction Lemma, cf. Beeson 1985, VI.2.2). *For every application term* $t[x]$ *there exists an application term* $\lambda x.t[x]$ *with* $\text{FV}(\lambda x.t[x]) := \{x_1, \ldots, x_n\} \subseteq \text{FV}(t[x]) \backslash \{x\}$ *such that the following holds:*

$$\forall x_1 \ldots \forall x_n (\lambda x.t[x]{\downarrow} \wedge \forall y\, (\lambda x.t[x])y \simeq t[y]).$$

Proof. (i) $\lambda x.x$ is **skk**;
(ii) $\lambda x.t$ is $\mathbf{k}t$ for t a constant or a variable other than x;
(iii) $\lambda x.uv$ is $\big(\mathbf{s}(\lambda x.u)\big)(\lambda x.v)$. □

Lemma 15.5 (Recursion Theorem, cf. Beeson 1985, VI.2.7). *There exists a closed application term* rec *such that for any* f, x,

$$\mathsf{rec}\,f \downarrow \;\wedge\; \mathsf{rec}\,f\,x \simeq f(\mathsf{rec}\,f)x.$$

Proof. Take rec to be $\lambda f.tt$, where t is $\lambda y\lambda x.f(yy)x$. □

Corollary 15.1. *For any* $E_{\wp}$*-recursive partial function* Υ *there exists a closed application term* τ_{fix} *such that* $\tau_{fix}\downarrow$ *and for all* **a**,

$$\Upsilon(\bar{e}, \mathbf{a}) \simeq \tau_{fix}(\mathbf{a}),$$

where $\tau_{fix} \simeq \bar{e}$. *Moreover,* τ_{fix} *can be effectively (e.g. primitive recursively) constructed from an index for* Υ.

15.3 Defining Realizability with Sets of Witnesses for Set Theory

Realizability semantics are a crucial tool in the study of intuitionistic theories (see Troelstra 1998, Rathjen 2006b). We introduce a form of realizability based on general set recursive functions where a realizer for an existential statement provides a set of witnesses for the existential quantifier rather than a single witness. Realizability based on indices of general set recursive functions was introduced in Rathjen (2006c) and employed to prove, inter alia, metamathematical properties for **CZF** augmented by strong forms of the axiom of choice in Rathjen and Tupailo (2006, Theorems 8.3 and 8.4). There are points of contact with a notion of realizability used by Tharp (1971) who employed (indices of) Σ_1 definable partial (class) functions as realizers, though there are important differences, too, as Tharp works in a classical context and assumes a definable search operation on the universe which basically amounts to working under the hypothesis $V = L$. Moreover, there are connections with Lifschitz' realizability (Lifschitz 1979) where a realizer for an existential arithmetical statement provides a finite co-recursive set of witnesses (see van Oosten 1990; Chen and Rathjen 2012 for extensions to analysis and set theory).

We adopt the conventions and notations from the previous section. However, we prefer to write $\jmath_0 e$ and $\jmath_1 e$ rather than $(e)_0$ and $(e)_1$, respectively, and instead of $[a](b) \simeq c$ we shall write $a \bullet b \simeq c$.

Definition 15.7. Bounded quantifiers will be treated as quantifiers in their own right, i.e., bounded and unbounded quantifiers are treated as syntactically different kinds of quantifiers.

We use the expression $a \neq \emptyset$ to convey that the set a is inhabited, that is $\exists x\, x \in a$.

We define a relation $a \Vdash_{\mathfrak{w}} B$ between sets a and formulae of set theory. $a \bullet f \Vdash_{\mathfrak{w}} B$ will be an abbreviation for $\exists x[a \bullet f \simeq x \,\wedge\, x \Vdash_{\mathfrak{w}} B]$.

$$
\begin{aligned}
a \Vdash_{\mathfrak{w}} A &\text{ iff } A \text{ holds for atomic formulae } A\\
a \Vdash_{\mathfrak{w}} A \wedge B &\text{ iff } \jmath_0 a \Vdash_{\mathfrak{w}} A \;\wedge\; \jmath_1 a \Vdash_{\mathfrak{w}} B\\
a \Vdash_{\mathfrak{w}} A \vee, \exists B &\text{ iff } a \neq \emptyset \,\wedge\, (\forall d \in a)(\big[\jmath_0 d = 0 \,\wedge\, \jmath_1 d \Vdash_{\mathfrak{w}} A\big] \vee, \exists\\
&\qquad\qquad\qquad\qquad \big[\jmath_0 d = 1 \,\wedge\, \jmath_1 d \Vdash_{\mathfrak{w}} B\big])\\
a \Vdash_{\mathfrak{w}} \neg A &\text{ iff } \forall c\, \neg c \Vdash_{\mathfrak{w}} A\\
a \Vdash_{\mathfrak{w}} A \rightarrow B &\text{ iff } \forall c \big[c \Vdash_{\mathfrak{w}} A \;\rightarrow\; a \bullet c \Vdash_{\mathfrak{w}} B\big]\\
a \Vdash_{\mathfrak{w}} (\forall x \in b)\, A &\text{ iff } (\forall c \in b)\; a \bullet c \Vdash_{\mathfrak{w}} A[x/c]\\
a \Vdash_{\mathfrak{w}} (\exists x \in b)\, A &\text{ iff } a \neq \emptyset \,\wedge\, (\forall d \in a)[\jmath_0 d \in b \,\wedge\, \jmath_1 d \Vdash_{\mathfrak{w}} A[x/\jmath_0 d]]\\
a \Vdash_{\mathfrak{w}} \forall x A &\text{ iff } \forall c\;\, a \bullet c \Vdash_{\mathfrak{w}} A[x/c]\\
a \Vdash_{\mathfrak{w}} \exists x A &\text{ iff } a \neq \emptyset \,\wedge\, (\forall d \in a)\; \jmath_1 d \Vdash_{\mathfrak{w}} A[x/\jmath_0 d]\\
\Vdash_{\mathfrak{w}} B &\text{ iff } \exists a\, a \Vdash_{\mathfrak{w}} B.
\end{aligned}
$$

In the course of proving that certain formulae are realized, e.g.

$$(A \vee, \exists B) \rightarrow [(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C)]$$

or the rule for introducing an existential quantifier in the antecendent of an implication, we will be faced with the problem that we have a non-empty set of realizers where a single realizer is required. The next Lemma shows that we can effectively pass from a set of realizers to a single realizer.

Lemma 15.6. *Let* $\mathbf{x} = x_1, \ldots, x_r$ *and* $\mathbf{a} = a_1, \ldots, a_r$. *To each formula* $A(\mathbf{x})$ *of* **CZF** *(with all free variables among* $\mathbf{x}$*) we can effectively assign (a code of) an* $E_{\wp}$*-recursive partial function* χ_A *such that*

$$\mathbf{IKP}(\mathcal{P}) \vdash \forall \mathbf{a}\, \forall c \neq \emptyset\, [\, (\forall d \in c)\, d \Vdash_{\mathfrak{w}} A(\mathbf{a})) \rightarrow \chi_A(\mathbf{a}, c) \Vdash_{\mathfrak{w}} A(\mathbf{a})].$$

Proof. We use induction on the buildup of A.

If A is atomic, let $\chi_A(\mathbf{a}, c) := 0$.

Let $A(\mathbf{x})$ be $B(\mathbf{x}) \wedge C(\mathbf{x})$ and χ_B and χ_C be already defined. Then

$$\chi_A(\mathbf{a}, c) := \jmath(\chi_B(\mathbf{a}, \{\jmath_0 x \mid x \in c\}), \chi_C(\mathbf{a}, \{\jmath_1 x \mid x \in c\}))$$

will do the job.

Let $A(\mathbf{x})$ be $B(\mathbf{x}) \rightarrow C(\mathbf{x})$ and suppose χ_B and χ_C have already been defined. Assume that $c \neq \emptyset$ and $(\forall d \in c)\, d \Vdash_{\mathfrak{w}} [B(\mathbf{a}) \rightarrow C(\mathbf{a})]$. Suppose $e \Vdash_{\mathfrak{w}} B(\mathbf{a})$. Define the $E_{\wp}$-recursive partial function ϑ by

$$\vartheta(c, e) \simeq \{d \bullet e \mid d \in c\}.$$

Then $\vartheta(c, e) \neq \emptyset$ and hence, by the inductive assumption, $\chi_C(\mathbf{a}, \vartheta(c, e)) \Vdash_{\mathfrak{w}} C(\mathbf{a})$, so that

$$\lambda e.\chi_C(\mathbf{a}, \vartheta(c, e)) \Vdash_{\mathfrak{w}} A(\mathbf{a}).$$

Now let $A(\mathbf{x})$ be of the form $\forall y\, B(\mathbf{x}, y)$. Suppose that $c \neq \emptyset$ and $(\forall d \in c)\, d \Vdash_{\mathfrak{w}} A(\mathbf{a})$. Fixing b, we then have $(\forall d \in c)\, d \bullet b \Vdash_{\mathfrak{w}} B(\mathbf{a}, b)$, thus, $\forall d' \in \vartheta(c, b)\ d' \Vdash_{\mathfrak{w}} B(\mathbf{a}, b)$, and therefore, by the inductive assumption, $\chi_B(\mathbf{a}, \vartheta(c, b)) \Vdash_{\mathfrak{w}} B(\mathbf{a}, b)$. As a result

$$\lambda b.\chi_B(\mathbf{a}, \vartheta(c, b)) \Vdash_{\mathfrak{w}} A(\mathbf{a}).$$

The case of $A(\mathbf{x})$ starting with a bounded universal quantifier is similar to the previous case.

In all the remaining cases, $\chi_A(\mathbf{a}, c) := \bigcup c$ will work owing to the definition of realizability in these cases. □

Lemma 15.7 (IKP($\mathcal{P}$)). *Realizers for equality laws:*

(i) $0 \Vdash_{\mathfrak{w}} x = x$.
(ii) $\lambda u.u \Vdash_{\mathfrak{w}} x = y \rightarrow y = x$.
(iii) $\lambda u.u \Vdash_{\mathfrak{w}} (x = y \wedge y = z) \rightarrow x = z$.
(iv) $\lambda u.u \Vdash_{\mathfrak{w}} (x = y \wedge y \in z) \rightarrow x \in z$.
(v) $\lambda u.u \Vdash_{\mathfrak{w}} (x = y \wedge z \in x) \rightarrow z \in y$.
(vi) $\lambda u.\jmath_1 u \Vdash_{\mathfrak{w}} (x = y \wedge A(x)) \rightarrow A(y)$ *for any formula* A.

Proof. (i)–(v) are obvious. (vi) follows by a trivial induction on the buildup of A. □

Lemma 15.8 (IKP($\mathcal{P}$)). *Realizers for logical axioms: Below we use the $E_{\wp}$-recursive function* $\mathfrak{sg}(a) := \{a\}$.

(IPL1) $\mathbf{k} \Vdash_{\mathfrak{w}} A \rightarrow (B \rightarrow A)$.
(IPL2) $\mathbf{s} \Vdash_{\mathfrak{w}} [A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$.
(IPL3) $\lambda e.\lambda d.\jmath(e, d))\rangle \Vdash_{\mathfrak{w}} A \rightarrow (B \rightarrow A \wedge B)$.
(IPL4) $\lambda e.\jmath_0 \Vdash_{\mathfrak{w}} A \wedge B \rightarrow A$.
(IPL5) $\lambda e.\jmath_1 e \Vdash_{\mathfrak{w}} A \wedge B \rightarrow B$.
(IPL6) $\lambda e.\mathfrak{sg}(\jmath(0, e)) \Vdash_{\mathfrak{w}} A \rightarrow A \vee, \exists B$.
(IPL7) $\lambda e.\mathfrak{sg}(\jmath(1, e)))\rangle \Vdash_{\mathfrak{w}} B \rightarrow A \vee, \exists B$.
(IPL8) $\mathfrak{k}(\mathbf{a}) \Vdash_{\mathfrak{w}} (A \vee, \exists B) \rightarrow [(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C)]$, *for some $E_{\wp}$-recursive partial function $\mathfrak{k}$, where* $\mathbf{a}$ *comprises all parameters appearing in the formula.*

(IPL9) $\lambda e.\lambda d.0 \Vdash_{\mathfrak{w}} (A \to B) \to ((A \to \neg B) \to \neg A)$.
(IPL10) $\lambda e.0 \Vdash_{\mathfrak{w}} A \to (\neg A \to B)$.
(IPL11) $\lambda e.e \bullet b \Vdash_{\mathfrak{w}} \forall x A(x) \to A(b)$.
(IPL12) $\lambda e.\mathfrak{sg}(e) \Vdash_{\mathfrak{w}} A(a) \to \exists x A(x)$.

Proof. As for IPL1 and IPL2, this justifies the combinators **s** and **k**. Combinatory completeness of these two combinators is equivalent to the fact that these two laws together with modus ponens generate the full set of theorems of propositional implicational intuitionistic logic.

Except for IPL8, one easily checks that the proposed realizers indeed realize the pertaining formulae.

So let's check IPL8. $A\vee,\exists B \to ((A \to C) \to ((B \to C) \to C))$. Suppose $e \Vdash_{\mathfrak{w}} A\vee,\exists B$. Then $e \neq \emptyset$. Let $d \in e$. Then $\jmath_0 d = 0 \wedge \jmath_1 d \Vdash_{\mathfrak{w}} A$ or $\jmath_0 d = 1 \wedge \jmath_1 d \Vdash_{\mathfrak{w}} B$. Suppose $f \Vdash_{\mathfrak{w}} A \to C$ and $g \Vdash_{\mathfrak{w}} B \to C$. Define an $E_{\wp}$-recursive partial function $\mathfrak{f}$ by

$$\mathfrak{f}(d', f', g') = [\mathbf{d_N}](\jmath_0 d', 0, f' \bullet (\jmath_1 d'), g' \bullet (\jmath_1 d')).$$

Then

$$\mathfrak{f}(d', f', g') = \begin{cases} f' \bullet (\jmath_1 d') & \text{if } \jmath_0 d' = 0 \\ g' \bullet (\jmath_1 d') & \text{if } \jmath_0 d' = 1 \end{cases}$$

As a result, $\mathfrak{f}(d, f, g) \Vdash_{\mathfrak{w}} C$ and hence $\lambda f.\lambda g.\mathfrak{f}(d, f, g) \Vdash_{\mathfrak{w}} (A \to C) \to ((B \to C) \to C)$. Thus, $\Phi(e, \lambda d.\lambda f.\lambda g.\mathfrak{f}(d, f, g)) \neq \emptyset$ and for all $p \in \Phi(e, \lambda d.\lambda f.\lambda g.\mathfrak{f}(d, f, g))$ we have

$$p \Vdash_{\mathfrak{w}} (A \to C) \to ((B \to C) \to C).$$

Let $E(\mathbf{a}) := (A \to C) \to ((B \to C) \to C)$, where $\mathbf{a}$ comprises all parameters appearing in the formula on the right hand side. The upshot is that by Lemma 15.6 we can conclude

$$\chi_E(\mathbf{a}, \Phi(e, \lambda d.\lambda f.\lambda g.\mathfrak{f}(d, f, g))) \Vdash_{\mathfrak{w}} E(\mathbf{a}).$$

And consequently we have

$$\mathfrak{k}(\mathbf{a}) := \lambda e.\chi_E(\mathbf{a}, \Phi(e, \lambda d.\lambda f.\lambda g.\mathfrak{f}(d, f, g))) \Vdash_{\mathfrak{w}} A\vee,\exists B \to E(\mathbf{a}). \qquad \square$$

Theorem 15.2. *Let $D(u_1, \ldots, u_r)$ be a formula of $\mathcal{L}_\in$ all of whose free variables are among $u_1, \ldots, u_r$. If*

$$\mathbf{CZF} + \mathbf{Pow} \vdash D(u_1, \ldots, u_r),$$

then one can effectively construct an index of an $E_{\wp}$-recursive function g such that

$$\mathbf{IKP}(\mathcal{P}) \vdash \forall a_1, \ldots, a_r\, g(a_1, \ldots, a_r) \Vdash_{\mathfrak{w}} D(a_1, \ldots, a_r).$$

Proof. We use a standard Hilbert-type systems for intuitionistic predicate logic. The proof proceeds by induction on the derivation. For the logical axioms and the equality axioms we have already produced appropriate $E_{\wp}$-recursive functions in Lemmata 15.7 and 15.8. It remains to deal with logical inferences and set-theoretic axioms. We start with the rules.

The only rule from propositional logic is modus ponens. Suppose that we have $E_{\wp}$-recursive functions g_0 and g_1 such that for all $\mathbf{a}$, $g_0(\mathbf{a}) \Vdash_{\mathfrak{w}} A(\mathbf{a}) \to B(\mathbf{a})$ and $g_1(\mathbf{a}) \Vdash_{\mathfrak{w}} A(\mathbf{a})$. Then $g(\mathbf{a}) \Vdash_{\mathfrak{w}} B(\mathbf{a})$ holds with the $E_{\wp}$-recursive function $g(\mathbf{a}) := g_0(\mathbf{a}) \bullet g_1(\mathbf{a})$.

For the $\forall$ quantifier we have the rule: from $B(\mathbf{u}) \to A(x, \mathbf{u})$ infer $B(\mathbf{u}) \to \forall x A(x, \mathbf{u})$ if x is not free in $B(\mathbf{u})$. Inductively we have an $E_{\wp}$-recursive function $\mathfrak{h}$ such that for all $b, \mathbf{a}$,

$$\mathfrak{h}(b, \mathbf{a}) \Vdash_{\mathfrak{w}} B(\mathbf{a}) \to A(b, \mathbf{a}).$$

Suppose $d \Vdash_{\mathfrak{w}} B(\mathbf{a})$. Then $\mathfrak{h}(b, \mathbf{a}) \bullet d \Vdash_{\mathfrak{w}} A(b, \mathbf{a})$ holds for all b, whence $\lambda x.(\mathfrak{h}(x, \mathbf{a}) \bullet d) \Vdash_{\mathfrak{w}} \forall x A(x, \mathbf{a})$. As a result,

$$\lambda d.\lambda x.(\mathfrak{h}(x, \mathbf{a}) \bullet d) \Vdash_{\mathfrak{w}} B(\mathbf{a}) \to \forall x A(x, \mathbf{a}).$$

For the $\exists$ quantifier we have the rule: from $A(x, \mathbf{u}) \to B(\mathbf{u})$ infer $\exists x A(x, \mathbf{u}) \to B(\mathbf{u})$ if x is not free in $B(\mathbf{u})$. Inductively we then have an $E_{\wp}$-recursive function $\mathfrak{g}$ such that for all $b, \mathbf{a}$,

$$\mathfrak{g}(b, \mathbf{a}) \Vdash_{\mathfrak{w}} A(b, \mathbf{a}) \to B(\mathbf{a}).$$

Suppose $e \Vdash_{\mathfrak{w}} \exists x A(x, \mathbf{a})$. Then $e \neq \emptyset$ and for all $d \in e$, $\jmath_1 d \Vdash_{\mathfrak{w}} A(\jmath_0 d, \mathbf{a})$. Consequently, $(\forall d \in e)\, \mathfrak{g}(\jmath_0 d, \mathbf{a}) \bullet \jmath_1 d \Vdash_{\mathfrak{w}} B(\mathbf{a})$. We then have $\Phi(e, \lambda d.\mathfrak{g}(\jmath_0 d, \mathbf{a}) \bullet \jmath_1 d) \neq \emptyset$ and

$$(\forall y \in \Phi(e, \lambda d.\mathfrak{g}(\jmath_0 d, \mathbf{a}) \bullet \jmath_1 d)\; y \Vdash_{\mathfrak{w}} B(\mathbf{a})\,.$$

Using Lemma 15.6 we arrive at $\chi_B(\mathbf{a}, \Phi(e, \lambda d.\mathfrak{g}(\jmath_0 d, \mathbf{a}) \bullet \jmath_1 d)) \Vdash_{\mathfrak{w}} B(\mathbf{a})$; whence

$$\lambda e.\chi_B(\mathbf{a}, \Phi(e, \lambda d.\mathfrak{g}(\jmath_0 d, \mathbf{a}) \bullet \jmath_1 d)) \Vdash_{\mathfrak{w}} \exists x A(x, \mathbf{a}) \to B(\mathbf{a})\,.$$

Next we show that every axiom of **CZF** + **Pow** is realized by an $E_{\wp}$-recursive function. We treat the axioms one after the other.

(Extensionality): Since $e \Vdash_{\mathfrak{w}} \forall x(x \in a \;\leftrightarrow\; x \in b)$ implies $a = b$, and hence $0 \Vdash_{\mathfrak{w}} a = b$, it follows that

$$\lambda u.0 \Vdash_{\mathfrak{w}} [\forall x(x \in a \;\leftrightarrow\; x \in b) \to a = b\,].$$

(Pair): There is an $E_\wp$-recursive function ℓ such that

$$\ell(a,b,c) := \{\jmath(0,a) \mid c = a\} \cup \{\jmath(1,b) \mid c = b\}.$$

We have $\forall u \in \{a,b\}\ \ell(a,b,u) \Vdash_{\mathfrak{w}} (u = a \vee, \exists u = b)$ and hence, letting $c := \{a,b\}$,

$$\lambda u.\ell(a,b,u) \Vdash_{\mathfrak{w}} \forall x \in c(x = u \vee, \exists x = b)\,.$$

We also have $\jmath(0,0) \Vdash_{\mathfrak{w}} (a \in c \wedge b \in c)$, so that

$$\jmath(\lambda u.\ell(a,b,u), \jmath(0,0)) \Vdash_{\mathfrak{w}} \forall x \in c(x = a \vee, \exists x = b) \wedge (a \in c \wedge b \in c).$$

Thus we arrive at

$$\mathfrak{sg}(\jmath(\mathfrak{p}(a,b), \jmath(\lambda u.\ell(a,b,u), \jmath(0,0))) \Vdash_{\mathfrak{w}}$$
$$\exists y\, [\forall x \in y(x = a \vee, \exists x = b) \wedge (a \in y \wedge b \in y)].$$

(Union): Let ℓ_U be the $E_\wp$-recursive function defined by

$$\ell_U(a,u) = \{\jmath(x, \jmath(0,0)) \mid x \in a \wedge u \in x\}.$$

For $u \in \bigcup a$ we then have $\ell_U(a,u) \Vdash_{\mathfrak{w}} \exists x \in a\ u \in x$, and therefore

$$\lambda u.\ell_U(u,a) \Vdash_{\mathfrak{w}} (\forall u \in \bigcup a)(\exists x \in a)\ u \in x.$$

Obviously $\lambda u.\lambda v.0 \Vdash_{\mathfrak{w}} (\forall x \in a)(\forall y \in x)\ y \in \bigcup a$. Therefore we have

$$\mathfrak{sg}(\jmath(\bigcup a, \jmath(\lambda u.\ell_U(u,a), \lambda u.\lambda v.0))) \Vdash_{\mathfrak{w}}$$
$$\exists w\, [(\forall u \in w)(\exists x \in a)\ u \in x \wedge (\forall x \in a)(\forall y \in x)\ y \in w].$$

(Empty Set): Obviously $\mathfrak{sg}(\jmath(\emptyset, \lambda v.0)) \Vdash_{\mathfrak{w}} \exists x\, (\forall u \in x) u \neq u.$

(Binary Intersection): Let $c := a \cap b$. As

$$\lambda v.\jmath(0,0) \Vdash_{\mathfrak{w}} \forall x \in c\, (x \in a \wedge x \in b)$$

and $\lambda u.0 \Vdash_{\mathfrak{w}} \forall x\, (x \in a \wedge x \in b \to x \in c)$ hold, we conclude that

$$\mathfrak{sg}(\jmath(a \cap b, \jmath(\lambda v.\jmath(0,0), \lambda u.0))) \Vdash_{\mathfrak{w}} \exists y\, [\forall x \in y\, (x \in a \wedge x \in b) \wedge$$
$$\forall x\, (x \in a \wedge x \in b \to x \in y)].$$

(Powerset): It suffices to find a realizer for the formula

$$\exists y\, \forall x (x \subseteq a \to x \in y)]$$

since realizability of the power set axiom follows then with the help of Δ_0 Separation. One easily verifies that $e \Vdash_{\mathfrak{w}} \forall u(u \in b \to u \in a)$ implies $b \subseteq a$ and consequently $b \in \mathcal{P}(a)$. Therefore we have

$$\lambda u.\lambda v.0 \Vdash_{\mathfrak{w}} \forall x[x \subseteq a \to x \in \mathcal{P}(a)],$$

thus $\mathfrak{sg}(\jmath(\mathcal{P}(a), \lambda u.\lambda v.0)) \Vdash_{\mathfrak{w}} \exists y\, \forall x[x \subseteq a \to x \in y]$.

(Set Induction): Suppose $e \Vdash_{\mathfrak{w}} \forall x[\forall y(y \in x \to A(y)) \to A(x)]$. Then, for all a,

$$e \bullet a \Vdash_{\mathfrak{w}} [\forall y(y \in a \to A(y)) \to A(a)]\,.$$

Suppose we have an index e^* such that for all $b \in a$, $e^* \bullet b \Vdash_{\mathfrak{w}} A(b)$. As $v \Vdash_{\mathfrak{w}} b \in a$ entails $b \in a$, we get

$$\lambda u.\lambda v.e^* \bullet u \Vdash_{\mathfrak{w}} \forall y(y \in a \to A(y)),$$

and hence

$$(e \bullet a) \bullet (\lambda u.\lambda v.e^* \bullet u) \Vdash_{\mathfrak{w}} A(a). \tag{15.1}$$

By the recursion theorem we can effectively cook up an index $\mathfrak{q}$ such that

$$(\mathfrak{q} \bullet e) \bullet a \simeq (e \bullet a) \bullet (\lambda u.\lambda v.(\mathfrak{q} \bullet e) \bullet u).$$

In view of the above it follows by set induction that for all a, $(\mathfrak{q} \bullet e) \bullet a \downarrow$ and $(\mathfrak{q} \bullet e) \bullet a \Vdash_{\mathfrak{w}} A(a)$. As a result we have $\lambda w.(\mathfrak{q} \bullet e) \bullet w \Vdash_{\mathfrak{w}} \forall x\, A(x)$, yielding

$$\lambda e \lambda w.(\mathfrak{q} \bullet e) \bullet w \Vdash_{\mathfrak{w}} \forall x[\forall y(y \in x \to A(y)) \to A(x)] \to \forall x\, A(x).$$

(Strong Collection): Suppose

$$e \Vdash_{\mathfrak{w}} \forall u(u \in a \to \exists y\, B(u, y)). \tag{15.2}$$

Then we have, for all $b \in a$, $(e \bullet b) \bullet 0 \Vdash_{\mathfrak{w}} \exists y\, B(b, y)$, and so $(e \bullet b) \bullet 0 \neq \emptyset$ and

$$(\forall d \in (e \bullet b) \bullet 0)\ \ \jmath_1 d \Vdash_{\mathfrak{w}} B(b, \jmath_0 d). \tag{15.3}$$

Let

$$C^* := \{\jmath_0 d \mid (\exists x \in a)[d \in (e \bullet x) \bullet 0]\}.$$

C^* is a set in our background theory, using Replacement or Strong Collection.

Now assume $e' \Vdash_{\mathfrak{w}} b \in a$. Then $b \in a$ and hence, by the above, $(e \bullet b) \bullet 0 \neq \emptyset$ and

$$(\forall d \in (e \bullet b) \bullet 0) \;\; \jmath(0, \jmath_1 d) \Vdash_{\mathfrak{w}} [\jmath_0 d \in C^* \wedge B(b, \jmath_0 d)]. \tag{15.4}$$

There is an $E_{\wp}$-recursive function ℓ_2 defined by

$$\ell_2(e, b) \simeq \{\jmath(\jmath_0 d, \jmath(0, \jmath_1 d)) \mid d \in (e \bullet b) \bullet 0\}.$$

From (15.4) we can infer that $\ell_2(e, b) \Vdash_{\mathfrak{w}} \exists y\, [y \in C^* \wedge B(b, y)]$ and hence

$$\lambda u.\lambda v.\ell_2(e, u) \Vdash_{\mathfrak{w}} \forall x(x \in a \rightarrow \exists y\, [y \in C^* \wedge B(x, y)]). \tag{15.5}$$

Now assume $c \in C^*$. Then there exists $b \in a$ and $d \in (e \bullet b) \bullet 0$ such that $c = \jmath_0 d$. Moreover, by (15.3), whenever $b \in a$, $d \in (e \bullet b) \bullet 0$ and $\jmath_0 d = c$, then $\jmath_1 d \Vdash_{\mathfrak{w}} B(b, c)$. Letting ℓ_3 be the $E_{\wp}$-recursive function defined by

$$\ell_3(a, c, e) \simeq \{\jmath(b, \jmath(0, \jmath_1 d)) \mid b \in a \wedge \exists d \in (e \bullet b) \bullet 0 \;\; \jmath_0 d = c\},$$

we then have

$$\ell_3(a, c, e) \Vdash_{\mathfrak{w}} \exists x(x \in a \wedge B(x, c)), \tag{15.6}$$

thus

$$\lambda u.\lambda v.\ell_3(a, u, e) \Vdash_{\mathfrak{w}} \forall y[y \in C^* \rightarrow \exists x(x \in a \wedge B(x, y))]. \tag{15.7}$$

Finally observe that there is an $E_{\wp}$-recursive function $\mathfrak{l}$ such that

$$\mathfrak{l}(a, e) := \{\jmath_0 d \mid d \in \bigcup_{x \in a}((e \bullet x) \bullet 0)\} = \{\jmath_0 d \mid (\exists x \in a)[d \in (e \bullet x) \bullet 0]\} = C^*.$$

Thus in view of (15.5) and (15.7) we arrive at

$$\begin{aligned} &\mathfrak{sg}(\jmath(\mathfrak{l}(a, e), \jmath(\lambda u.\lambda v.\ell_2(e, u), \lambda u.\lambda v.\ell_3(a, u, e)))) \\ &\qquad \Vdash_{\mathfrak{w}} \exists z\, [\forall x(x \in a \rightarrow \exists y\, [y \in z \wedge B(x, y)]) \\ &\qquad \wedge \forall y[y \in z \rightarrow \exists x(x \in a \wedge B(x, y))]]. \end{aligned}$$

As a result, $\lambda w.\lambda q.\mathfrak{sg}(\jmath(\mathfrak{l}(w, q), \jmath(\lambda u.\lambda v.\ell_2(q, u), \lambda u.\lambda v.\ell_3(w, u, q))))$ is a realizer for each instance of Strong Collection.

(Infinity): By Aczel and Rathjen (2010, Lemma 9.2.2) it suffices to find a realizer for the formula

$$\exists z\, \forall x(x \in z \leftrightarrow [x = \emptyset \vee, \exists\, \exists y \in z\, x = y \cup \{y\}]).$$

Here $x = \emptyset$ is an abbreviation for $\forall y(y \in x \to y \neq y)$ and $(\exists y \in z)\, x = y \cup \{y\}$ is an abbreviation for

$$\exists y(y \in z \wedge [\forall w(w \in x \to [w \in y \vee, \exists w = y]) \wedge [\forall w(w \in y \to w \in x) \wedge y \in x]]).$$

We have

$$\lambda u'.\lambda v'.0 \Vdash_{\mathfrak{w}} \forall y(y \in \emptyset \to y \neq y). \tag{15.8}$$

For $n+1 \in \omega$ we have

$$\ell_4(n+1) \Vdash_{\mathfrak{w}} \forall w(w \in n+1 \to (w \in n \vee, \exists w = n)) \tag{15.9}$$

for the $E_{\wp}$-recursive function

$$\ell_4(u) := \lambda w.\lambda v'.\{\jmath(0,0) \mid w \in [\mathbf{p_N}](u)\} \cup \{\jmath(1,0) \mid w = [\mathbf{p_N}](u)\}.$$

We also have $\jmath(\lambda w'.\lambda v'.0, 0) \Vdash_{\mathfrak{w}} \forall w(w \in n \to w \in n+1) \wedge n \in n+1$. Thus

$$\begin{aligned} \ell_5(n+1) \Vdash_{\mathfrak{w}} n \in \omega \wedge [\forall w(w \in n+1 \to (w \in n \vee, \exists w = n)) \\ \wedge [\forall w(w \in n \to w \in n+1) \wedge n \in n+1]]. \end{aligned} \tag{15.10}$$

with $\ell_5(n+1) := \jmath(0, \jmath(\ell_4(n+1), \jmath(\lambda w'.\lambda v'.0, 0)))$. From (15.10) we conclude that

$$\ell_6(n+1) \Vdash_{\mathfrak{w}} (\exists y \in \omega)(n+1 = y \cup \{y\}), \tag{15.11}$$

where $\ell_6(m) := \mathfrak{sg}(\jmath([\mathbf{p_N}](m), \ell_5(m)))$. Now from (15.8) and (15.11) we conclude that for every $m \in \omega$:

$$\mathfrak{sg}([\mathbf{d_N}](0, m, \jmath(0, \lambda u'.\lambda v'.0), \jmath(1, \ell_6(m)))) \Vdash_{\mathfrak{w}} m = \emptyset \vee, \exists \exists y \in \omega\, m = y \cup \{y\}.$$

If $e \Vdash_{\mathfrak{w}} a \in \omega$ then $a \in \omega$, and hence with $\ell_7(\omega) := \lambda u.\mathfrak{sg}([\mathbf{d_N}]$ $(0, u, \jmath(0, \lambda u'.\lambda v'.0), \jmath(1, \ell_6(u))))$,

$$\ell_7(\omega) \Vdash_{\mathfrak{w}} (\forall x \in \omega)[x = \emptyset \vee, \exists \exists y \in \omega\, x = y \cup \{y\}]. \tag{15.12}$$

Conversely, if $e \Vdash_{\mathfrak{w}} \forall y(y \in a \to y \neq y)$, then really $\forall y \in a\, y \neq y$, and hence $a = \emptyset$, so that $a \in \omega$. Also, if $e' \Vdash_{\mathfrak{w}} \exists y \in \omega\; a = y \cup \{y\}$ then by unraveling this definition it turns out that $a \in \omega$ holds. As a result, if $d \Vdash_{\mathfrak{w}} [a = \emptyset \vee, \exists \exists y \in \omega\, a = y \cup \{y\}]$ then there exists $f \in d$ such that $\jmath_0 f = 0$ and $\jmath_1 f \Vdash_{\mathfrak{w}} a = \emptyset$ or $\jmath_0 f = 1$ and $\jmath_1 f \Vdash_{\mathfrak{w}} \exists y \in \omega\, a = y \cup \{y\}$. In either case we have $a \in \omega$, and so

$$\lambda x.\lambda e.0 \Vdash_{\mathfrak{w}} \forall x([x = \emptyset \vee, \exists \exists y \in \omega\, x = y \cup \{y\}] \to x \in \omega). \tag{15.13}$$

Combining (15.12) and (15.13), we have

$$\mathfrak{sg}(\jmath(\omega, \lambda v.\jmath(\lambda d.(\ell_7(\omega) \bullet v), \lambda e.0))) \Vdash_{\mathfrak{w}} \exists z \forall x(x \in z \leftrightarrow [x = \emptyset \vee \exists y \in z\, x = y \cup \{y\}]). \tag{15.14}$$

□

We would like to show that **KP**($\mathcal{P}$) also realizes every theorem of Tharp's quasi-intuitionistic set theory without **Ord-Im**. This requires a special Lemma about realizability of bounded formulae.

Definition 15.8. To each $\Delta_0^{\mathcal{P}}$ formula $D(x_1, \ldots, x_r)$ of $\mathcal{L}_\in$ all of whose free variables are among $\mathbf{x} = x_1, \ldots, x_r$, we assign a total $E_\wp$-recursive function $\mathfrak{k}_D$ of arity r as follows:

1. $\mathfrak{k}_D(\mathbf{x}) = \{0\}$ if $D(\mathbf{x})$ is atomic.
2. $\mathfrak{k}_D(\mathbf{x}) = \{\{\langle 0, z\rangle\} \mid z \in \mathfrak{k}_A(\mathbf{x}) \wedge A(\mathbf{x})\} \cup \{\{\langle 1, z\rangle\} \mid z \in \mathfrak{k}_B(\mathbf{x}) \wedge B(\mathbf{x})\}$ if $D(\mathbf{x})$ is of the form $A(\mathbf{x}) \vee, \exists\, B(\mathbf{x})$.
3. $\mathfrak{k}_D(\mathbf{x}) = \{\langle z, w\rangle \mid z \in \mathfrak{k}_A(\mathbf{x}) \wedge w \in \mathfrak{k}_B(\mathbf{x})\}$ if $D(\mathbf{x})$ is of the form $A(\mathbf{x}) \wedge B(\mathbf{x})$.
4. $\mathfrak{k}_D(\mathbf{x}) = \{\lambda v.\chi_B(\mathbf{x}, \mathfrak{k}_B(\mathbf{x}))\}$ if $D(\mathbf{x})$ is of the form $A(\mathbf{x}) \to B(\mathbf{x})$.
5. $\mathfrak{k}_D(\mathbf{x}) = \{\{\langle z, v\rangle\} \mid z \in x_i \wedge v \in \mathfrak{k}_A(\mathbf{x}, z) \wedge A(\mathbf{x}, z)\}$ if $D(\mathbf{x})$ is of the form $\exists z \in x_i\; A(\mathbf{x}, z)$.
6. $\mathfrak{k}_D(\mathbf{x}) = \{\lambda z.\chi_A(\mathbf{x}, z, \mathfrak{k}_A(\mathbf{x}, z))\}$ if $D(\mathbf{z})$ is of the form $\forall z \in x_i\; A(\mathbf{x}, z)$.
7. $\mathfrak{k}_D(\mathbf{x}) = \{\{\langle z, \langle \lambda y.0, v\rangle\rangle\} \mid z \in \mathcal{P}(x_i) \wedge v \in \mathfrak{k}_A(\mathbf{x}, z) \wedge A(\mathbf{x}, z)\}$ if $D(\mathbf{x})$ is of the form $\exists z \subseteq x_i\; A(\mathbf{x}, z)$.
8. $\mathfrak{k}_D(\mathbf{x}) = \{\lambda y.\lambda z.\chi_A(\mathbf{x}, z, \mathfrak{k}_A(\mathbf{x}, z))\}$ if $D(\mathbf{z})$ is of the form $\forall z \subseteq x_i\; A(\mathbf{x}, z)$.

In the above, we tacitly used the fact that for every $\Delta_0^{\mathcal{P}}$ formula $A(\mathbf{x}, u)$ there is an $E_\wp$-recursive function $\mathfrak{f}_A$ such that $\mathfrak{f}_A(\mathbf{x}, a) = \{u \in a \mid A(\mathbf{x}, u)\}$. This is proved in Rathjen (2012, Lemma 2.20).

For Δ_0-formulae realizability and truth coincide as the following Proposition shows.

Proposition 15.1. *Let $D(\mathbf{x})$ be a $\Delta_0^{\mathcal{P}}$ formula whose free variables are among $\mathbf{x} = x_1, \ldots, x_r$. Then the following are provable in* **IKP**($\mathcal{P}$)*:*

(i) $D(\mathbf{x}) \to \mathfrak{k}_D(\mathbf{x}) \neq \emptyset \wedge \forall u \in \mathfrak{k}_D(\mathbf{x})\; u \Vdash_{\mathfrak{w}} D(\mathbf{x})$.
(ii) $(\exists e\; e \Vdash_{\mathfrak{w}} D(\mathbf{x})) \to D(\mathbf{x})$.

Proof. We show (i) and (ii) simultaneously by induction on the complexity of D.

1. For atomic D this is obvious.
2. Let $D(\mathbf{x})$ be of the form $A(\mathbf{x}) \vee, \exists\, B(\mathbf{x})$. First suppose that $D(\mathbf{x})$ holds. Then the induction hypothesis entails that $A(\mathbf{x})$ and $\mathfrak{k}_A(\mathbf{x}) \neq \emptyset$ or $B(\mathbf{x})$ and $\mathfrak{k}_B(\mathbf{x}) \neq \emptyset$. In every case we have $\mathfrak{k}_D(\mathbf{x}) \neq \emptyset$.

   If $u \in \mathfrak{k}_D(\mathbf{x})$ then either $u = \{\langle 0, z\rangle\}$ and $A(\mathbf{x})$ for some $z \in \mathfrak{k}_A(\mathbf{x})$ or $u = \{\langle 1, z\rangle\}$ and $B(\mathbf{x})$ for some $z \in \mathfrak{k}_B(\mathbf{x})$. In the first case the inductive assumption yields $z \Vdash_{\mathfrak{w}} A(\mathbf{x})$ and hence $u \Vdash_{\mathfrak{w}} D(\mathbf{x})$. In the second case the inductive assumption yields $z \Vdash_{\mathfrak{w}} B(\mathbf{x})$ and hence also $u \Vdash_{\mathfrak{w}} D(\mathbf{x})$. This shows (i).

   As to (ii), suppose that $e \Vdash_{\mathfrak{w}} D(\mathbf{x})$. Then there exists $u \in e$ such that $u = \langle 0, d\rangle \wedge d \Vdash_{\mathfrak{w}} A(\mathbf{x})$ or $u = \langle 1, d\rangle \wedge d \Vdash_{\mathfrak{w}} B(\mathbf{x})$ for some d. The induction hypothesis yields $A(\mathbf{x})$ or $B(\mathbf{x})$, thus $D(\mathbf{x})$.
3. Let $D(\mathbf{x})$ be of the form $A(\mathbf{x}) \wedge B(\mathbf{x})$. Then (i) and (ii) are immediate by the induction hypothesis.
4. Let $D(\mathbf{x})$ be of the form $A(\mathbf{x}) \to B(\mathbf{x})$. By definition, $\mathfrak{k}_D(\mathbf{x}) = \{\lambda v.\chi_B(\mathbf{x}, \mathfrak{k}_B(\mathbf{x}))\} \neq \emptyset$. As to (i), assume that $D(\mathbf{x})$ holds and $e \Vdash_{\mathfrak{w}} A(\mathbf{x})$. Then the induction hypothesis (ii) applied to $A(\mathbf{x})$ yields that $A(\mathbf{x})$ holds, which implies that $B(\mathbf{x})$ holds. The induction hypothesis (i) for the latter formula yields that $\mathfrak{k}_B(\mathbf{x}) \neq \emptyset$ and $\forall u \in \mathfrak{k}_B(\mathbf{x})\ u \Vdash_{\mathfrak{w}} B(\mathbf{x})$. An application of Lemma 15.6 thus yields $\chi_B(\mathbf{x}, \mathfrak{k}_B(\mathbf{x})) \Vdash_{\mathfrak{w}} B(\mathbf{x})$. As a result, $\lambda v.\chi_B(\mathbf{x}, \mathfrak{k}_B(\mathbf{x})) \Vdash_{\mathfrak{w}} D(\mathbf{x})$ confirming (i).

   For (ii), suppose $e \Vdash_{\mathfrak{w}} (A(\mathbf{x}) \to B(\mathbf{x}))$ and $A(\mathbf{x})$ holds. By the induction hypothesis (i) for the latter formula, $\mathfrak{k}_A(\mathbf{x}) \neq \emptyset$ and $\forall u \in \mathfrak{k}_A(\mathbf{x})\ u \Vdash_{\mathfrak{w}} A(\mathbf{x})$. Thus, picking $u_0 \in \mathfrak{k}_A(\mathbf{x})$ we have $e \bullet u_0 \Vdash_{\mathfrak{w}} B(\mathbf{x})$, and hence the induction hypothesis (ii) for the latter formula yields that $B(\mathbf{x})$ holds.
5. Let $D(\mathbf{x})$ be of the form $\exists z \in x_i\ A(\mathbf{x}, z)$. To verify (i), suppose $\exists z \in x_i\ A(\mathbf{x}, z)$ holds. Then there is $z \in x_i$ such that $A(\mathbf{x}, z)$. The induction hypothesis (i) for the latter formula yields that $\mathfrak{k}_A(\mathbf{x}, z) \neq \emptyset$, and hence $\mathfrak{k}_D(\mathbf{x}) \neq \emptyset$. Now suppose $u \in \mathfrak{k}_D(\mathbf{x})$. Then $u = \{\langle z, v\rangle\}$ for some $z \in x_i$ and $v \in \mathfrak{k}_A(\mathbf{x}, z)$. As $A(\mathbf{x}, z)$ holds the induction hypothesis (i) yields that $v \Vdash_{\mathfrak{w}} A(\mathbf{x}, z)$, whence $u \Vdash_{\mathfrak{w}} \exists z \in x_i\ A(\mathbf{x}, z)$.

   For (ii), assume $e \Vdash_{\mathfrak{w}} \exists z \in x_i\ A(\mathbf{x}, z)$. Then $e \neq \emptyset$. Picking $d \in e$ we have $\jmath_0 d \in x_i$ and $\jmath_1 d \Vdash_{\mathfrak{w}} A(\mathbf{x}, \jmath_0 d)$, thus $A(\mathbf{x}, \jmath_0 d)$ by the induction hypothesis (ii), thence $\exists z \in x_i\ A(\mathbf{x}, z)$ holds.
6. Let $D(\mathbf{x})$ be of the form $\forall z \in x_i\ A(\mathbf{x}, z)$. To verify (i), suppose $\forall z \in x_i\ A(\mathbf{x}, z)$ is true. By definition, $\mathfrak{k}_D(\mathbf{x}) = \{\lambda z.\chi_A(\mathbf{x}, z, \mathfrak{k}_A(\mathbf{x}, z))\} \neq \emptyset$. If $z_0 \in x_i$ we have $A(\mathbf{x}, z_0)$, so that inductively $\mathfrak{k}_A(\mathbf{x}, z_0) \neq \emptyset$ and $\forall d \in \mathfrak{k}_A(\mathbf{x}, z_0)\ d \Vdash_{\mathfrak{w}} A(\mathbf{x}, z_0)$. Whence, by Lemma 15.6, $\chi_A(\mathbf{x}, z_0, \mathfrak{k}_A(\mathbf{x}, z_0)) \Vdash_{\mathfrak{w}} A(\mathbf{z}, z_0)$. As a result, $\lambda z.\chi_A(\mathbf{x}, z, \mathfrak{k}_A(\mathbf{x}, z)) \Vdash_{\mathfrak{w}} D(\mathbf{x})$.

   As for (ii), suppose $e \Vdash_{\mathfrak{w}} \forall z \in x_i\ A(\mathbf{x}, z)$. Thus $e \bullet z \Vdash_{\mathfrak{w}} A(\mathbf{x}, z)$ for all $z \in x_i$, so that inductively $\forall z \in x_i\ A(\mathbf{x}, z)$ holds.
7. Let $D(\mathbf{x})$ be of the form $\exists z \subseteq x_i\ A(\mathbf{x}, z)$. To verify (i), suppose $\exists z \subseteq x_i\ A(\mathbf{x}, z)$ holds. Then there is $z \in \mathcal{P}(x_i)$ such that $A(\mathbf{x}, z)$. The induction hypothesis (i) for the latter formula yields that $\mathfrak{k}_A(\mathbf{x}, z) \neq \emptyset$, and hence $\mathfrak{k}_D(\mathbf{x}) \neq \emptyset$. Now suppose

$u \in \mathfrak{k}_D(\mathbf{x})$. Then $u = \{\langle z, \langle \lambda y.0, v\rangle\rangle\}$ for some $z \subseteq x_i$ and $v \in \mathfrak{k}_A(\mathbf{x}, z)$. As $A(\mathbf{x}, z)$ holds the induction hypothesis (i) yields that $v \Vdash_{\mathfrak{w}} A(\mathbf{x}, z)$. Also $\lambda y.0 \Vdash_{\mathfrak{w}} z \subseteq x_i$. Whence $u \Vdash_{\mathfrak{w}} \exists z\,(z \subseteq x_i \;\wedge\; A(\mathbf{x}, z))$.

For (ii), assume $e \Vdash_{\mathfrak{w}} \exists z\,[z \subseteq x_i \wedge A(\mathbf{x}, z)]$. Then $e \neq \emptyset$. Picking $d \in e$ we have $\jmath_1 d \Vdash_{\mathfrak{w}} [\jmath_0 d \subseteq x_i \wedge A(\mathbf{x}, \jmath_0 d)]$. This entails $\jmath_0 d \subseteq x_i$ and $\jmath_1(\jmath_1 d) \Vdash_{\mathfrak{w}} A(\mathbf{x}, \jmath_0 d)$. Thus $A(\mathbf{x}, \jmath_0 d)$ by the induction hypothesis (ii), thence $\exists z \subseteq x_i\; A(\mathbf{x}, z)$ holds.

8. Let $D(\mathbf{x})$ be of the form $\forall z \in x_i\;\; A(\mathbf{x}, z)$. To verify (i), suppose $\forall z \in x_i\;\; A(\mathbf{x}, z)$ is true. By definition, $\mathfrak{k}_D(\mathbf{x}) = \{\lambda y.\lambda z.\chi_A(\mathbf{x}, z, \mathfrak{k}_A(\mathbf{x}, z))\} \neq \emptyset$. If $y \Vdash_{\mathfrak{w}} z_0 \subseteq x_i$, then $z_0 \subseteq x_i$ holds and we have $A(\mathbf{x}, z_0)$, so that inductively $\mathfrak{k}_A(\mathbf{x}, z_0) \neq \emptyset$ and $\forall d \in \mathfrak{k}_A(\mathbf{x}, z_0)\;\; d \Vdash_{\mathfrak{w}} A(\mathbf{x}, z_0)$. Whence, by Lemma 15.6, $\chi_A(\mathbf{x}, z_0, \mathfrak{k}_A(\mathbf{x}, z_0)) \Vdash_{\mathfrak{w}} A(\mathbf{z}, z_0)$. As a result, $\lambda y.\lambda z.\chi_A(\mathbf{x}, z, \mathfrak{k}_A(\mathbf{x}, z)) \Vdash_{\mathfrak{w}} D(\mathbf{x})$.

   As for (ii), suppose $e \Vdash_{\mathfrak{w}} \forall z \subseteq x_i\;\; A(\mathbf{x}, z)$. Thus $e \bullet z \Vdash_{\mathfrak{w}} [z \subseteq x_i \rightarrow A(\mathbf{x}, z)]$ for all z. If $z \subseteq x_i$, then $\lambda y.0 \Vdash_{\mathfrak{w}} z \subseteq x_i$, so that $(e \bullet z) \bullet (\lambda y.0) \Vdash_{\mathfrak{w}} A(\mathbf{x}, z)$, and therefore, by the inductive assumption, $A(\mathbf{x}, z)$ holds. As a result, $\forall z \in x_i\;\; A(\mathbf{x}, z)$ holds. □

Theorem 15.3. *Let $\mathcal{T}^-$ denote Tharp's (1971) quasi-intuitionistic set theory without* **Ord-Im**. *Let $D(u_1, \ldots, u_r)$ be a formula of $\mathcal{L}_\in$ all of whose free variables are among $u_1, \ldots, u_r$. If*

$$\mathcal{T}^- \vdash D(u_1, \ldots, u_r),$$

then one can effectively construct an index of an $E_\wp$-recursive function g such that

$$\mathbf{KP}(\mathcal{P}) \vdash \forall a_1, \ldots, a_r\; g(a_1, \ldots, a_r) \Vdash_{\mathfrak{w}} D(a_1, \ldots, a_r)\,.$$

Proof. Note that with the exception of excluded middle for power bounded formulae, the axioms of $\mathcal{T}^-$ are axioms of $\mathbf{CZF}_{\mathcal{P}}$, too. Let $D(\mathbf{u})$ be $\Delta_0^{\mathcal{P}}$. Define

$$\mathfrak{d}_D(\mathbf{a}) := \{\langle 0, u\rangle \mid u \in \mathfrak{k}_D(\mathbf{a})\} \;\cup\; \{\langle 1, u\rangle \mid u \in \mathfrak{k}_{\neg D}(\mathbf{a})\},$$

with $\mathfrak{k}_D, \mathfrak{k}_{\neg D}$ defined as in Definition 15.8. Note that $\mathfrak{d}_D$ is E-recursive. By Proposition 15.1(i) and classical logic we have that $\mathfrak{d}_D(\mathbf{a}) \neq \emptyset$. Moreover, if $\langle i, u\rangle \in \mathfrak{d}_D(\mathbf{a})$ then either $i = 0$ and $u \Vdash_{\mathfrak{w}} D(\mathbf{a})$ or $i = 1$ and $u \Vdash_{\mathfrak{w}} \neg D(\mathbf{a})$. Thus $\mathfrak{d}_D(\mathbf{a}) \Vdash_{\mathfrak{w}} D(\mathbf{a}) \vee, \exists \neg D(\mathbf{a})$.

In view of the previous Theorem 15.2 we thus found realizers for all theorems of $\mathcal{T}^-$. □

Lemma 15.4. $\mathbf{CZF}_{\mathcal{P}}$ *is a subtheory of $\mathcal{T}^-$.*

Proof. The only axioms of $\mathbf{CZF}_{\mathcal{P}}$ that do not already belong to $\mathcal{T}^-$ are the instances of Bounded Separation. Let $A(u)$ be bounded. We shall reason in $\mathcal{T}^-$. Using excluded middle for bounded formulae, Pairing and Emptyset, we have

$$\forall u \in a\, \exists z\, [(A(u) \wedge z = \{u\}) \vee, \exists\, (\neg A(u) \wedge z = 0)].$$

Thus, by Strong Collection, there exists a set b such that

$$\forall u \in a\, \exists z \in b\, [(A(u) \wedge z = \{u\}) \vee, \exists\, (\neg A(u) \wedge z = 0)]$$
$$\wedge\, \forall z \in b\, \exists u \in a\, (A(u) \wedge z = \{u\}) \vee, \exists\, (\neg A(u) \wedge z = 0)]. \tag{15.15}$$

By Union, $\bigcup b$ is a set, and by (15.15), $\bigcup b = \{u \in a \mid A(u)\}$. □

15.4 A Type Theory Pertaining to $\mathbf{CZF}_{\mathcal{P}}$

Let $\mathbf{ML}_1$ be Martin-Löf's type theory with a single universe $\mathbf{U}$ but without any W-types (cf. Martin-Löf 1984). The type $\mathbf{U}$ of small types reflects the basic forms of type. These are $\mathbf{N}_0$ (empty type), $\mathbf{N}$ (type of naturals), $(\Pi x : A)F(x)$, $(\Sigma x : A)F(x)$, $A + B$ and $I(A, b, c)$ where A and B are types, F is a family of types over A and $b, c : A$.

$\mathbf{ML_1V}$ is the extension of $\mathbf{ML}_1$ with Aczel's type of iterative sets $\mathbf{V}$ (cf. Aczel 1978). $\mathbf{V}$ is inductively specified by the rule

$$\frac{A : \mathbf{U} \quad x : A \Rightarrow F : \mathbf{V}}{\mathbf{sup}(x : A)F : \mathbf{V}}$$

It is this type $\mathbf{V}$ with the above introduction rule and a corresponding elimination rule (or rule of transfinite recursion on $\mathbf{V}$) that has been used in Aczel (1978) to give an interpretation of constructive set theory (for more details see Aczel 1982; Rathjen 1994).

Remark 15.5. $\mathbf{V}$ can be viewed as a single W-type on top of $\mathbf{U}$. $\mathbf{V}$ should certainly not be construed as an additional universe on top of $\mathbf{U}$. As it turns out, adding $\mathbf{V}$ amounts to the same as adding an elimination rule to $\mathbf{U}$ which renders $\mathbf{U}$ an inductively defined type. $\mathbf{V}$ can then be explicitly defined from $\mathbf{U}$ in extensional $\mathbf{ML}_1$ augmented by the principle of transfinite recursion on $\mathbf{U}$ as has been shown by Palmgren in (1993).

We extend the syntax of $\mathbf{ML_1V}$ with a type constant $\mathbf{P}$ and several other constants pertaining to it. The rules for $\mathbf{P}$ render it an impredicatively Π-closed type universe inside $\mathbf{U}$. The rules governing $\mathbf{P}$ are given by the schemes

$$0_{\mathbf{P}} : \mathbf{P} \qquad \mathbf{P} : \mathbf{U} \qquad \frac{a : \mathbf{P}}{\mathbf{T_P}(a) : \mathbf{U}} \qquad \frac{a : \mathbf{P} \quad b_1 : \mathbf{T_P}(a) \quad b_2 : \mathbf{T_P}(a)}{b_1 = b_2 : \mathbf{T_P}(a)}$$

$$\frac{A : \mathbf{U} \quad x : A \Rightarrow B : \mathbf{P}}{(\pi x : A)B : \mathbf{P}} \qquad \frac{A : \mathbf{U} \quad x : A \Rightarrow B_1 = B_2 : \mathbf{P}}{(\pi x : A)B_1 = (\pi x : A)B_2 : \mathbf{P}}$$

$$\mathbf{T_P}(0_{\mathbf{P}}) = N_0 \qquad \frac{A : \mathbf{U} \quad x : A \Rightarrow B : \mathbf{P}}{s_{A,B} : \mathbf{T_P}((\pi x : A)B) \leftrightarrow (\Pi x : A)\mathbf{T_P}(B)} \quad (\star).$$

The formulation of the rules for the type **P**, embodies the principle that elements of **P** are only codes for types, hence the need for a decoding function $\mathbf{T_P}$ and the π-binder. $0_\mathbf{P}$ represents the false proposition and thus $\mathbf{T_P}(0_\mathbf{P})$ should be the empty type.

With these rules the type **P** behaves like the impredicative type of propositions of the calculus of constructions, with the additional property that all propositions in **P** are proof-irrelevant. The equivalence in the rule ($\star$) was already introduced in Coquand (1990). This type theory will be denoted by $\mathbf{MLV_P}$.

15.5 Reducing $\mathbf{MLV_P}$ to $\mathbf{CZF}_\mathcal{P}$

Here we build on the types-as-classes interpretation from Rathjen (2006c) and Rathjen and Tupailo (2006, Definition 6.7) that uses classes of indices of generalized set recursive functions to interpret large Π-types. Rathjen and Tupailo (2006, Theorem 6.8) shows that this provides a translation of $\mathbf{ML_1V}$ into **CZF**. In this interpretation the type **U** is emulated by the inductively defined class $\mathbf{Y}^*$ introduced in Rathjen and Tupailo (2006, Definition 2.8). A larger class $\mathbf{Y}^{**}$ is obtained by adding a fifth clause to the definition of $\mathbf{Y}^*$ which just says that the powerset of $\{0\}$ and every set $x \subseteq \{0\}$ is in $\mathbf{Y}^{**}$. To deal with $\mathbf{MLV_P}$, **U** will be interpreted as $\mathbf{Y}^{**}$ and the type **V** will then be interpreted as the class $\mathbf{V}(\mathbf{Y}^{**})$ which is defined in the same vein as $\mathbf{V}(\mathbf{Y}^*)$ in Rathjen and Tupailo (2006, Definition 3.1). The type **P** will be interpreted by $\mathcal{P}(\{0\})$, the powerset of $\{0\}$. For sets A and a function $F : A \to \mathcal{P}(\{0\})$ let $\pi(A, F) := \{y \in \{0\} \mid \forall x \in A\, F(x) = \{0\}\}$. This is the way we interpret the π-binder. $\mathbf{T_P}$ will be interpreted as the identity function while $s_{A,B}$ is the unique 1–1 correspondence between the sets $\pi(A, F)$ and $\Pi_{x\in A} F(x)$.

Theorem 15.4. *The types-as-classes translation provides an interpretation of* $\mathbf{MLV_P}$ *in* $\mathbf{CZF}_\mathcal{P}$.

Proof. For details see Rathjen and Tupailo (2006, Theorem 6.8) and Rathjen (1994, Theorem 4.11). ☐

15.6 Reducing $\mathbf{CZF} + \mathbf{Pow}^{\neg\neg}$ to $\mathbf{MLV_P}$

Recall that the negative power set axiom, $\mathbf{Pow}^{\neg\neg}$, asserts that for every set a there exists a set c containing all the subsets x of a for which $\forall u \in a(\neg\neg u \in x \to u \in x)$ holds. The latter set will be denoted by $\mathcal{P}^{\neg\neg}(a)$.

Lemma 15.10. *The theory obtained from* **CZF** *by adding the axiom '*$\mathcal{P}^{\neg\neg}(\{0\})$ is a set*' is equivalent to* $\mathbf{CZF} + \mathbf{Pow}^{\neg\neg}$.

Proof. Gambino (1999, Lemma 4.3.2). □

Theorem 15.5. *The theory* **CZF**+$\mathbf{Pow}^{\neg\neg}$ *can be justified in the type theory* $\mathbf{MLV_P}$.

Proof. For the axioms of **CZF** this is due to Aczel (1978). The validity of the negative power set axiom in a type theory with **P** was shown by Gambino (1999, Lemma 4.3.7). □

15.7 Completing the Circle: The Proof of Theorem 15.1

The main thing we know so far is that $\mathbf{CZF}_{\mathcal{P}}$ is proof-theoretically no stronger than $\mathbf{KP}(\mathcal{P})$ (Theorem 15.2). As for the proof-theoretic equivalence of ((i) and (ii) in Theorem 15.1, we need to show that $\mathbf{CZF}_{\mathcal{P}} + \mathbf{RDC} + \mathbf{\Pi\,\Sigma{-}AC}$ is no stronger than $\mathbf{CZF}_{\mathcal{P}}$. We shall draw on the formulae-as-classes interpretation of Rathjen (2006c) to achieve this.

Theorem 15.6. $\mathbf{CZF}_{\mathcal{P}}+\mathbf{RDC}+\mathbf{\Pi\,\Sigma{-}AC}$ *has a formulae-as-classes interpretation in* $\mathbf{CZF}_{\mathcal{P}}$.

Proof. The interpretation of $\mathbf{CZF}+\mathbf{RDC}+\mathbf{\Pi\,\Sigma{-}AC}$ into **CZF** of Rathjen (2006c, Theorem 4.13) can be lifted to the theories with **Pow** added on both sides if one uses the stronger notion of computability introduced in Definition 15.3. One just needs to show that the power set axiom is validated in this interpretation if one has it in the background theory and uses the stronger notion of computability. This is not very difficult. □

To get back from $\mathbf{KP}(\mathcal{P})$ to $\mathbf{CZF}_{\mathcal{P}}$ we shall rely on Rathjen (2012). Let $\mathcal{OT}(\vartheta) = (\mathrm{BH}, \prec)$ be the primitive recursive ordinal representation system for the Bachmann-Howard ordinal given in Rathjen and Weiermann (1993, Lemma 1.3); here $\mathrm{OT}(\vartheta)$ is a primitive recursive set of naturals equipped with a primitive recursive well-ordering $\prec$ and

$$\mathrm{BH} := \{\alpha \in \mathrm{OT}(\vartheta) \mid \alpha \prec \Omega\}.$$

For $\tau \in \mathrm{BH}$ let

$$V_\tau := \bigcup_{\nu \prec \tau} \mathcal{P}(V_\nu) \tag{15.16}$$

$$V_\tau^{\neg\neg} := \bigcup_{\nu \prec \tau} \mathcal{P}^{\neg\neg}(V_\nu^{\neg\neg}). \tag{15.17}$$

Let 'V_τ *exists*' be the statement

$$\exists F\,[F \text{ function} \wedge \mathrm{dom}(f) = \{\nu \in \mathrm{BH} \mid \nu \prec \tau\} \wedge \forall \nu \prec \tau\; F(\nu) = \bigcup_{\xi \prec \nu} \mathcal{P}(F(\xi))]\,.$$

Lemma 15.11. *For every (meta) $\tau \in \mathrm{BH}$,* **CZF** *proves the scheme of transfinite induction up to τ, i.e.,*

$$\forall \nu \prec \tau[(\forall \mu \prec \nu\, \varphi(\mu)) \to \varphi(\nu)] \to \forall \nu \prec \tau\, \varphi(\nu)$$

for all formulae $\varphi(\nu)$.

Proof. This is a consequence of Rathjen (2005, Lemma 4.3, Theorem 4.13). □

Lemma 15.12. *Let $\tau \in \mathrm{BH}$. The following are provable in* **CZF** + **Pow**$^{\neg\neg}$ *for all $\beta \preceq \alpha \preceq \tau$:*

(i) *'$V_\alpha^{\neg\neg}$ exists'.*
(ii) $V_0^{\neg\neg} = \emptyset$.
(iii) *If α is a limit, then $V_\alpha^{\neg\neg} = \bigcup_{\xi \prec \alpha} V_\xi^{\neg\neg}$.*
(iv) $V_{\alpha+1}^{\neg\neg} = V_\alpha^{\neg\neg} \cup \mathcal{P}^{\neg\neg}(V_\alpha^{\neg\neg})$.
(v) $V_\beta^{\neg\neg} \subseteq V_\alpha^{\neg\neg}$.
(vi) *$V_\alpha^{\neg\neg}$ is transitive.*
(vii) $u \in x \in V_\alpha^{\neg\neg} \to \exists \xi \prec \alpha\; u \in V_\xi^{\neg\neg}$.

Proof. (i) Follows by transfinite recursion on α using Lemma 15.11 and Replacement.

(ii) Holds because $V_0^{\neg\neg} = \bigcup_{\xi \prec 0} V_\xi^{\neg\neg} = \emptyset$.

(iii) : $V_\alpha^{\neg\neg} = \bigcup_{\xi \prec \alpha} \mathcal{P}^{\neg\neg}(V_\xi^{\neg\neg}) = \bigcup_{\xi \prec \alpha} \bigcup_{\zeta \prec \xi} \mathcal{P}^{\neg\neg}(V_\xi^{\neg\neg}) = \bigcup_{\xi \prec \alpha} V_\xi^{\neg\neg}$ when α is a limit.

(iv) :

$$\begin{aligned} V_{\alpha+1}^{\neg\neg} &= \bigcup_{\xi \prec \alpha+1} \mathcal{P}^{\neg\neg}(V_\xi^{\neg\neg}) \\ &= \mathcal{P}^{\neg\neg}(V_\alpha^{\neg\neg}) \cup \bigcup_{\xi \prec \alpha} \mathcal{P}^{\neg\neg}(V_\xi^{\neg\neg}) \\ &= \mathcal{P}^{\neg\neg}(V_\alpha^{\neg\neg}) \cup V_\alpha^{\neg\neg}. \end{aligned}$$

(v) : Suppose $\beta \prec \alpha$. It suffices to show that $V_\beta^{\neg\neg} \in \mathcal{P}^{\neg\neg}(V_\beta^{\neg\neg})$. But this is clearly the case since $V_\beta^{\neg\neg} \subseteq V_\beta^{\neg\neg}$ and (trivially)

$$\forall y \in V_\beta^{\neg\neg}(\neg\neg y \in V_\beta^{\neg\neg} \to y \in V_\beta^{\neg\neg}).$$

(vi) and (vii): Let $u \in x \in V_\alpha^{\neg\neg}$. Then $u \in x \in \mathcal{P}^{\neg\neg}(V_\xi^{\neg\neg})$ for some $\xi \prec \alpha$. Hence $u \in V_\xi^{\neg\neg}$ for some $\xi \prec \alpha$, so that $u \in V_\alpha^{\neg\neg}$ by (v).

□

Theorem 15.7. (i) *The following theories are proof-theoretically equivalent:*

1. **KP**$(\mathcal{P})$
2. $\mathbf{Z} + \{\text{'}V_\tau \textit{ exists'}\}_{\tau\in\mathrm{BH}}$.

(ii) *The following theories are proof-theoretically equivalent:*

1. **IKP**$(\mathcal{P})$
2. $\mathbf{IZ} + \{\text{'}V_\tau \textit{ exists'}\}_{\tau\in\mathrm{BH}}$.

Proof. This is shown in Rathjen (2012). □

15.7.1 Reducing $\mathbf{Z} + \{\text{'}V_\tau \textit{ Exists'}\}_{\tau\in\mathrm{BH}}$ to $\mathbf{CZF} + \mathbf{Pow}^{\neg\neg}$

The next step is to employ a double negation interpretation to reduce $\mathbf{Z} + \{\text{'}V_\tau \text{ exists'}\}_{\tau\in\mathrm{BH}}$ to an intuitionistic theory. Here we don't follow Friedman's approach in Friedman (1973c). Instead we use two new relations $=_\infty$ and $\in_\infty$ to interpret $=$ and $\in$, respectively. Moreover, these relations are designed to be stable under double negation. This Ansatz was inspired by a double negation interpretation of Zermelo set theory in $V^{\neg\neg}_{\omega+\omega}$ due to Gambino (see Gambino 1999, Proposition 2.3.21). In it he uses Aczel's a-relations, which combine the idea of bisumulation with stability of doubly negated formulae, to interpret set-theoretic equality (for details see Gambino 1999, Definition 2.2.14). Our interpretation, however, does not employ a-relations since our background theory has only Bounded Separation. Instead it uses an equivalence relation defined by transfinite recursion on the ordinal representations of BH.

Theorem 15.8. *For every $\rho \in \mathrm{BH}$, the theory* $\mathbf{Z} + \text{'}V^{\neg\neg}_\rho$ exists' *has an interpretation in* $\mathbf{CZF} + \mathbf{Pow}^{\neg\neg}$.

The proof of 15.8 will occupy the remainder of this subsection. Given $\rho \in \mathrm{BH}$ one can effectively find $\rho^* \in \mathrm{BH}$ such that $\rho \prec \rho^*$ and ρ^* is a limit ordinal bigger than ω. In view of Theorem 15.7 we also know that $\mathbf{CZF} + \mathbf{Pow}^{\neg\neg}$ proves *'*$V^{\neg\neg}_{\rho^*}$ *exists'*. We would like to use the set $V^{\neg\neg}_{\rho^*}$ to provide a model for the theory $\mathbf{Z}$ + *'*V_ρ *exists'*. The idea is, of course, to use some kind of double negation interpretation. But as is well known, the extensionality axiom creates a problem when one uses the usual Gödel-Gentzen translation. To overcome this problem we define an equivalence relation $=_\infty$ on $V^{\neg\neg}_{\rho^*}$ which will be used to interpret set-theoretic equality and thereby also membership.

Definition 15.9. Let $x, y \in V^{\neg\neg}_{\rho^*}$. By transfinite recursion on $\alpha \prec \rho^*$ define

$$\begin{aligned} x =_\alpha y \text{ iff } x, y \in V^{\neg\neg}_\alpha \; & \wedge \forall u \in x\, \neg\neg\exists v \in y\, \exists\beta \prec \alpha\; u =_\beta v \\ & \wedge \forall v \in y\, \neg\neg\exists u \in x\, \exists\beta \prec \alpha\; u =_\beta v \end{aligned}$$

$$x =_\infty y \text{ iff } \neg\neg\exists\alpha \prec \rho^* \, x =_\alpha y$$

$$x \in_\infty y \text{ iff } \neg\neg\exists\alpha \prec \rho^* \, \exists u \in y \; x =_\alpha u \, .$$

Lemma 15.13. *Let* $x, y \in V^{\neg\neg}_{\rho^*}$ *and* $\alpha \prec \beta \prec \rho^*$. *Then we have*

(i) $x =_\alpha y \to x =_\beta y$.
(ii) $x, y \in V^{\neg\neg}_\alpha \wedge x =_\beta y \to x =_\alpha y$.
(iii) $=_\alpha$ *is a symmetric and transitive relation.* $=_\alpha$ *is a reflexive relation on* $V^{\neg\neg}_\alpha$.

Proof. (i) Suppose $x =_\alpha y$. Then $x, y \in V^{\neg\neg}_\alpha$, thus $x, y \in V^{\neg\neg}_\beta$ by Lemma 15.12(v). Clearly we have $\exists v \in y \, \exists \xi \prec \alpha \; u =_\xi v \to \exists v \in y \, \exists \xi \prec \beta \; u =_\xi v$, thus

$$\neg\neg\exists v \in y \, \exists \xi \prec \alpha \; u =_\xi v \to \neg\neg\exists v \in y \, \exists \xi \prec \beta \; u =_\xi v \, ,$$

and hence

$$\forall u \in x \, \neg\neg\exists v \in y \, \exists \xi \prec \alpha \; u =_\xi v \to \forall u \in x \, \neg\neg\exists v \in y \, \exists \xi \prec \beta \; u =_\xi v \, .$$

Likewise, $\forall v \in y \, \neg\neg\exists u \in x \, \exists \xi \prec \alpha \; u =_\xi v \to \forall v \in y \, \neg\neg\exists u \in x \, \exists \xi \prec \beta \; u =_\xi v$. As a result, $x =_\beta y$.

(ii) : We use induction on α. Suppose that $x, y \in V^{\neg\neg}_\alpha$ and $x =_\beta y$. If $u \in x$ and $v \in y$, then $u \in x \in \mathcal{P}^{\neg\neg}(V^{\neg\neg}_{\xi_0})$ and $v \in y \in \mathcal{P}^{\neg\neg}(V^{\neg\neg}_{\xi_1})$ for some $\xi_0, \xi_1 \prec \alpha$. Hence $u \in V^{\neg\neg}_{\xi_0}$ and $v \in V^{\neg\neg}_{\xi_1}$. Due to the linearity of $\prec$ and in view of Lemma 15.12(v), there exists $\alpha_0 \prec \alpha$ such that $u, v \in V^{\neg\neg}_{\alpha_0}$. As a result, if $u \in x$ and $\exists v \in y \, \exists \zeta \prec \beta \; u =_\zeta v$, then the induction hypothesis yields $\exists v \in y \, \exists \zeta \prec \alpha \, u =_\zeta v$. Thus $u \in x$ and $\neg\neg\exists v \in y \, \exists \zeta \prec \beta \, u =_\zeta v$ imply $\neg\neg\exists v \in y \, \exists \zeta \prec \alpha \, u =_\zeta v$. Consequently,

$$\forall u \in x \, \neg\neg\exists v \in y \, \exists \zeta \prec \beta \, u =_\zeta v \to \forall u \in x \, \neg\neg\exists v \in y \, \exists \zeta \prec \alpha \, u =_\zeta v. \quad (15.18)$$

Likewise one proves

$$\forall v \in y \, \neg\neg\exists x \in u \, \exists \zeta \prec \beta \, u =_\zeta v \to \forall v \in y \, \neg\neg\exists u \in x \, \exists \zeta \prec \alpha \, u =_\zeta v. \quad (15.19)$$

Hence, since we assumed that $x =_\beta y$ we get $x =_\alpha y$ from (15.18) and (15.19).

(iii) Follows by induction on α. As for transitivity, suppose $x, y, z \in V^{\neg\neg}_\alpha$, $x =_\alpha y$, and $y =_\alpha z$. Assume that $u \in x$, $v \in y$, $w \in z$ and $u =_{\xi_0} v$ and $v =_{\xi_1} w$ hold for some $\xi_0, \xi_1 \prec \alpha$. Then, using (i) and the linearity of $\prec$, we find $\xi \prec \alpha$ such that $u =_\xi v$ and $v =_\xi w$, so that, by the induction hypothesis, we get $u =_\xi w$. As a result, letting A be $u \in x \wedge v \in y \wedge \exists \xi_0 \prec \alpha \, u =_{\xi_0} v$,

$$A \to (\exists w \in z\, \exists \xi_1 \prec \alpha\, v =_{\xi_0} w \to \exists w \in z\, \exists \xi \prec \alpha\, u =_\xi w)$$
$$A \to (\neg\neg\exists w \in z\, \exists \xi_1 \prec \alpha\, v =_{\xi_0} w \to \neg\neg\exists w \in z\, \exists \xi \prec \alpha\, u =_\xi w)$$
$$A \to \neg\neg\exists w \in z\, \exists \xi \prec \alpha\, u =_\xi w \quad (\text{since } y =_\alpha z)$$
$$u \in x \to (\exists v \in y\, \exists \xi_0 \prec \alpha\, u =_{\xi_0} v \to \neg\neg\exists w \in z\, \exists \xi \prec \alpha\, u =_\xi w)$$
$$u \in x \to (\neg\neg\exists v \in y\, \exists \xi_0 \prec \alpha\, u =_{\xi_0} v \to \neg\neg\exists w \in z\, \exists \xi \prec \alpha\, u =_\xi w)$$
$$u \in x \to \neg\neg\exists w \in z\, \exists \xi \prec \alpha\, u =_\xi w \quad (\text{since } x =_\alpha y)$$

and hence $\forall u \in x\, \neg\neg\exists w \in z\, \exists \xi \prec \alpha\, u =_\xi w$. Likewise one shows that $\forall w \in z\, \neg\neg\exists u \in x\, \exists \xi \prec \alpha\, u =_\xi w$. Thus $x =_\alpha z$.

Symmetry and reflexivity are established similarly. □

Corollary 15.2. *Let $x, y \in V^{\neg\neg}_{\rho^*}$. Then:*

$$x =_\infty y \;\leftrightarrow\; \forall u \in V^{\neg\neg}_{\rho^*}\,(u \in_\infty x \;\leftrightarrow\; u \in_\infty y)\,.$$

Proof. "→": Suppose $\alpha, \beta \prec \rho^*$, $v \in x$, $u =_\alpha v$, $w \in y$, and $v =_\beta w$. Letting $\gamma := \max(\alpha, \beta)$, we obtain $u =_\gamma v$ and $v =_\gamma w$ by Lemma 15.13(i), and hence $u =_\gamma w$ by Lemma 15.13(iii). Thus, letting B stand for the conjunction of $\alpha \prec \rho^*$, $v \in x$, and $u =_\alpha v$, we have the following implications:

$$B \;\wedge\; \exists \beta' \prec \rho^*\, \exists w' \in y\; v =_{\beta'} w' \to \exists \gamma' \prec \rho^*\, \exists w' \in y\; u =_{\gamma'} w'$$
$$B \;\wedge\; \neg\neg\exists \beta' \prec \rho^*\, \exists w' \in y\; v =_{\beta'} w' \to \neg\neg\exists \gamma' \prec \rho^*\, \exists w' \in y\; u =_{\gamma'} w'$$
$$B \;\wedge\; \exists \eta \prec \rho^*\; x =_\eta y \to u \in_\infty y$$
$$B \;\wedge\; \neg\neg\exists \eta \prec \rho^*\; x =_\eta y \to u \in_\infty y$$
$$B \;\wedge\; x =_\infty y \to u \in_\infty y$$
$$\exists \alpha \prec \rho^*\, \exists v \in x\; u =_\alpha v \;\wedge\; x =_\infty y \to u \in_\infty y$$
$$\neg\neg\exists \alpha \prec \rho^*\, \exists v \in x\; u =_\alpha v \;\wedge\; x =_\infty y \to u \in_\infty y$$
$$u \in_\infty x \;\wedge\; x =_\infty y \to u \in_\infty y\,.$$

In the above, we used several times that $C \to \neg\neg C$ and

$$(A \to \neg\neg C) \to (\neg\neg A \to \neg\neg C)$$

are intuitionistically valid propositions.

"←": Assume that $\forall u \in V^{\neg\neg}_{\rho^*}\,(u \in_\infty x \;\leftrightarrow\; u \in_\infty y)$. Choose $\alpha \prec \rho^*$ such that $x, y \in V^{\neg\neg}_\alpha$. Let $u \in x$. Then $u \in V^{\neg\neg}_\xi$ for some $\xi \prec \alpha$ by Lemma 15.11(vii). By Lemma 15.13(iii), we have $u =_\xi u$, which implies $u \in_\infty x$, and hence $u \in_\infty y$ by our standing assumption. We also have

$$\exists\eta \prec \rho^* \; \exists w \in y \; u =_\eta w \;\rightarrow\; \exists\eta \prec \alpha \; \exists w \in y \; u =_\eta w\,,$$

and hence

$$\neg\neg\exists\eta \prec \rho^* \; \exists w \in y \; u =_\eta w \;\rightarrow\; \neg\neg\exists\eta \prec \alpha \; \exists w \in y \; u =_\eta w\,,$$

using Lemma 15.13(ii). Thence, as $u \in_\infty y$, we can conclude that $\neg\neg\exists\eta \prec \alpha \; \exists w \in y \; u =_\eta w$. As a result,

$$\forall u \in x \, \neg\neg\exists\eta \prec \alpha \; \exists w \in y \; u =_\eta w$$

Likewise, we can conclude that $\forall v \in y \, \neg\neg\exists\eta \prec \alpha \; \exists u \in x \; v =_\eta u$, so that $x =_\alpha y$, and consequently $x =_\infty y$. □

Corollary 15.3. *Let $x, y, z \in V_{\rho^*}^{\neg\neg}$. Then:*

$$x =_\infty y \;\wedge\; x \in_\infty z \rightarrow y \in_\infty z\,.$$

Proof. Suppose $x =_\alpha y$, $x =_\beta u$, and $u \in z$ for some $\alpha, \beta \prec \rho^*$. Pick $\delta \prec \rho^*$ such that $x, y, z \in V_\delta^{\neg\neg}$. By Lemma 15.13 we have $x =_\delta y$, $x =_\delta u$, and thus $y =_\delta u$, which entails $y \in_\infty z$. As a result of the foregoing we have

$$\exists\alpha \prec \rho^* \; x =_\alpha y \;\wedge\; \exists\delta' \prec \rho^* \, \exists u \in z \; x = u \rightarrow y \in_\infty z,$$

$$x =_\infty y \;\wedge\; x \in_\infty z \rightarrow y \in_\infty z\,,$$

exploiting (again) that $y \in_\infty z$ is a twice negated formula. □

Next we will show in **CZF** + **Pow**$^{\neg\neg}$ that the structure $(V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty)$ models the double negation translation of all the axioms of **Z** + *'V_ρ exists'* when the elementhood and equality symbols are interpreted as $\in_\infty$ and $=_\infty$, respectively.

Definition 15.10 (N-translation). Let the map $(.)^N$ from the language of set theory into itself be defined as follows:

$$\begin{aligned}
(x \in y)^N &:= x \in_\infty y \\
(x = y)^N &:= x =_\infty y \\
(A \wedge B)^N &:= A^N \wedge B^N \\
(A \vee, \exists B)^N &:= \neg(\neg A^N \wedge \neg B^N) \\
(A \rightarrow B)^N &:= A^N \rightarrow B^N \\
(\neg A)^N &:= \neg\, A^N \\
(\forall x A)^N &:= \forall x\, A^N \\
(\exists x A)^N &:= \neg\forall x\, \neg A^N\,.
\end{aligned}$$

Note that the formulae $x \in_\infty y$ and $x =_\infty y$ are already doubly negated, so that there is no need to put double negations in front of them.

Lemma 15.14. **CZF** + **Pow**$^{\neg\neg} \vdash (V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty) \models (Extensionality)^N$.

Proof. Observe that (Extensionality)^N is

$$\forall x, \forall y\, [x =_\infty y \;\leftrightarrow\; \forall u\, (u \in_\infty x \;\leftrightarrow\; u \in_\infty y)]\,.$$

So the claimed assertion is a consequence of Corollary 15.2. □

Corollary 15.4. **CZF** + **Pow**$^{\neg\neg} \vdash (V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty) \models \forall x \forall y\, [A(x) \wedge x = y \to A(y)]$.

Proof. This follows from Lemma 15.14 and Corollary 15.3 by formula induction on $A(x)$. □

Lemma 15.15. **CZF** + **Pow**$^{\neg\neg} \vdash (V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty) \models (Pairing)^N$.

Proof. Let $a, b \in V_{\rho^*}^{\neg\neg}$. Pick $\alpha \prec \rho^*$ such that $a, b \in V_\alpha^{\neg\neg}$ and let

$$c := \{x \in V_\alpha^{\neg\neg} \mid \neg\neg(x =_\infty a \vee, \exists\, x =_\infty b)\}\,.$$

Note that $c \subseteq V_\alpha^{\neg\neg}$. If $u \in V_\alpha^{\neg\neg}$ and $\neg\neg u \in c$, then $\neg\neg(\neg\neg(u =_\infty a \vee, \exists\, u =_\infty b))$, hence $\neg\neg(u =_\infty a \vee, \exists\, u =_\infty b)$, so that $u \in c$. This shows that $c \in \mathcal{P}^{\neg\neg}(V_\alpha^{\neg\neg})$, thus $c \in V_{\alpha+1}^{\neg\neg}$.

Now suppose $z =_\infty x$ and $x \in c$. Then $\neg\neg(x =_\infty a \vee, \exists\, x =_\infty b)$, and thus, by Corollary 15.4, $\neg\neg(z =_\infty a \vee, \exists\, z =_\infty b)$. Hence, as $z =_\beta x$ implies $z =_\infty x$,

$$\begin{aligned} \beta \prec \rho^* \wedge x \in c \wedge z =_\beta x &\to \neg\neg(z =_\infty a \vee, \exists\, z =_\infty b) \\ \exists \beta \prec \rho^*\, \exists x \in c\; z =_\beta x &\to \neg\neg(z =_\infty a \vee, \exists\, z =_\infty b) \\ \neg\neg\exists \beta \prec \rho^*\, \exists x \in c\; z =_\beta x &\to \neg\neg(z =_\infty a \vee, \exists\, z =_\infty b) \\ z \in_\infty c &\to \neg\neg(z =_\infty a \vee, \exists\, z =_\infty b)\,. \end{aligned}$$

Conversely, $z =_\infty a \vee, \exists\, z =_\infty b$ implies $z \in_\infty c$ by Corollary 15.4 since $a \in_\infty c$ and $b \in_\infty c$. Thus $\neg\neg(z =_\infty a \vee, \exists\, z =_\infty b)$ implies $z \in_\infty c$ since the latter formula starts with a negation. □

Lemma 15.16. **CZF** + **Pow**$^{\neg\neg} \vdash (V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty) \models (Union)^N$.

Proof. Let $a \in V_{\rho^*}^{\neg\neg}$. Pick $\alpha \prec \rho^*$ such that $a \in V_\alpha^{\neg\neg}$ and let

$$c := \{v \in V_\alpha^{\neg\neg} \mid \neg\neg\exists z \in a\; v \in_\infty z\}\,.$$

Note that $c \subseteq V_\alpha^{\neg\neg}$. If $v \in V_\alpha^{\neg\neg}$ and $\neg\neg v \in c$, then $\neg\neg(\neg\neg\exists z \in a\; v \in_\infty z)$, hence $\neg\neg\exists z \in a\; v \in_\infty z$, so that $v \in c$. This shows that $c \in \mathcal{P}^{\neg\neg}(V_\alpha^{\neg\neg})$, thus $c \in V_{\alpha+1}^{\neg\neg}$.

For $x \in V^{\neg\neg}_{\rho^*}$ we have:

$$\beta \prec \rho^* \wedge v \in c \wedge x =_\beta v \to x =_\infty v \wedge \neg\neg\exists z \in a\; v \in_\infty z$$
$$\beta \prec \rho^* \wedge v \in c \wedge x =_\beta v \to x =_\infty v \wedge \neg\neg\exists z \in V^{\neg\neg}_{\rho^*}\,(z \in_\infty a \wedge v \in_\infty z)$$
$$\beta \prec \rho^* \wedge v \in c \wedge x =_\beta v \to \neg\neg\exists y \in V^{\neg\neg}_{\rho^*}\,(y \in_\infty a \wedge x \in_\infty y)$$
(by Corollary 15.4)
$$\neg\neg\exists\gamma \prec \rho^*\, \exists u \in c\; x =_\beta u \to \neg\neg\exists y \in V^{\neg\neg}_{\rho^*}\,(y \in_\infty a \wedge x \in_\infty y)$$
$$x \in_\infty c \to \neg\neg\exists y \in V^{\neg\neg}_{\rho^*}\,(y \in_\infty a \wedge x \in_\infty y)\,.$$

Conversely, let $x, y, z \in V^{\neg\neg}_{\rho^*}$ and $\beta, \delta \prec \rho^*$. Then:

$$y \in a \wedge u \in y \to u \in V^{\neg\neg}_\alpha \wedge \exists z \in a\; u \in_\infty z$$
$$y \in a \wedge u \in y \to u \in V^{\neg\neg}_\alpha \wedge \neg\neg\exists z \in a\; u \in_\infty z$$
$$y \in a \wedge u \in y \to u \in_\infty c$$
$$y \in a \wedge u \in y \wedge x =_\beta u \to x \in_\infty c \quad \text{(by Corollary 15.4)}$$
$$y \in a \wedge \neg\neg\exists\beta' \prec \rho^*\, \exists u' \in y\; x =_{\beta'} u' \to x \in_\infty c$$
$$y \in a \wedge x \in_\infty y \to x \in_\infty c$$
$$y \in a \wedge z =_\delta y \wedge x \in_\infty z \to x \in_\infty c \quad \text{(by Corollary 15.4)}$$
$$\exists\delta' \prec \rho^*\, \exists y' \in a\; z =_\delta y' \wedge x \in_\infty z \to x \in_\infty c$$
$$\neg\neg\exists\delta' \prec \rho^*\, \exists y' \in a\; z =_\delta y' \wedge x \in_\infty z \to x \in_\infty c$$
$$z \in_\infty a \wedge x \in_\infty z \to x \in_\infty c$$
$$\neg\neg\exists z' \in V^{\neg\neg}_{\rho^*}(z' \in_\infty a \wedge x \in_\infty z') \to x \in_\infty c\,.$$

From the above we conclude that $(V^{\neg\neg}_{\rho^*}, \in_\infty, =_\infty) \models (\text{Union})^N$. □

Lemma 15.17. CZF + Pow$^{\neg\neg}$ $\vdash (V^{\neg\neg}_{\rho^*}, \in_\infty, =_\infty) \models$ *(full Separation)*^{N}.

Proof. Let $a \in V^{\neg\neg}_{\rho^*}$ and let $A(v)$ be a formula with parameters from $V^{\neg\neg}_{\rho^*}$ and at most the free variable v. Let $A^*(v)$ arise from $A(v)$ by first applying the N-translation and subsequently restricting all unbounded quantifiers to $V^{\neg\neg}_{\rho^*}$. Pick $\alpha \prec \rho^*$ such that $a \in V^{\neg\neg}_\alpha$ and let

$$c := \{x \in V^{\neg\neg}_\alpha \mid x \in_\infty a \wedge (V^{\neg\neg}_{\rho^*}, \in_\infty, =_\infty) \models A^*(x)\}\,.$$

c is a set by bounded Separation in our background theory. Obviously $c \subseteq V^{\neg\neg}_\alpha$. Suppose $u \in V^{\neg\neg}_\alpha$ and $\neg\neg u \in c$. Then $\neg\neg u \in_\infty a$ and $\neg\neg A^*(u)$, thus $u \in_\infty a$ and $A^*(u)$ since both formulae are negative. As a result, $c \in V^{\neg\neg}_{\alpha+1}$.

Now let $x \in V^{\neg\neg}_{\rho^*}$ and $\beta \prec \rho^*$. Then:

$$u \in c \to u \in_\infty a \wedge A^*(u)$$
$$u \in c \wedge x =_\beta u \to x \in_\infty a \wedge A^*(x) \quad \text{(by Corollary 15.4)}$$
$$\exists \beta' \prec \rho^* \, \exists u \in c \; x =_\beta u \to x \in_\infty a \wedge A^*(x)$$
$$\neg\neg\exists \beta' \prec \rho^* \, \exists u \in c \; x =_\beta u \to x \in_\infty a \wedge A^*(x) \quad \text{(succedent is negative)}$$
$$x \in_\infty c \to x \in_\infty a \wedge A^*(x).$$

$$u \in a \wedge A^*(u) \to u \in c$$
$$u \in a \wedge A^*(u) \to u \in_\infty c$$
$$u \in a \wedge x =_\beta u \wedge A^*(x) \to x \in_\infty c \quad \text{(by Corollary 15.4)}$$
$$\exists \beta' \prec \rho^* \, \exists u \in a \; x =_\beta u \wedge A^*(x) \to x \in_\infty c$$
$$x \in_\infty a \wedge A^*(x) \to x \in_\infty c\,.$$

As a result of the above we have

$$(V^{\neg\neg}_{\rho^*}, \in_\infty, =_\infty) \models \exists z \, \forall x [x \in_\infty z \leftrightarrow (x \in_\infty a \wedge A^N(x))]. \qquad \square$$

Lemma 15.18. $\mathbf{CZF} + \mathbf{Pow}^{\neg\neg} \vdash (V^{\neg\neg}_{\rho^*}, \in_\infty, =_\infty) \models (\textit{Set Induction})^N$.

Proof. Let $A(v)$ be a formula with parameters from $V^{\neg\neg}_{\rho^*}$ and at most the free variable v. Let $A^*(v)$ arise from $A(v)$ by first applying the N-translation and subsequently restricting all unbounded quantifiers to $V^{\neg\neg}_{\rho^*}$. Assume that

$$\forall z \in V^{\neg\neg}_{\rho^*} [\forall y \in V^{\neg\neg}_{\rho^*} (y \in_\infty z \to A^*(y)) \to A^*(z)]. \tag{15.20}$$

Let $a \in V^{\neg\neg}_\alpha$ where $\alpha \prec \rho^*$. The aim is to show that $A^*(a)$ holds. To this end we proceed by induction on α. If $u \in a$ then $u \in V^{\neg\neg}_\xi$ for some $\xi \prec \alpha$ by Lemma 15.12(vii), thus $A^*(u)$ holds by the inductive assumption. For $x \in V^{\neg\neg}_{\rho^*}$ we thus have

$$x =_\beta u \wedge u \in a \to A^*(x) \quad \text{(by Corollary 15.4)}$$
$$\exists \beta \prec \rho^* \, \exists u \in a \; x =_\beta u \to A^*(x)$$
$$x \in_\infty a \to A^*(x) \quad (A^*(x) \text{ being negative})\,.$$

In view of our assumption (15.20) we thus have $A^*(a)$. $\square$

Lemma 15.19. $\mathbf{CZF} + \mathbf{Pow}^{\neg\neg} \vdash (V^{\neg\neg}_{\rho^*}, \in_\infty, =_\infty) \models (\mathit{Power\ Set})^N$.

Proof. For $x, y \in V^{\neg\neg}_{\rho^*}$ define $x \subseteq_\infty y$ as $\forall u \in V^{\neg\neg}_{\rho^*}\,(u \in_\infty x \to u \in_\infty y)$.

Let $a \in V^{\neg\neg}_{\alpha+1}$ for some $\alpha \prec \rho^*$. Let

$$c := \{x \in V^{\neg\neg}_{\alpha+1} \mid x \subseteq_\infty a\}.$$

Then $c \subseteq V^{\neg\neg}_{\alpha+1}$ and we have

$$\begin{aligned}
w \in V^{\neg\neg}_{\alpha+1} \wedge \neg\neg\, w \in c &\to \neg\neg\, w \subseteq_\infty a \\
&\to \neg\neg\, \forall u \in V^{\neg\neg}_{\rho^*}\,(u \in_\infty w \to u \in_\infty a) \\
&\to \forall u \in V^{\neg\neg}_{\rho^*}\,\neg\neg\,(u \in_\infty w \to u \in_\infty a) \\
&\to \forall u \in V^{\neg\neg}_{\rho^*}\,(\neg\neg\, u \in_\infty w \to \neg\neg\, u \in_\infty a) \\
&\to \forall u \in V^{\neg\neg}_{\rho^*}\,(u \in_\infty w \to u \in_\infty a) \\
&\to w \subseteq_\infty a \\
&\to w \in c\,.
\end{aligned}$$

This shows that $c \in V^{\neg\neg}_{\alpha+2}$.

Now suppose $y \subseteq_\infty a$. Let

$$y^* := \{v \in V^{\neg\neg}_{\alpha} \mid v \in_\infty y\}.$$

Then $y^* \subseteq_\infty y$. Let $u \in_\infty y$. Then $u \in a$ and hence

$$\begin{aligned}
\beta \prec \rho^* \wedge u =_\beta v \wedge v \in a &\to v \in_\infty y \\
\beta \prec \rho^* \wedge u =_\beta v \wedge v \in a &\to v \in_\infty y^* \\
\beta \prec \rho^* \wedge u =_\beta v \wedge v \in a &\to u \in_\infty y^* \\
\exists \beta \prec \rho^*\, \exists v \in a\; u =_\beta v &\to u \in_\infty y^* \\
\neg\neg\, \exists \beta \prec \rho^*\, \exists v \in a\; u =_\beta v &\to u \in_\infty y^* \\
u \in_\infty y &\to u \in_\infty y^*\,.
\end{aligned}$$

So $y \subseteq_\infty y^*$, which together with $y^* \subseteq_\infty y$ yields $y =_\infty y^*$, and hence $y \in_\infty c$. As a result,

$$y \subseteq_\infty a \to y \in_\infty c\,. \tag{15.21}$$

Conversely, suppose $y \in_\infty c$. Then we have

$$\begin{aligned}
\beta \prec \rho^* \wedge y =_\beta z \wedge z \in c &\rightarrow z \subseteq_\infty a \\
\beta \prec \rho^* \wedge y =_\beta z \wedge z \in c &\rightarrow y \subseteq_\infty a \\
\exists \beta \prec \rho^* \, \exists z \in c \; y =_\beta z &\rightarrow y \subseteq_\infty a \\
\neg\neg \, \exists \beta \prec \rho^* \, \exists z \in c \; y =_\beta z &\rightarrow y \subseteq_\infty a \\
y \in_\infty c &\rightarrow y \subseteq_\infty a \,.
\end{aligned} \tag{15.22}$$

Equations 15.21 and 15.22 imply that the Powerset axiom holds in $(V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty)$. □

Lemma 15.20. **CZF** + **Pow**$^{\neg\neg}$ $\vdash (V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty) \models$ *(Infinity)*^{N}.

Proof. By recursion on $n \in \omega$ define

$$\begin{aligned}
0^* &:= \emptyset \\
(n+1)^* &:= \{u \in V_{n+1}^{\neg\neg} \mid \neg\neg(u \in_\infty n^* \vee, \exists\, u =_\infty n^*)\} \\
\omega^* &:= \{u \in V_{\omega}^{\neg\neg} \mid \neg\neg \, \exists n \in \omega \, u =_\infty n^*\} \,.
\end{aligned}$$

By induction on n one readily verifies that $n^* \in V_{n+1}^{\neg\neg}$. Also $\omega^* \in V_{\omega+1}^{\neg\neg}$. Moreover, it is by now routine (though tedious) to verify that the following statement holds in $(V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty)$:

$$\begin{aligned}
&\forall x \, [x \in \omega^* \leftrightarrow \neg\neg \, (\neg\neg \, \exists u \, u \in x \vee, \exists \, \neg\neg \, \exists y \, [y \in \omega^* \wedge \forall v \\
&\quad (v \in_\infty x \leftrightarrow \neg\neg(v \in y \vee, \exists \, v = y))])] \,.
\end{aligned} \tag{15.23}$$

It is a consequence of (15.23) that the N-translation of the Infinity axiom holds in $(V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty)$. □

Lemma 15.21. **CZF** + **Pow**$^{\neg\neg}$ $\vdash (V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty) \models$ *(V_ρ exists)*^{N}.

Proof. The elements of the ordinal representation system $\mathcal{OT}(\vartheta) = (\mathrm{OT}(\vartheta) \cap \Omega, \prec)$ are elements of ω. In the proof of Lemma 15.20 we defined the internalization $n^* \in V_{n+1}^{\neg\neg}$ of $n \in \omega$ in the structure $(V_{\rho^*}^{\neg\neg}, \in_\infty, =_\infty)$. We will now define the internalization $H(\alpha)$ of the ordered pair $\langle \alpha^*, V_\alpha^{\neg\neg} \rangle$ for each $\alpha \prec \rho$. Recall that we chose ρ to be of the form ω^{ρ_0} for some $\rho_0 \succ 1$, so that for all $\alpha \prec \rho$, $\alpha + \omega \prec \rho$. For $x \in V_{\rho^*}^{\neg\neg}$ we use $x \in_\infty OP(\alpha^*, V_\alpha^{\neg\neg})$ to abbreviate the following formula:

$$\begin{aligned}
&\neg\neg \, [\forall v \in V_{\rho^*}^{\neg\neg} (v \in_\infty x \leftrightarrow v =_\infty \alpha^*) \vee, \exists \, \forall v \in V_{\rho^*}^{\neg\neg} \\
&\quad (v \in_\infty x \leftrightarrow \neg\neg \, (v =_\infty \alpha^* \vee, \exists \, v =_\infty V_\alpha^{\neg\neg}))] \,.
\end{aligned}$$

For $\alpha \prec \rho$ define

$$H(\alpha) := \{x \in V^{\neg\neg}_{\omega+\alpha+2} \mid x \in_\infty OP(\alpha^*, V^{\neg\neg}_\alpha)\}$$
$$V^*_\rho := \{z \in V^{\neg\neg}_\rho \mid \neg\neg\, \exists\alpha \prec \rho\; z =_\infty H(\alpha)\}\,.$$

One readily checks that $H(\alpha) \in V^{\neg\neg}_{\omega+\alpha+3}$ and $V^*_\rho \in V^{\neg\neg}_{\rho+1}$. It remains to show that V^*_ρ is the set witnessing that $(V^{\neg\neg}_{\rho^*}, \in_\infty, =_\infty) \models (V_\rho \text{ exists})^N$ holds. This is so by design of V^*_ρ but it is rather tedious to check in detail. □

15.7.2 Proof of Theorem 15.1

We use $\le$ and $\equiv$ for the relations of being proof-theoretically reducible and proof-theoretically equivalent, respectively. We have $\mathbf{CZF}_{\mathcal{P}} + \mathbf{RDC} + \mathbf{\Pi\Sigma-AC} \le \mathbf{CZF}_{\mathcal{P}} \le \mathbf{IKP}(\mathcal{P})$ using Theorem 15.6 and Theorem 15.2. By Lemma 15.4 and Theorem 15.3 we get $\mathbf{CZF}_{\mathcal{P}} \le \mathcal{T}^{-} \le \mathbf{KP}(\mathcal{P})$. $\mathbf{CZF} + \mathbf{Pow}^{\neg\neg} \le \mathbf{MLV_P} \le \mathbf{CZF}_{\mathcal{P}}$ holds by Theorems 15.5 and 15.4. Theorems 15.8 and 15.7 yield $\mathbf{KP}(\mathcal{P}) \le \mathbf{Z} + \{\text{‘}V_\tau \text{ exists’}\}_{\tau\in\mathrm{BH}} \le \mathbf{CZF} + \mathbf{Pow}^{\neg\neg}$. Moreover, $\mathbf{IKP}(\mathcal{P}) \equiv \mathbf{IZ} + \{\text{‘}V_\tau \text{ exists’}\}_{\tau\in\mathrm{BH}}$ holds by Theorem 15.7. The upshot of these results is thus that all theories of Theorem 15.1 are proof-theoretically equivalent. □

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