Lifschitz Realizability for Intuitionistic Zermelo-Fraenkel Set Theory

RAY-MING CHEN, School of Mathematics, University of Leeds
Leeds LS2 9JT, UK, E-mail: rmc@maths.leeds.ac.uk

MICHAEL RATHJEN, School of Mathematics, University of Leeds
Leeds LS2 9JT, UK, E-mail: rathjen@maths.leeds.ac.uk

Abstract

A variant of realizability for Heyting arithmetic which validates Church's thesis with uniqueness condition, but not the general form of Church's thesis, was introduced by V. Lifschitz in [15]. A Lifschitz counterpart to Kleene's realizability for functions (in Baire space) was developed by van Oosten [19]. In that paper he also extended Lifschitz' realizability to second order arithmetic. The objective here is to extend it to full intuitionistic Zermelo-Fraenkel set theory, **IZF**. The machinery would also work for extensions of **IZF** with large set axioms. In addition to separating Church's thesis with uniqueness condition from its general form in intuitionistic set theory, we also obtain several interesting corollaries. The interpretation repudiates a weak form of countable choice, $\mathbf{AC}_{\omega,\omega}$, asserting that a countable family of inhabited sets of natural numbers has a choice function. $\mathbf{AC}_{\omega,\omega}$ is validated by ordinary Kleene realizability and is of course provable in **ZF**. On the other hand, a pivotal consequence of $\mathbf{AC}_{\omega,\omega}$, namely that the sets of Cauchy reals and Dedekind reals are isomorphic, remains valid in this interpretation.

Another interesting aspect of this realizability is that it validates the *lesser limited principle of omniscience.*

MSC:03E25,03E35,03E70,03F25, 03F35,03F50,03F55,03F60

Keywords: Intuitionistic set theory, Lifschitz realizability, Church's thesis, countable axiom of choice, lesser limited principle of omniscience

1 Introduction

In the constructive context, Church's thesis refers to the viewpoint that quantifier combinations $\forall x \exists y$ can be replaced by recursive functions getting y from x. Dragalin [8] pointed out that there are two formal versions of Church's thesis one could consider adding to Heyting arithmetic **HA**:

$$\mathbf{CT}_0 \qquad \forall x \exists y\, A(x,y) \to \exists z \forall x\, [z \bullet x \downarrow \wedge A(x, z \bullet x)]$$

$$\mathbf{CT}_0! \qquad \forall x \exists ! y\, A(x,y) \to \exists z \forall x\, [z \bullet x \downarrow \wedge A(x, z \bullet x)]$$

(we write $z \bullet x$ for $\{z\}(x)$), and he posed the question whether the latter version is actually weaker than the former. The question was answered affirmatively in 1979 by Vladimir Lifschitz [15]. He introduced a modification of Kleene's realizability that validates $\mathbf{CT}_0!$ but falsifies instances of $\mathbf{CT}_0$. A Lifschitz counterpart to Kleene's realizability for functions (in Baire space) was developed by van Oosten [19]. In that paper he also extended Lifschitz' realizability to second order arithmetic. The objective here is to extend Lifschitz' realizability to full intuitionistic Zermelo-Fraenkel set theory, **IZF**. In addition to separating Church's thesis with uniqueness condition from its general form in intuitionistic set theory, we also obtain several interesting corollaries. The interpretation repudiates a weak form of countable choice, $\mathbf{AC}_{\omega,\omega}$, asserting that a countable family of inhabited sets of natural numbers has a choice function. $\mathbf{AC}_{\omega,\omega}$ is validated by ordinary Kleene realizability and is of course provable in **ZF**.

Definition: 1.1 Before we can describe the pivotal features of Lifschitz' notion of realizability we need to introduce some terminology. Variables $n, m, l, i, j, k, l, e, d, f, g, p, q$ range over numbers. We assume a bijective primitive recursive pairing function $\jmath : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and inverses $\jmath_1$ and $\jmath_2$. The symbol $\bullet$ denotes partial recursive application, T is Kleene's predicate (so $n \bullet k \downarrow$ iff $\exists m\, \mathrm{T}(n, k, m)$, read $n \bullet k$ is *defined*), and U the result-extracting function. $e \bullet k \simeq l$ stands for $\exists m\, \mathrm{T}(e, k, m)$ and $l = \mathrm{U}(\mu m.\mathrm{T}(n, k, m))$, where μ is the minimalization operator. If X is a set we write $e \bullet k \in X$ instead of $\exists l\,(e \bullet k \simeq l \wedge l \in X)$.

If f is an $n+1$-ary partial recursive function, we use $\lambda x.f(x, k_1, \ldots, k_n)$ to denote an index (usually provided by the S-m-n theorem) of the function $m \mapsto f(m, k_1, \ldots, k_n)$.

The main idea behind separating $\mathbf{CT}_0$ from $\mathbf{CT}_0!$ is to find a property P of pairs of numbers so that if there is a unique n such that $P(e, n)$ holds then there is an effective procedure to find n from e, while in general there is no such procedure if $\{m \mid P(e, m)\}$ contains more than one element. Lifschitz singled out the property $n \leq \jmath_2 e \wedge \forall m \neg \mathrm{T}(\jmath_1 e, n, m)$.

Lemma: 1.2 *Letting*

$$\mathrm{D}_e \quad := \quad \{n \leq \jmath_2 e \mid \forall m \neg \mathrm{T}(\jmath_1 e, n, m)\} \tag{1}$$

there is no index g of a partial recursive function such that, for all e,

$$\mathrm{D}_e \neq \emptyset \;\Rightarrow\; g \bullet e \in \mathrm{D}_e. \tag{2}$$

Proof: This can be seen as follows. Let W_f and W_h be two disjoint, recursively inseparable r.e. sets. There is a total recursive function F such that

$$\forall n[F(n) \bullet 0 \simeq f \bullet n \wedge F(n) \bullet 1 \simeq h \bullet n],$$

letting $F(n) := \lambda x.H(n, x)$ where

$$H(n,x) \quad \simeq \quad \begin{cases} f \bullet n \text{ if } x = 0 \\ h \bullet n \text{ if } x > 0 \end{cases}$$

Then for all x, $\mathrm{D}_{\jmath(F(x),1)} \neq \emptyset$. As a result, if (2) held, $g \bullet \jmath(F(x), 1) \in \mathrm{D}_{\jmath(F(x),1)}$ and g would provide a recursive separation of W_f and W_h. $\square$

If, on the other hand, we know that D_e is a singleton, then we try to compute $(\jmath_1 e) \bullet 0, (\jmath_1 e) \bullet 1, \ldots, (\jmath_1 e) \bullet (\jmath_2 e)$ simultaneously and as soon as the $(\jmath_2 e) - 1$ many (guaranteed) successes have been recorded we know that the remaining one failure is the unique element of D_e.

1.1 Realizability for set theories

Realizability semantics for intuitionistic theories were first proposed by Kleene in 1945 [12]. Inspired by Kreisel's and Troelstra's [14] definition of realizability for higher order Heyting arithmetic, realizability was first applied to systems of set theory by Myhill [18] and Friedman [9]. More recently, realizability models of set theory were investigated by Beeson [3, 5] (for non-extensional set theories) and McCarty [16, 17] (directly for extensional set theories). Rathjen [22] adapted realizability to the context of constructive Zermelo-Fraenkel set theory, **CZF**, and developed hybrids [23, 24] which combine realizability for extensional set theory with truth in order to prove metamathematical properties of intuitionistic set theories such as the disjunction and the numerical existence property.

The authors of the present paper had problems making up their mind as to whether to present **IZF** as a pure system of set theory or to opt for a language with urelements as it is done in Friedman's and Beeson's work (cf. [10, 5]). Both approaches have advantages and disadvantages. The disadvantage of pure set theory is that the natural numbers have to be encoded as finite ordinals, rendering the presentation of the basic parts of Lifschitz' realizability for atomic formulas, which are trivial in the arithmetic context, very cumbersome. The disadvantage of having a sorted language with numbers and sets is that realizability for those theories has never been worked out properly in the extensional cases. In the end we went for the latter choice.

1.2 IZF with urelements

We will formalize **IZF** in a similar manner as in [5, chap.viii] by having two unary predicates for natural numbers and for sets. We shall however eschew terms other than variables and constants by avoiding symbols for primitive recursive functions. Instead we will have symbols for primitive recursive relations. This makes the axiomatization of the arithmetic part a bit awkward (albeit still a straightforward affair) but relieves us from the burden of having to deal with complex terms in the realizability interpretation.

1.3 Logic and language

IZF is based on first-order intuitionistic predicate calculus with equality $=$. The language consists of the following. A binary predicate $\in$; unary predicates N and S (for numbers and sets); for each natural number n a constant $\bar{n}$ (but we omit the bar when $n = 0$); a 2-place relation symbol SUC (for the successor relation), two 3-place relation symbols

ADD, MULT (for the graphs of addition and multiplication), and further relation symbols for all primitive recursive relations.

To alleviate the burden of syntax we shall use variables n, m, k, l, i, j to range over natural numbers, so $\exists n \ldots$ and $\forall n \ldots$ will be abbreviations for $\exists x(\mathrm{N}(x) \wedge \ldots)$ and $\forall x(\mathrm{N}(x) \to \ldots)$, respectively. $\exists ! n A(n)$ is short for $\exists n\, A(n) \wedge \forall n \forall m[A(n) \wedge A(m) \to n = m]$. $x \notin y$ stands for $\neg(x \in y)$. $x \subseteq y$ abbreviates $\forall z(z \in x \to z \in y)$. We use $\forall x \in y \ldots$ and $\exists x \in y \ldots$ for $\forall x(x \in y \to \ldots)$ and $\exists x(x \in y \wedge \ldots)$, respectively.

Definition: 1.3 We list the axioms of **IZF** in groups:

A. Axioms on Numbers and Sets

1. $\forall x\, \neg(\mathrm{N}(x) \wedge \mathrm{S}(x))$
2. $\forall x \forall y(x \in y \to \mathrm{S}(y))$
3. $\mathrm{N}(\bar{n})$ for all natural numbers n.

B. Number-Theoretic Axioms

1. $\mathrm{SUC}(\bar{n}, \overline{n+1})$ for all naturals n.
2. $\forall n\, \exists ! m\, \mathrm{SUC}(n, m)$
3. $\forall n \forall m[\mathrm{SUC}(n, m) \to m \neq 0]$
4. $\forall m\, [m = 0 \vee \exists n\, \mathrm{SUC}(n, m)]$
5. $\forall n \forall m \forall k\, (\mathrm{SUC}(m, n) \wedge \mathrm{SUC}(k, n) \to m = k)$
6. $\forall n \forall m\, \exists ! k\, \mathrm{ADD}(n, m, k)$
7. $\forall n\, \mathrm{ADD}(n, 0, n)$
8. $\forall n \forall k \forall m \forall l \forall i\, [\mathrm{ADD}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{SUC}(m, i) \to \mathrm{ADD}(n, l, i)]$
9. $\forall n \forall m\, \exists ! k\, \mathrm{MULT}(n, m, k)$
10. $\forall n\, \mathrm{MULT}(n, 0, 0)$
11. $\forall n \forall k \forall m \forall l \forall i\, [\mathrm{MULT}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{ADD}(m, n, i) \to \mathrm{MULT}(n, l, i)]$
12. Defining axioms for all symbols of primitive recursive relations R. These are similar to the above. We spare the reader the details.
13. $A(0) \wedge \forall n \forall m[A(n) \wedge \mathrm{SUC}(n, m) \to A(m)] \to \forall n A(n)$

C. Set-Theoretic Axioms

1. Extensionality. $\forall x \forall y(\mathrm{S}(x) \wedge \mathrm{S}(y) \to [\forall z(z \in x \leftrightarrow z \in y) \to x = y])$
2. Pairing. $\forall x \forall y(\exists u[\mathrm{S}(u) \wedge x \in u \wedge y \in u])$

3. Union. $\forall x \exists u[\mathrm{S}(u) \wedge \forall z(z \in u \leftrightarrow \exists y(y \in x \wedge z \in y))]$

4. Separation. $\forall x \exists u[\mathrm{S}(u) \wedge \forall z(z \in u \leftrightarrow z \in x \wedge A(z))]$
   (u not free in $A(z)$)

5. Power set. $\forall x \exists u[\mathrm{S}(u) \wedge \forall z(z \in u \leftrightarrow (\mathrm{S}(z) \wedge z \subseteq x))]$

6. Infinity. $\exists u(\mathrm{S}(u) \wedge \forall z[z \in u \leftrightarrow \mathrm{N}(u)])$.

7. $\in$-induction. $\forall x[\forall y(y \in x \to A(y)) \to A(x)] \to \forall x A(x)$.

8. Collection. $\forall y \in x \exists z\, A(x, z) \to \exists u[\mathrm{S}(u) \wedge \forall y \in x\, \exists z \in u\, A(y, z)]$

Remark 1.4 The theory **IZF** in [5] comes with the additional axiom $\forall x[\mathrm{N}(x) \vee \mathrm{S}(x)]$. We could have adopted this axiom as well. The reason for not including it is that on the one hand this axioms does not make the theory stronger but on the other hand it would force us to define a more complicated realizability structure in which all objects carry a label which tells one whether it denotes a set or a number. This would have to be done in a hereditary way and would thus burden us with an extra layer of coding. A proof that $\mathbf{IZF} + \forall x[\mathrm{N}(x) \vee \mathrm{S}(x)]$ can be interpreted in **IZF** using hereditarily labelled sets is sketched in [5, VIII.1]. Moreover, the same techniques can also be used to interpret **IZF** in pure **IZF** without urelements, $\mathbf{IZF_0}$ (cf. [5, VIII.1]). $\mathbf{IZF_0}$ has only the binary predicate $\in$ (no N, no S and no symbols for primitive recursive relations). In **IZF** we define the *pure sets* as those whose transitive closure contains only sets. Let Pure be the class of pure sets. To every formula A of $\mathbf{IZF_0}$ we assign a formula A^{Pure} of **IZF** which is obtained by relativizing all quantifiers to Pure. Then the exact relationship between the two theories is that

$$\mathbf{IZF_0} \vdash A \Leftrightarrow \mathbf{IZF} \vdash A^{\mathrm{Pure}}.$$

2 The realizability structure

In what follows we shall be arguing informally in a classical set theory with urelements where the urlements are the natural numbers (e.g. **IZF** plus classical logic). The unique set of natural numbers provided by the Infinity axiom will be denoted by $\mathbb{N}$.

Definition: 2.1 Ordinals are transitive sets whose elements are transitive also. We use lower case Greek letters to range over ordinals. By recursion on α define

$$\mathrm{V}^{set}_{\alpha} = \bigcup_{\beta \in \alpha} \mathcal{P}(\mathbb{N} \times (\mathrm{V}^{set}_{\beta} \cup \mathbb{N})). \tag{3}$$

$$\mathrm{V}^{set} = \bigcup_{\alpha} \mathrm{V}^{set}_{\alpha}. \tag{4}$$

$$\mathrm{V}(L) = \mathbb{N} \cup \mathrm{V}^{set} \tag{5}$$

where $\mathcal{P}(x)$ denotes the power set of x.

Lemma: 2.2 *(i) The hierarchy* V^{set} *is cumulative: if* $\alpha \leq \beta$ *then* $\mathrm{V}^{set}_{\alpha} \subseteq \mathrm{V}^{set}_{\beta}$.

(ii) If $x \subseteq \mathrm{V}(L)$ *and* $\mathrm{S}(x)$ *then* $x \in \mathrm{V}^{set}$.

(iii) Every $x \in \mathrm{V}^{set}$ *is a set, i.e.* $\mathrm{S}(x)$ *holds.*

(iv) $\forall x \in \mathrm{V}(L)\,[\mathrm{N}(x) \vee \mathrm{S}(x)]$.

Proof: (i) is immediate by (3). For (iii) note that if $x \in \mathrm{V}^{set}$ then $x \in \mathcal{P}(\mathbb{N} \times (\mathrm{V}^{set}_{\beta} \cup \mathbb{N}))$ for some β. So the claim follows from our rendering of the power set axiom which ensures that $\mathcal{P}(y)$ consists only of sets. (iv) follows from (iii).

(ii): If $x \subseteq \mathbb{N} \times \mathrm{V}(L)$ then, using strong collection and (i), there is an α such that $x \subseteq \mathbb{N} \times \mathrm{V}^{set}_{\alpha} \cup \mathbb{N}$, so $x \in \mathrm{V}^{set}_{\alpha+1}$, thus $x \in \mathrm{V}^{set}$. For a more detailed proof see [22, Lemma 3.5]. □

3 Defining Lifschitz realizability for set theory

We adopt the conventions and notations from Definition 1.1.

Definition: 3.1 Let $a, a_i, b \in \mathrm{V}(L)$ and $e \in \mathbb{N}$. Below R is a symbol for an n-ary primitive recursive relation. Recall that $\mathrm{D}_e = \{n \leq \jmath_2 e \mid \forall m \neg \mathrm{T}(\jmath_1 e, n, m)\}$.

We define a relation $e \Vdash_L B$ between naturals e and sentences of **IZF** with parameters from $\mathrm{V}(L)$. $e \bullet f \Vdash_L B$ will be an abbreviation for $\exists k[e \bullet f \simeq k \wedge k \Vdash_L B]$.

$$
\begin{array}{rcl}
e \Vdash_L \mathrm{R}(a_1, \ldots, a_n) & \text{iff} & a_1, \ldots, a_n \in \mathbb{N} \wedge \mathrm{R}(a_1, \ldots, a_n) \\
e \Vdash_L \mathrm{N}(a) & \text{iff} & a \in \mathbb{N} \wedge e = a \\
e \Vdash_L \mathrm{S}(a) & \text{iff} & \mathrm{S}(a) \quad (\text{iff } a \in \mathrm{V}^{set}) \\
e \Vdash_L a \in b & \text{iff} & \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e) \exists c\, [\langle \jmath_1 d, c \rangle \in b \wedge \jmath_2 d \Vdash_L a = c] \\
e \Vdash_L a = b & \text{iff} & (a, b \in \mathbb{N} \wedge a = b) \text{ or } (\mathrm{D}_e \neq \emptyset \wedge \mathrm{S}(a) \wedge \mathrm{S}(b) \wedge \\
 & & (\forall d \in \mathrm{D}_e) \forall f, c\, [\langle f, c \rangle \in a \rightarrow (\jmath_1 d) \bullet f \Vdash_L c \in b] \wedge \\
 & & (\forall d \in \mathrm{D}_e) \forall f, c\, [\langle f, c \rangle \in b \rightarrow (\jmath_2 d) \bullet f \Vdash_L c \in a]) \\
e \Vdash_L A \wedge B & \text{iff} & \jmath_1 e \Vdash_L A \wedge \jmath_2 e \Vdash_L B \\
e \Vdash_L A \vee B & \text{iff} & \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e)([\jmath_1 d = 0 \wedge \jmath_2 d \Vdash_L A] \vee \\
 & & \qquad [\jmath_1 d \neq 0 \wedge \jmath_2 d \Vdash_L B]) \\
e \Vdash_L \neg A & \text{iff} & (\forall f \in \mathbb{N})\, \neg f \Vdash_L A \\
e \Vdash_L A \rightarrow B & \text{iff} & (\forall f \in \mathbb{N})\, [f \Vdash_L A \rightarrow e \bullet f \Vdash_L B] \\
e \Vdash_L \forall x A & \text{iff} & \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e)(\forall c \in \mathrm{V}(L))\, d \Vdash_L A[x/c] \\
e \Vdash_L \exists x A & \text{iff} & \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e)(\exists c \in \mathrm{V}(L))\, d \Vdash_L A[x/c] \\
\\
\mathrm{V}(L) \models B & \text{iff} & (\exists e \in \mathbb{N})\, e \Vdash_L B.
\end{array}
$$

Notice that the definitions of $e \Vdash a \in b$ and $e \Vdash_L a = b$ fall under the scope of definition by transfinite recursion.

4 Recursion-theoretic preliminaries

Before we can prove the soundness of Lifschitz' realizability for **IZF** we need to recall some recursion-theoretic facts, mainly Lemmata 1–5 from Lifschitz' paper [15]. Van Oosten has carried out a detailed analysis of these results by singling out the extra amount of classical logic one has to add to intuitionistic first-order arithmetic **HA** to prove them.

Definition: 4.1 $\mathrm{MP}_{\mathrm{pr}}$ is Markov's principle for primitive recursive formulae A:

$$\neg\neg\exists n\, A(n) \to \exists n\, A(n).$$

$\mathrm{B}\Sigma^0_2$-MP is Markov's principle for bounded Σ^0_2-formulae:

$$\neg\neg\exists n \leq m\, \forall k\, A(n,k,e) \to \exists n \leq m\, \forall k\, A(n,k,e)$$

for A primitive recursive.

Lemma: 4.2 *There is a total recursive function* $\mathfrak{sg}$ *such that*

$$\mathbf{HA} \vdash \forall n \forall m (m \in \mathrm{D}_{\mathfrak{sg}(n)} \leftrightarrow m = n).$$

Proof: [15, Lemma 2] and [19, Lemma 2.2]. □

Lemma: 4.3 *There is a partial recursive function* ϕ *such that*

$$\mathbf{HA} + \mathrm{MP}_{\mathrm{pr}} \vdash \forall e [\exists n \forall m (m \in \mathrm{D}_e \leftrightarrow m = n) \to \phi(e)\downarrow \wedge\, \phi(e) \in \mathrm{D}_e].$$

Proof: [15, Lemma 1] and [19, Lemma 2.3]. □

Lemma: 4.4 *There is a partial recursive function* Φ *such that* $\mathbf{HA} + \mathrm{MP}_{\mathrm{pr}} + \mathrm{B}\Sigma^0_2$-MP *proves that for all* e *and* f *whenever* $(\forall g \in \mathrm{D}_e)\, f \bullet g\downarrow$ *then* $\Phi(e,f)\downarrow$ *and*

$$\forall h [h \in \mathrm{D}_{\Phi(e,f)} \leftrightarrow (\exists g \in \mathrm{D}_e)\, h = f \bullet g].$$

Proof: [15, Lemma 4] and [19, Lemma 2.4]. □

Lemma: 4.5 *There is a total recursive function* $\mathfrak{un}$ *such that* $\mathbf{HA} + \mathrm{MP}_{\mathrm{pr}} + \mathrm{B}\Sigma^0_2$-MP *proves that*

$$\forall e \forall h [h \in \mathrm{D}_{\mathfrak{un}(e)} \leftrightarrow (\exists g \in \mathrm{D}_e)(h \in \mathrm{D}_g)].$$

In other words, $\mathrm{D}_{\mathfrak{un}(e)} = \bigcup_{g \in \mathrm{D}_e} \mathrm{D}_g$.

Proof: [15, Lemma 3] and [19, Lemma 2.5]. □

Lemma: 4.6 *Let $\vec{x} = x_1, \ldots, x_r$ and $\vec{a} = a_1, \ldots, a_r$. To each formula $A(\vec{x})$ of* **IZF** *(with all free variables among $\vec{x}$) we can effectively assign (a code of) a partial recursive function χ_A such that, letting* $\mathbf{IZF}' := \mathbf{IZF} + \mathrm{MP}_{\mathrm{pr}} + \mathrm{B}\Sigma^0_2\text{-MP}$,

$$\mathbf{IZF}' \vdash (\forall e \in \mathbb{N})(\forall \vec{a} \in \mathrm{V}(L))[\mathrm{D}_e \neq \emptyset \wedge ((\forall d \in \mathrm{D}_e)\, d \Vdash_L A(\vec{a})) \rightarrow \chi_A(e) \Vdash_L A(\vec{a})].$$

Proof: This is similar to [15, Lemma 5] and [19, Lemma 2.6]. However, due to the vastly more complicated setting we are dealing with here, we provide a detailed proof. We use induction on the buildup of A.

If $A(\vec{x})$ is of the form $\mathrm{N}(x_i)$, define $\chi_A(e) := \phi(e)$, where ϕ is from Lemma 4.3. To see that this works note that $\mathrm{D}_e \neq \emptyset$ and for all $(\forall d \in \mathrm{D}_e)\, d \Vdash_L \mathrm{N}(a_i)$ entails that $\mathrm{N}(a_i)$ and $\mathrm{D}_e = \{a_i\}$, thus $\phi(e) = a_i$ and $\phi(e) \Vdash_L \mathrm{N}(a_i)$ follow by Lemma 4.3.

If $A(\vec{x})$ is of either form $\mathrm{S}(x_j)$ or $R(\vec{t})$ let $\chi_A(e) := 0$.

If $A(\vec{x})$ is of the form $x_i = x_j$ let $\chi_A(e) := \mathfrak{un}(e)$, where $\mathfrak{un}$ stems from Lemma 4.5. Note that $\mathfrak{un}$ is a total recursive function. To see that this works assume that $\mathrm{D}_e \neq \emptyset$ and for all $(\forall d \in \mathrm{D}_e)\, d \Vdash_L a_i = a_j$. Now, either $a_i, a_j \in \mathbb{N}$ or a_i and a_j are both sets. In the former case we then have $a_i = a_j$ and for any $n \in \mathbb{N}$, $n \Vdash_L a_i = a_j$, so in particular $\mathfrak{un}(e) \Vdash_L a_i = a_j$. If both a_i and a_j are sets, then $\mathfrak{un}(e) \Vdash_L a_i = a_j$ holds owing to Lemma 4.5 and the definition of realizability in this case.

Let $A(\vec{x})$ be $B(\vec{x}) \wedge C(\vec{x})$ and χ_B and χ_C be already defined. Let $\jmath_1^*$ and $\jmath_2^*$ be indices for $\jmath_1$ and $\jmath_2$, respectively. Consider the set $\mathrm{D}_{\Phi(\jmath_1^*, e)} = \{\jmath_1 n \mid n \in \mathrm{D}_e\}$ with Φ as in Lemma 4.4. If D_e is non-empty then so is $\mathrm{D}_{\Phi(\jmath_1^*, e)}$. If every element of D_e realizes $A(\vec{a})$ then every element of $\mathrm{D}_{\Phi(\jmath_1^*, e)}$ realizes $B(\vec{a})$. Hence under these assumptions $\chi_B(\Phi(\jmath_1^*, e))$ realizes $B(\vec{a})$. Similarly, $\chi_C(\Phi(\jmath_2^*, e))$ realizes $C(\vec{a})$. Hence the claim follows with $\chi_A(e) := \jmath(\chi_B(\Phi(\jmath_1^*, e)), \chi_C(\Phi(\jmath_2^*, e)))$.

Let $A(\vec{x})$ be $B(\vec{x}) \rightarrow C(\vec{x})$ and χ_B and χ_C be already defined. Let θ be a partial recursive function such that $(\theta(m)) \bullet k \simeq k \bullet m$. Assume that $\mathrm{D}_e \neq \emptyset$. Suppose $m \Vdash_L B(\vec{a})$. Then $d \bullet m \downarrow$ and $d \bullet m \Vdash_L C(\vec{a})$ for all $d \in \mathrm{D}_e$. Thus, by Lemma 4.4, we have $\mathrm{D}_{\Phi(e, \theta(m))} = \{d \bullet m \mid d \in \mathrm{D}_e\}$. Moreover, $\mathrm{D}_{\Phi(e, \theta(m))}$ is non-empty (since $\mathrm{D}_e \neq \emptyset$) and every of its elements realizes $C(\vec{a})$, hence, by the inductive assumption, $\chi_C(\Phi(e, \theta(m)))$ realizes $C(\vec{a})$. Thus we may define $\chi_A(e) := \lambda m.\chi_C(\Phi(e, \theta(m)))$.

In all the remaining cases $\chi_A(e) := \mathfrak{un}(e)$ will work owing to Lemma 4.5 and the definition of realizability in these cases. □

The next result shows that our definition of realizability for arithmetic formulae coincides with the one given by Lifschitz [15].

Lemma: 4.7 *For every formula $A(u, \vec{x})$ there are partial recursive functions ψ_1 and ψ_2 such that provably in* $\mathbf{IZF}'$ *we have for all $e \in \mathbb{N}$ and $\vec{a} \in \mathrm{V}(L)$:*

(i) $e \Vdash_L \forall x[\mathrm{N}(x) \rightarrow A(x, \vec{a})] \;\rightarrow\; \forall n\, \psi_1(e) \bullet n \Vdash_L A(n, \vec{a})$;

(ii) $\forall n\, e \bullet n \Vdash_L A(n, \vec{a}) \;\rightarrow\; \psi_2(e) \Vdash_L \forall x[\mathrm{N}(x) \rightarrow A(x, \vec{a})]$;

(iii) $e \Vdash_L \exists x[\mathrm{N}(x) \wedge A(x, \vec{a})] \;\leftrightarrow\; \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e) \jmath_2 d \Vdash_L A(\jmath_1 d, \vec{a})$.

Proof: (i). Suppose $e \Vdash_L \forall x[\mathrm{N}(x) \to A(x, \vec{a})]$. Then $\mathrm{D}_e \neq \emptyset$ and for all $d \in \mathrm{D}_e$ and $n \in \mathbb{N}$, $d \bullet n \Vdash_L A(n, \vec{a})$. Thus, if we define f_n such that $f_n \bullet d \simeq d \bullet n$, we conclude with the aid of Lemma 4.4 that for all $n \in \mathbb{N}$ and $h \in \mathrm{D}_{\Phi(e,f_n)}$, $h \Vdash_L A(n, \vec{a})$. Hence, by Lemma 4.6, $(\forall n \in \mathbb{N})\, \chi_A(\Phi(e, f_n)) \Vdash_L A(n, \vec{a})$. So we can define ψ_1 by letting $\psi_1(e) := \lambda n.\chi_A(\Phi(e, f_n))$.

(ii). Suppose $\forall n\, e \bullet n \Vdash_L A(n, \vec{a})$. Then $e \Vdash_L \mathrm{N}(x) \to A(x, \vec{a})$ for all $x \in \mathrm{V}(L)$, hence $\mathfrak{sg}(e) \Vdash_L \forall x[\mathrm{N}(x) \to A(x, \vec{a})]$, so $\psi_2(n) := \mathfrak{sg}(n)$ will work.

(iii). Suppose $e \Vdash_L \exists x[\mathrm{N}(x) \wedge A(x, \vec{a})]$. Then $\mathrm{D}_e \neq \emptyset$ and for all $d \in \mathrm{D}_e$ there exists $c \in \mathrm{V}(L)$ such that $\jmath_1 d \Vdash_L \mathrm{N}(c)$ and $\jmath_2 d \Vdash_L A(c, \vec{a})$. But $\jmath_1 d \Vdash_L \mathrm{N}(c)$ entails that $c = \jmath_1 d$, thus $\jmath_2 d \Vdash_L A(\jmath_1 d, \vec{a})$. The converse is obvious. □

4.1 The soundness theorem for intuitionistic predicate logic with equality

Lemma: 4.8 *There are* $\mathbf{i_r}, \mathbf{i_s}, \mathbf{i_t}, \mathbf{i_0}, \mathbf{i_1} \in \mathbb{N}$ *such that for all* $x, y, z \in \mathrm{V}(L)$,

1. $\mathbf{i}_r \Vdash_L x = x$.
2. $\mathbf{i}_s \Vdash_L x = y \to y = x$.
3. $\mathbf{i}_t \Vdash_L (x = y \wedge y = z) \to x = z$.
4. $\mathbf{i}_0 \Vdash_L (x = y \wedge y \in z) \to x \in z$.
5. $\mathbf{i}_1 \Vdash_L (x = y \wedge z \in x) \to z \in y$.
6. *Moreover, for each formula* $A(v, u_1, \ldots, u_r)$ *of* **IZF** *all of whose free variables are among* $v, u_1, \ldots, u_r$ *there exists* $\mathbf{i}_A \in \mathbb{N}$ *such that for all* $x, y, z_1, \ldots, z_r \in \mathrm{V}(L)$,

$$\mathbf{i}_A \Vdash_L x = y \wedge A(x, \vec{z}) \to A(y, \vec{z}),$$

where $\vec{z} = z_1, \ldots, z_r$.

Proof: (1) Note that $n \Vdash_L x = x$ holds for all $n, x \in \mathbb{N}$. Let $x \in \mathbb{N}$ and $a \in \mathrm{V}^{set}_\alpha$. Suppose $e \bullet 0 \downarrow$ and $e \bullet 0 \Vdash_L b = b$ holds for all $b \in \mathbb{N} \cup \bigcup_{\beta \in \alpha} \mathrm{V}^{set}_\beta$. Then we have $(\forall \langle f, b\rangle \in a)\, \mathfrak{sg}(\jmath(f, e \bullet 0)) \Vdash_L b \in a$. There is a recursive function ℓ such that $(\ell(e \bullet 0)) \bullet f \simeq \mathfrak{sg}(\jmath(f, e \bullet 0))$, and hence, by the foregoing,

$$(\forall \langle f, b\rangle \in a)\, (\jmath_1 d) \bullet f \Vdash_L b \in a$$

with $d = \jmath(\ell(e \bullet 0), \ell(e \bullet 0))$. As a result, $\mathfrak{sg}(\jmath(\ell(e \bullet 0), \ell(e \bullet 0))) \Vdash_L a = a$. By the recursion theorem there exists an e^* such that

$$e^* \bullet 0 \simeq \mathfrak{sg}(\jmath(\ell(e^* \bullet 0), \ell(e^* \bullet 0))).$$

By induction on α it therefore follows that $e^* \bullet 0 \Vdash_L a = a$ holds for all $a \in \mathrm{V}^{set}$. So we may put $\mathbf{i}_r := e^* \bullet 0$. As $\mathbf{i}_r \Vdash_L n = n$ (trivially) holds for all $n \in \mathbb{N}$, too, we get $\mathbf{i}_r \Vdash_L z = z$

for all $z \in \mathrm{V}(L)$.

(2): It is routine to check that

$$\mathbf{i}_s := \lambda e.\Phi(e, \lambda d.\jmath(\jmath_2 d, \jmath_1 d)) \Vdash_L x = y \to y = x,$$

with Φ from Lemma 4.4.

(3) and (4): We prove these simultaneously. Let $\mathbf{TC}(a)$ denote the transitive closure of a. We employ (transfinite) induction on the ordering $\lhd$ which is the transitive closure of the ordering $\lhd_1$ on ordered triples:

$$\begin{aligned} \langle x, y, z\rangle \lhd_1 \langle a, b, c\rangle \quad \text{iff} \quad & (x = a \wedge y = b \wedge z \in \mathbf{TC}(c)) \vee (x = a \wedge y \in \mathbf{TC}(b) \wedge z = c) \\ & \vee (x \in \mathbf{TC}(a) \wedge y = b \wedge z = c). \end{aligned}$$

$\lhd$-induction follows from the usual $\in$-induction.

Now suppose $a, b, c \in \mathrm{V}(L)$ and inductively assume that for all $\langle x, y, z\rangle \lhd \langle a, b, c\rangle$,

$$e^{\#} \bullet 0 \Vdash_L (x = y \wedge y = z) \to x = z \tag{6}$$

$$e^{\#} \bullet 1 \Vdash_L (x = y \wedge y \in z) \to x \in z. \tag{7}$$

Suppose $e \Vdash_L a = b \wedge b = c$. Then $\jmath_1 e \Vdash_L a = b$ and $\jmath_2 e \Vdash_L b = c$. Then either $a, b, c \in \mathbb{N}$ and for any $n \in \mathbb{N}$ we have $n \Vdash_L b = c$, or $a, b, c \in \mathrm{V}^{set}_{\alpha}$. So let's assume $a, b, c \in \mathrm{V}^{set}$. Let $d \in \mathrm{D}_{\jmath_1 e}$ and $d' \in \mathrm{D}_{\jmath_2 e}$. If $\langle f, u\rangle \in a$, then $(\jmath_1 d) \bullet f \Vdash_L u \in b$, and hence, for all $g \in \mathrm{D}_{(\jmath_1 d) \bullet f}$ there exists v such that $\langle \jmath_1 g, v\rangle \in b$ and $\jmath_2 g \Vdash_L u = v$. Moreover, $(\jmath_1 d') \bullet (\jmath_1 g) \Vdash_L v \in c$. As $\langle u, v, c\rangle \lhd \langle a, b, c\rangle$ we can employ (7) to conclude that

$$\ell_1(e^{\#}, g, d') := (e^{\#} \bullet 1) \bullet \jmath(\jmath_2 g, (\jmath_1 d') \bullet (\jmath_1 g)) \Vdash_L u \in c.$$

Using Lemmata 4.4 and 4.6 repeatedly we get

$$\ell_2(e^{\#}, f, d, d') := \chi_1(\Phi((\jmath_1 d) \bullet f, \lambda g.\ell_1(e^{\#}, g, d'))) \Vdash_L u \in c$$

$$\ell_3(e^{\#}, f, d) := \chi_2(\Phi(\jmath_2 e, \lambda d'.\ell_2(e^{\#}, f, d, d'))) \Vdash_L u \in c$$

$$\ell^*(e^{\#}, e, f) := \chi_3(\Phi(\jmath_1 e, \lambda d.\ell_3(e^{\#}, f, d))) \Vdash_L u \in c \tag{8}$$

for appropriate partial recursive functions χ_i.

Similarly one distills a partial recursive function ℓ^{**} such that for $\langle f, u\rangle \in c$,

$$\ell^{**}(e^{\#}, e, f) := \chi_3(\Phi(\jmath_2 e, \lambda d.\ell_3(e^{\#}, f, d))) \Vdash_L u \in a. \tag{9}$$

As a result of (8) and (9) we have with

$$\wp_1(e^{\#}) := (\lambda e.\mathfrak{sg}(\jmath(\lambda f.\ell^*(e^{\#}, e, f), \lambda f.\ell^{**}(e^{\#}, e, f))),$$

$$\wp_1(e^{\#}) \Vdash_L a = b \wedge b = c \to a = c. \tag{10}$$

Next suppose $e \Vdash_L a = b \wedge b \in c$. Then $\jmath_1 e \Vdash_L a = b$ and $\jmath_2 e \Vdash_L b \in c$. Hence $\mathrm{D}_{\jmath_2 e} \neq \emptyset$ and for all $d \in \mathrm{D}_{\jmath_2 e}$ there exists v such that $\langle \jmath_1 d, v\rangle \in c$ and $\jmath_2 d \Vdash_L b = v$, thus $\jmath(\jmath_1 e, \jmath_2 d) \Vdash_L a = b \wedge b = v$. As $\langle a, b, v\rangle \lhd \langle a, b, c\rangle$ we can employ (6) to conclude

$$\ell_4(e^{\#}, e, d) := (e^{\#} \bullet 0) \bullet \jmath(\jmath_1 e, \jmath_2 d) \Vdash_L a = v.$$

Letting $\ell_4(e^\#, e, d) := \jmath(\jmath_1 d, (e^\# \bullet 0) \bullet \jmath(\jmath_1 e, \jmath_2 d))$, we thus have $\langle \jmath_1(\ell_4(e^\#, e, d)), v\rangle \in c$ and $\jmath_2(\ell_4(e^\#, e, d)) \Vdash_L a = v$. Hence, by Lemma 4.4, $\Phi(\jmath_1 e, \lambda d.\ell_4(e^\#, e, d)) \Vdash_L a \in c$. So the upshot is that

$$\wp_2(e^\#) := \Phi(\jmath_1 e, \lambda d.\ell_4(e^\#, e, d)) \quad \Vdash_L \quad a = b \wedge b \in c \to a \in c. \tag{11}$$

Finally we use the recursion theorem to find an index $e^\#$ such that

$$\begin{aligned} e^\# \bullet 0 \quad &\simeq \quad \wp_1(e^\#) \\ e^\# \bullet 1 \quad &\simeq \quad \wp_2(e^\#). \end{aligned}$$

With $\mathbf{i}_t := e^\# \bullet 0$ and $\mathbf{i}_0 := e^\# \bullet 1$ the above shows that (3) and (4) are satisfied.

(5). Suppose $e \Vdash_L a = b \wedge c \in a$. Then $\jmath_1 e \Vdash_L a = b$ and $\jmath_2 e \Vdash_L c \in a$. From the latter we get that $\mathrm{D}_{\jmath_2 e} \neq \emptyset$ and for all $d \in \mathrm{D}_{\jmath_2 e}$ there exists v such that $\langle \jmath_1 d, v\rangle \in a$ and $\jmath_2 d \Vdash_L c = v$. Thus, $\mathrm{D}_{\jmath_1 e} \neq \emptyset$ and since $\jmath_1 e \Vdash_L a = b$, it follows that for all $h \in \mathrm{D}_{\jmath_1 e}$, $(\jmath_1 h) \bullet (\jmath_1 d) \Vdash_L v \in b$, so that by (4),

$$\ell_5(d, h) := \mathbf{i}_0(\jmath(\jmath_2 d, (\jmath_1 h) \bullet (\jmath_1 d)) \Vdash_L c \in b.$$

Using Lemmata 4.4 and 4.6 repeatedly we get

$$\begin{aligned} \ell_6(e, d) := \chi_3(\Phi(\jmath_1 e, \lambda h.\ell_5(d, h)) \quad &\Vdash_L \quad c \in b \\ \ell_7(e) := \chi_4(\Phi(\jmath_2 e, \lambda d.\ell_6(e, d))) \quad &\Vdash_L \quad c \in b \end{aligned}$$

for appropriate partial recursive functions χ_i. So we may put $\mathbf{i}_1 := \lambda e.\ell_7(e)$.

(6). This is shown by a routine induction on the complexity of A, the non-trivial atomic cases being provided by (2)-(5). $\square$

Corollary: 4.9 *There is a total recursive function θ such that for all $a \in \mathrm{V}(L)$,*

$$(\forall \langle f, u\rangle \in a)\, \theta(f) \Vdash_L u \in a.$$

Proof: Let

$$\theta(f) \quad := \quad \mathfrak{sg}(\jmath(f, \mathbf{i}_r)). \tag{12}$$

$\square$

Theorem: 4.10 *Let $\mathcal{D}$ be a proof in intuitionistic predicate logic with equality of a formula $A(u_1, \ldots, u_r)$ of* **IZF** *all of whose free variables are among $u_1, \ldots, u_r$. Then there is $e_{\mathcal{D}} \in \mathbb{N}$ such that* **IZF**$'$ *proves*

$$e_{\mathcal{D}} \Vdash_L \forall u_1 \ldots \forall u_r\, A(u_1, \ldots, u_r).$$

Proof: We use a standard Hilbert-type systems for intuitionistic predicate logic. The proof proceeds by induction on the derivation. The correctness of axioms and rules pertaining to the connectives $\wedge, \neg, \rightarrow$ is exactly the same as for Kleene's realizability. We have also shown realizability of the equality axioms in Lemma 4.8. So it remains to address the axioms and rules for $\vee, \forall, \exists$.

Axioms for $\vee$:

$A \rightarrow A \vee B$ or $A \rightarrow B \vee A$. Suppose $e \Vdash_L A$. As $\mathrm{D}_{\mathfrak{sg}(\jmath(0,e))} = \{\jmath(0,e)\}$ by Lemma 4.2, it follows that $\mathfrak{sg}(\jmath(0,e)) \Vdash_L A \vee B$ and hence $\lambda e.\mathfrak{sg}(\jmath(0,e)) \Vdash_L A \rightarrow A \vee B$. Similarly, $\lambda e.\mathfrak{sg}(\jmath(1,e)) \Vdash_L A \rightarrow B \vee A$.

$A \vee B \rightarrow ((A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C))$. Suppose $e \Vdash_L A \vee B$. Then $\mathrm{D}_e \neq \emptyset$. Let $d \in \mathrm{D}_e$. Then $\jmath_1 d = 0 \wedge \jmath_2 d \Vdash_L A$ or $\jmath_1 d \neq 0 \wedge \jmath_2 d \Vdash_L B$. Suppose $f \Vdash_L A \rightarrow C$ and $g \Vdash_L B \rightarrow C$. Define a partial recursive function $\mathfrak{f}$ by

$$\mathfrak{f}(d, f', g') = \begin{cases} f' \bullet (\jmath_2 d) & \text{if } \jmath_1 d = 0 \\ g' \bullet (\jmath_2 d) & \text{if } \jmath_1 d \neq 0 \end{cases}$$

Then $\mathfrak{f}(d, f, g) \Vdash_L C$ and hence $\lambda f.\lambda g.\mathfrak{f}(d, f, g) \Vdash_L (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C)$. With the aid of Lemma 4.4 we can thus conclude that $\mathrm{D}_{\Phi(e,\lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g))} \neq \emptyset$ and for all $p \in \mathrm{D}_{\Phi(e,\lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g))}$ we have

$$p \Vdash_L (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C).$$

Let $E := (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C)$. By Lemma 4.6 we can therefore conclude that

$$\chi_E(\Phi(e, \lambda d.\lambda f.\lambda g.\mathfrak{f}(d, f, g))) \Vdash_L E.$$

As a result, $\lambda e.\chi_E(\Phi(e, \lambda d.\lambda f.\lambda g.\mathfrak{f}(d, f, g))) \Vdash_L A \vee B \rightarrow E$.

Axioms and Rules for $\forall$:

If $e \Vdash_L \forall x A(x, \vec{a})$, then $\mathrm{D}_e \neq \emptyset$ and $(\forall b \in \mathrm{V}(L))(\forall d \in \mathrm{D}_e)\, d \Vdash_L A(b, \vec{a})$, and hence, by Lemma 4.6, $\chi_A(e) \Vdash_L A(b, \vec{a})$ for all $b \in \mathrm{V}(L)$. Consequently,

$$\lambda e.\chi_A(e) \Vdash_L \forall x A(x, \vec{a}) \rightarrow A(b, \vec{a})$$

for all $b, \vec{a} \in \mathrm{V}(L)$.

We also have the rule: from $B(\vec{u}) \rightarrow A(x, \vec{u})$ infer $B(\vec{u}) \rightarrow \forall x A(x, \vec{u})$ if x is not free in $B(\vec{u})$. Inductively we have a realizer $\mathfrak{h}$ such that for all $b, \vec{a} \in \mathrm{V}(L)$,

$$\mathfrak{h} \Vdash_L B(\vec{a}) \rightarrow A(b, \vec{a}).$$

Suppose $d \Vdash_L B(\vec{a})$. Then $\mathfrak{h} \bullet d \Vdash_L A(b, \vec{a})$ holds for all $b \in \mathrm{V}(L)$, whence $\mathfrak{sg}(\mathfrak{h} \bullet d) \Vdash_L \forall x A(x, \vec{a})$. As a result,

$$\lambda d.\mathfrak{sg}(\mathfrak{h} \bullet d) \Vdash_L B(\vec{a}) \rightarrow \forall x A(x, \vec{a})$$

for all $\vec{a} \in \mathrm{V}(L)$.

Axioms and Rules for $\exists$:

If $e \Vdash_L A(a)$ then $\mathfrak{sg}(e) \Vdash_L \exists x A(x)$, thus $\lambda e.\mathfrak{sg}(e) \Vdash_L A(a) \rightarrow \exists x A(x)$ for all $a \in \mathrm{V}(L)$.

Finally we have the rule: from $A(x, \vec{u}) \to B(\vec{u})$ infer $\exists x\, A(x, \vec{u}) \to B(\vec{u})$ if x is not free in $B(\vec{u})$. Inductively we have a realizer $\mathfrak{g}$ such that for all $b, \vec{a} \in \mathrm{V}(L)$,

$$\mathfrak{g} \Vdash_L A(b, \vec{a}) \to B(\vec{a}).$$

Suppose $e \Vdash_L \exists x A(x, \vec{a})$. Then $\mathrm{D}_e \neq \emptyset$ and for all $d \in \mathrm{D}_e$ exists $c \in \mathrm{V}(L)$ such that $d \Vdash_L A(c, \vec{a})$. Consequently, $(\forall d \in \mathrm{D}_e)\, \mathfrak{g} \bullet d \Vdash_L B(\vec{a})$. By Lemma 4.4 we then have $\mathrm{D}_{\Phi(e,\mathfrak{g})} \neq \emptyset$ and $(\forall g \in \mathrm{D}_{\Phi(e,\mathfrak{g})})\, g \Vdash_L B(\vec{a})$. Using Lemma 4.6 we arrive at $\chi_B(\Phi(e, \mathfrak{g})) \Vdash_L B(\vec{a})$; whence $\lambda e.\chi_B(\Phi(e, \mathfrak{g})) \Vdash_L \exists x A(x, \vec{a}) \to B(\vec{a})$. $\square$

Lemma: 4.11 *For every formula $A(u, \vec{x})$ there are partial recursive functions Υ_1, Υ_2, and Υ_3 (depending solely on the formula) such that provably in* **IZF**$'$ *we have for all $e \in \mathbb{N}$ and $b, \vec{a} \in \mathrm{V}(L)$:*

(i) $e \Vdash_L \forall x \in b\, A(x, \vec{a}) \;\to\; \forall \langle d, v\rangle \in b\;\; \Upsilon_1(e) \bullet d \Vdash_L A(v, \vec{a})$*;*

(ii) $\forall \langle d, v\rangle \in b\;\; e \bullet d \Vdash_L A(v, \vec{a}) \;\to\; \Upsilon_2(e) \Vdash_L \forall x \in b\, A(x, \vec{a})]$*;*

(iii) $\langle d, v\rangle \in b \wedge e \Vdash_L A(v, \vec{a})] \;\to\; \Upsilon(e, d) \Vdash_L \exists x \in b\, A(x, \vec{a})]$*.*

Proof: (i). Suppose $e \Vdash_L \forall x[x \in b \to A(x, \vec{a})]$. Then $\mathrm{D}_e \neq \emptyset$ and for all $d' \in \mathrm{D}_e$ and $c \in \mathrm{V}(L)$, $d' \Vdash_L c \in b \to A(c, \vec{a})$. Now, if $\langle d, v\rangle \in b$, then $\theta(d) \Vdash_L v \in b$ by Corollary 4.9, and hence $\forall d' \in \mathrm{D}_e\;\; d' \bullet \theta(d) \Vdash_L A(v, \vec{a})$. There is then also a partial recursive function θ' such that $\forall d' \in \mathrm{D}_e\;\; \theta'(d) \bullet d' \Vdash_L A(v, \vec{a})$, so that by Lemma 4.4, $\mathrm{D}_{\Phi(e,\theta'(d))} \neq \emptyset$ and $\forall h \in \mathrm{D}_{\Phi(e,\theta'(d))}\;\; h \Vdash_L A(v, \vec{a})$. Hence, using Lemma 4.6, $\chi_A(\Phi(e, \theta'(d))) \Vdash_L A(v, \vec{a})$. Put $\Upsilon_1(e, d) := \chi_A(\Phi(e, \theta'(d)))$.

(ii) Suppose $\forall \langle d, v\rangle \in b\;\; e \bullet d \Vdash_L A(v, \vec{a})$. Assume $f \Vdash_L c \in b$. Then $\mathrm{D}_f \neq \emptyset$ and $\forall h \in \mathrm{D}_f\; \exists v\, [\langle \jmath_1 h, v\rangle \in b \wedge \jmath_2 h \Vdash_L c = v]$. $\langle \jmath_1 h, v\rangle \in b \wedge \jmath_2 h \Vdash_L c = v$ implies $e \bullet \jmath_1 h \Vdash_L A(v, \vec{a})$, and furthermore with the help of Lemma 4.8(6), $\mathbf{i}_A \bullet \jmath(\jmath_2 h, e \bullet \jmath_1 h) \Vdash_L A(c, \vec{a})$. Therefore $\forall h' \in \mathrm{D}_{\Phi(f, \lambda y.\mathbf{i}_A \bullet \jmath(\jmath_2 y, e \bullet \jmath_1 y))}\;\; h' \Vdash_L A(c, \vec{a})$, and thus by Lemma 4.6, $\chi_A(\Phi(f, \lambda y.\mathbf{i}_A \bullet \jmath(\jmath_2 y, e \bullet \jmath_1 y))) \Vdash_L A(c, \vec{a})$. Hence $\Upsilon_2(e) \Vdash_L \forall x \in b\, A(x, \vec{a})$, where $\Upsilon_2(e) := \mathfrak{sg}(\lambda f.\chi_A(\Phi(f, \lambda y.\mathbf{i}_A \bullet \jmath(\jmath_2 y, e \bullet \jmath_1 y))))$.

(iii) Suppose $\langle d, v\rangle \in b$ and $e \Vdash_L A(v, \vec{a})$. Then $\theta(d) \Vdash_L v \in b$, thus $\jmath(\theta(d), e) \Vdash_L v \in b \wedge A(v, \vec{a})$, so that $\mathfrak{sg}(\jmath(\theta(d), e)) \Vdash_L \exists x \in b\, A(x, \vec{a})$. Put $\Upsilon_3(e, d) = \mathfrak{sg}(\jmath(\theta(d), e))$. $\square$

5 The soundness theorem for IZF

Lemma: 5.1 *There is a partial recursive function $\mathfrak{sub}$ such that for all α, $a \in \mathrm{V}^{set}_\alpha$ and $b \in \mathrm{V}^{set}$,*

$$e \Vdash_L b \subseteq a \to \exists b^* \in \mathrm{V}^{set}_\alpha\; \mathfrak{sub}(e) \Vdash_L b = b^*.$$

Proof: Suppose $a \in \mathrm{V}^{set}_\alpha$, $b \in \mathrm{V}^{set}$, and $e \Vdash_L b \subseteq a$. Then $\mathrm{D}_e \neq \emptyset$. Let

$$b^* := \{\langle \jmath(f', g'), u\rangle \mid (\exists \langle f', x\rangle \in b)[\langle \jmath_1 g', u\rangle \in a \wedge \jmath_2 g' \Vdash_L x = u]\}.$$

Clearly, $b^* \in \mathrm{V}^{set}_\alpha$. With θ from Corollary 4.9 we have:

$$\begin{array}{rcl}
\langle f, x\rangle \in b & \rightarrow & \theta(f) \Vdash_L x \in b \\
& \rightarrow & (\forall d \in \mathrm{D}_e)\; d \bullet \theta(f) \Vdash_L x \in a \\
& \rightarrow & (\forall d \in \mathrm{D}_e)\; (\mathrm{D}_{d\bullet\theta(f)} \neq \emptyset \;\wedge \\
& & \quad (\forall d' \in \mathrm{D}_{d\bullet\theta(f)})\, \exists u\, [\langle \jmath_1 d', u\rangle \in a \;\wedge\; \jmath_2 d' \Vdash_L x = u]) \\
& \rightarrow & (\forall d \in \mathrm{D}_e)\; (\forall d' \in \mathrm{D}_{d\bullet\theta(f)})\, \exists u\, [\langle \jmath(f, d'), u\rangle \in b^* \;\wedge\; \jmath_2 d' \Vdash_L x = u] \\
& \rightarrow & (\forall d \in \mathrm{D}_e)(\forall h \in \mathrm{D}_{\Phi(d\bullet\theta(f), \lambda d'.\jmath(\jmath(f,d'), \jmath_2 d'))}) \\
& & \quad \exists u\, [\langle \jmath_1 h, u\rangle \in b^* \;\wedge\; \jmath_2 h \Vdash_L x = u] \\
& \rightarrow & (\forall d \in \mathrm{D}_e)\; \Phi(d \bullet \theta(f), \lambda d'.\jmath(\jmath(f, d'), \jmath_2 d')) \Vdash_L x \in b^* \\
& \rightarrow & (\forall g \in \mathrm{D}_{\Phi(e, \lambda d.\Phi(d\bullet\theta(f), \lambda d'.\jmath(\jmath(f,d'), \jmath_2 d')))})\; g \Vdash_L x \in b^* \\
& \rightarrow & \chi_A(\Phi(e, \lambda d.\Phi(d \bullet \theta(f), \lambda d'.\jmath(\jmath(f, d'), \jmath_2 d')))) \Vdash_L x \in b^*
\end{array}$$

where the fifth and seventh arrow are justified by Lemma 4.4 and the last arrow follows by Lemma 4.6 with $A \equiv x_1 \in x_2$.

Conversely, we have

$$\begin{array}{rcl}
\langle h, u\rangle \in b^* & \rightarrow & \exists x\, [\langle \jmath_1 h, x\rangle \in b \;\wedge\; \langle \jmath_1(\jmath_2 h), u\rangle \in a \;\wedge\; \mathbf{i}_s((\jmath_2(\jmath_2 h))) \Vdash_L u = x] \\
& \rightarrow & \mathfrak{sg}(\jmath(\jmath_1 h, \mathbf{i}_s((\jmath_2(\jmath_2 h))))) \Vdash_L u \in b
\end{array}$$

with $\mathbf{i}_s$ from Lemma 4.8. The upshot of the foregoing is that with

$$\begin{array}{rcl}
\nu(e, f) & := & \chi_A(\Phi(e, \lambda d.\Phi(d \bullet \theta(f), \lambda d'.\jmath(\jmath(f, d'), \jmath_2 d'))))\,, \\
\mu(h) & := & \mathfrak{sg}(\jmath(\jmath_1 h, \mathbf{i}_s((\jmath_2(\jmath_2 h)))))\,, \\
\mathfrak{sub}(e) & := & \mathfrak{sg}(\jmath(\lambda f.\nu(e, f), \lambda h.\mu(h)))
\end{array}$$

we have $\mathfrak{sub}(e) \Vdash_L b = b^*$. □

Theorem: 5.2 *For every axiom A of* **IZF**, *one can effectively construct an index e such that*

$$\mathbf{IZF}' \vdash\; (\bar{e} \Vdash_L A).$$

Proof: We treat the axioms one after the other.

(**Arithmetic axioms**): There are several and they are very boring to validate. In view of Lemma 4.7 it's also obvious how to realize them. We do one case study. $0 \Vdash_L \mathrm{SUC}(n, n+1)$ holds for all $n \in \mathbb{N}$. Hence $\jmath(n+1, 0) \Vdash_L \mathrm{N}(n+1) \wedge \mathrm{SUC}(n, n+1)$, thus

$$\mathfrak{sg}(\jmath(n+1, 0)) \Vdash_L \exists k\, \mathrm{SUC}(n, k),$$

so $\forall n\ e^* \bullet n \Vdash_L \exists k\, \mathrm{SUC}(n,k)$ with e^* is chosen such that $e^* \bullet n = \mathfrak{sg}(\jmath(n+1,0))$. By Lemma 4.7 we then have

$$\psi_2(e^*) \quad \Vdash_L \quad \forall n \exists k\, \mathrm{SUC}(n,k). \tag{13}$$

Now suppose $e \Vdash_L \mathrm{SUC}(c,a) \wedge \mathrm{SUC}(c,b)$. Then $c,a,b \in \mathbb{N}$ and $c+1=a=b$, thus $0 \Vdash_L a=b$ and hence

$$\mathfrak{sg}(\mathfrak{sg}(\mathfrak{sg}(\lambda u.0))) \quad \Vdash_L \quad \forall x \forall y \forall z\, [\mathrm{SUC}(x,y) \wedge \mathrm{SUC}(x,z) \rightarrow y=z]. \tag{14}$$

From (13) and (14) we obtain a realizer for the first number-theoretic axiom.

(**Induction on** $\mathbb{N}$): Suppose

$$e \Vdash_L A(0) \wedge \forall x \forall y [\mathrm{N}(x) \wedge \mathrm{N}(y) \wedge A(x) \wedge \mathrm{SUC}(x,y) \rightarrow A(y)].$$

Then $\mathrm{D}_{\jmath_2 e} \neq \emptyset$ and $(\forall d \in \mathrm{D}_{\jmath_2 e})\, \mathrm{D}_d \neq \emptyset$. Moreover, if $d \in \mathrm{D}_{\jmath_2 e}$ then for all $h \in \mathrm{D}_d$, $h \Vdash_L \mathrm{N}(x) \wedge \mathrm{N}(y) \wedge A(x) \wedge \mathrm{SUC}(x,y) \rightarrow A(y)$ for all $x,y \in \mathrm{V}(L)$. Thus for all $h \in \mathrm{D}_{\mathfrak{un}(\jmath_2 e)}$ (with $\mathfrak{un}$ from Lemma 4.5) and all $x,y \in \mathrm{V}(L)$ we have

$$h \Vdash_L \mathrm{N}(x) \wedge \mathrm{N}(y) \wedge A(x) \wedge \mathrm{SUC}(x,y) \rightarrow A(y). \tag{15}$$

Clearly, $\jmath_1 e \Vdash_L A(0)$. Now suppose $n \in \mathbb{N}$ and $\mathrm{SUC}(n,m)$ and we have an index e^* such that

$$(\forall h \in \mathrm{D}_{\mathfrak{un}(\jmath_2 e)})\, e^* \bullet \jmath(h,n) \Vdash_L A(n).$$

Then $\jmath(n,m) \Vdash_L \mathrm{N}(n) \wedge \mathrm{N}(m)$, so $\jmath(\jmath(n,m), e^* \bullet \jmath(h,n)) \Vdash_L (\mathrm{N}(n) \wedge \mathrm{N}(m)) \wedge A(n)$, and finally $\jmath(\jmath(\jmath(n,m), e^* \bullet \jmath(h,n)),0) \Vdash_L ((\mathrm{N}(n) \wedge \mathrm{N}(m)) \wedge A(n)) \wedge \mathrm{SUC}(n,m)$. From the latter we get

$$\mathfrak{l}^{\#}(e^*,n,h) := h \bullet \jmath(\jmath(\jmath(n,m), e^* \bullet \jmath(h,n)),0) \Vdash_L A(m).$$

We suppressed m in $\mathfrak{l}^{\#}$ since m is computable from n ($m=n+1$). Now choose e^* by the recursion theorem in such a way that $e^* \bullet \jmath(h,0) = \jmath_1 e$ and

$$e^* \bullet \jmath(h,k+1) \simeq \mathfrak{l}^{\#}(e^*,k,h).$$

If we inductively assume that $e^* \bullet \jmath(h,n) \downarrow$ for all $h \in \mathrm{D}_{\mathfrak{un}(e)}$ then the foregoing showed that $e^* \bullet \jmath(h,m) \downarrow$ for all $h \in \mathrm{D}_{\mathfrak{un}(e)}$. Hence $(\forall g \in \mathrm{D}_{\Phi(\mathfrak{un}(e), \lambda h. e^* \bullet \jmath(h,m))})\, g \Vdash_L A(m)$ by Lemma 4.4 and thus with

$$\mathfrak{l}^{\diamond}(e,m) = \begin{cases} \jmath_1 e & \text{if } m=0 \\ \chi_A(\Phi(\mathfrak{un}(e), \lambda h. e^* \bullet \jmath(h,m))) & \text{if } m \neq 0 \end{cases}$$

(using Lemma 4.6) we have $\mathfrak{l}^{\diamond}(e,m) \Vdash_L A(m)$ for all $m \in \mathbb{N}$. As a result,

$$\lambda m. \mathfrak{l}^{\diamond}(e,m) \Vdash_L \mathrm{N}(a) \rightarrow A(a)$$

holds for all $a \in \mathrm{V}(L)$ since $d \Vdash_L \mathrm{N}(a)$ implies $d=a$. Thus

$$\mathfrak{sg}(\lambda m. \mathfrak{l}^{\diamond}(e,m)) \Vdash_L \forall x (\mathrm{N}(x) \rightarrow A(x)),$$

and hence

$$\lambda e.\mathfrak{sg}(\lambda m.\mathfrak{l}^{\diamond}(e,m)) \Vdash_L A(0) \wedge \forall n\forall m[A(n) \wedge \mathrm{SUC}(n,m) \to A(m)] \to \forall n A(n).$$

(**Extensionality**): Let $a, b \in \mathrm{V}(L)$. Also suppose that $\mathrm{S}(a) \wedge \mathrm{S}(b)$ and

$$e \Vdash_L \forall x(x \in a \leftrightarrow x \in b).$$

Then $(\forall d \in \mathrm{D}_e)(\forall u \in \mathrm{V}(L))\, d \Vdash_L (u \in a \leftrightarrow u \in b)$. Thus for all $d \in \mathrm{D}_e$ we have

$$(\forall \langle f, y\rangle \in a)\ (\jmath_1 d) \bullet \theta(f) \Vdash_L y \in b$$
$$(\forall \langle f, y\rangle \in b)\ (\jmath_2 d) \bullet \theta(f) \Vdash_L y \in a$$

with θ defined as in Corollary 4.9. Letting $\psi(d) := \jmath(\lambda f.(\jmath_1 d) \bullet \theta(f), \lambda f.(\jmath_2 d) \bullet \theta(f))$ we therefore have

$$(\forall \langle f, y\rangle \in a)\ (\jmath_1(\psi(d))) \bullet f \Vdash_L y \in b$$
$$(\forall \langle f, y\rangle \in b)\ (\jmath_2(\psi(d))) \bullet f \Vdash_L y \in a.$$

Thus, by Lemma 4.4, $\Phi(e, \lambda x.\psi(x))\downarrow$, $\mathrm{D}_{\Phi(e,\lambda x.\psi(x))} \neq \emptyset$ and every $h \in \mathrm{D}_{\Phi(e,\lambda x.\psi(x))}$ is of the form $(\lambda x.\psi(x)) \bullet d = \psi(d)$ for some $d \in \mathrm{D}_e$. Thus $\Phi(e, \lambda x.\psi(x)) \Vdash_L a = b$. Furthermore,

$$\lambda f.\lambda e.\Phi(e, \lambda x.\psi(x)) \Vdash_L \mathrm{S}(a) \wedge \mathrm{S}(b) \to (\forall x(x \in a \leftrightarrow x \in b) \to a = b)$$

and hence

$$\mathfrak{sg}(\mathfrak{sg}(\lambda f.\lambda e.\Phi(e, \lambda x.\psi(x)))) \Vdash_L \forall u \forall y[\mathrm{S}(u) \wedge \mathrm{S}(y) \to (\forall x(x \in u \leftrightarrow x \in y) \to u = y)].$$

(**Pair**): Let $u, v \in \mathrm{V}(L)$. Put $a = \{\langle 0, u\rangle, \langle 0, v\rangle\}$. Then $a \in \mathrm{V}^{set}$ and $\theta(0) \Vdash_L u \in a$ and $\theta(0) \Vdash_L v \in a$, whence $\jmath(0, \jmath(\theta(0), \theta(0))) \Vdash_L \mathrm{S}(a) \wedge u \in a \wedge v \in a$, so $\mathfrak{sg}(\jmath(0, \jmath(\theta(0), \theta(0)))) \Vdash_L \exists y[\mathrm{S}(y) \wedge u \in y \wedge v \in y]$.

(**Union**): For each $u \in \mathrm{V}(L)$, put

$$\mathrm{Un}(u) \quad = \quad \{\langle \jmath(f, h), y\rangle \mid \exists x(\langle f, x\rangle \in u \wedge \langle h, y\rangle \in x)\}.$$

Then $\mathrm{Un}(u) \in \mathrm{V}^{set}$. Suppose

$$e \Vdash_L \exists x(x \in u \wedge z \in x).$$

Then

$$(\forall d \in \mathrm{D}_e)(\exists x \in \mathrm{V}(L))\, [\jmath_1 d \Vdash_L x \in u \wedge \jmath_2 d \Vdash_L z \in x].$$

Fix $d \in \mathrm{D}_e$ and $x \in \mathrm{V}(L)$ such that $\jmath_1 d \Vdash_L x \in u \wedge \jmath_2 d \Vdash_L z \in x$. Then

$$(\forall f \in \mathrm{D}_{\jmath_1 d})\exists w\, [\langle \jmath_1 f, w\rangle \in u \wedge \jmath_2 f \Vdash_L x = w]\,.$$

Letting $\mathfrak{q}(f,d) := \mathbf{i}_1 \bullet \jmath(\jmath_2 f, \jmath_2 d)$ with $\mathbf{i}_1$ from Lemma 4.8 we get

$$(\forall f \in \mathrm{D}_{\jmath_1 d})\exists w\, [\langle \jmath_1 f, w\rangle \in u \,\wedge\, \mathfrak{q}(f,d) \Vdash_L z \in w]$$

and hence

$$(\forall f \in \mathrm{D}_{\jmath_1 d})\exists w\, [\, \langle \jmath_1 f, w\rangle \in u \,\wedge\, \exists v(\langle \jmath_1(\mathfrak{q}(f,d)), v\rangle \in w \,\wedge\, \jmath_2(\mathfrak{q}(f,d)) \Vdash_L z = v\,].$$

Since $\langle \jmath(\jmath_1 f, \jmath_1(\mathfrak{q}(f,d))), v\rangle \in \mathrm{Un}(u)$, we arrive at

$$(\forall f \in V_{\jmath_1 d})\, \mathfrak{l}(f,d) \Vdash_L z \in \mathrm{Un}(u),$$

where $\mathfrak{l}(f,d) := \mathfrak{sg}(\jmath(\jmath(\jmath_1 f, \jmath_1(\mathfrak{q}(f,d))), \jmath_2(\mathfrak{q}(f,d))))$. As a result,

$$(\forall h \in \mathrm{D}_{\Phi(\jmath_1 d, \lambda f.\mathfrak{l}(f,d))} \Vdash_L z \in \mathrm{Un}(u),$$

hence

$$\chi_A(\Phi(\jmath_1 d, \lambda f.\mathfrak{l}(f,d))) \Vdash_L z \in \mathrm{Un}(u)$$

where A is the formula $x_0 \in x_1$. Since the latter holds for all $d \in \mathrm{D}_e$ we get

$$(\forall g \in \mathrm{D}_{\Phi(e, \lambda d.\chi_A(\Phi(\jmath_1 d, \lambda f.\mathfrak{l}(f,d))))}) \Vdash_L z \in \mathrm{Un}(u)$$

so

$$\chi_A(\Phi(e, \lambda d.\chi_A(\Phi(\jmath_1 d, \lambda f.\mathfrak{l}(f,d))))) \Vdash_L z \in \mathrm{Un}(u).$$

The upshot is that $\mathfrak{sg}(\jmath(0, \mathfrak{sg}(\lambda e.\chi_A(\Phi(e, \lambda d.\chi_A(\Phi(\jmath_1 d, \lambda f.\mathfrak{l}(f,d))))))))$ realizes $\exists w[\mathrm{S}(w) \,\wedge\, \forall z(\exists x(x \in u \,\wedge\, z \in x) \to z \in w)]$ from which one gets a realizer for the union axiom via realizers for the separation axioms.

(**Infinity**): Let $M := \{\langle n,n\rangle \mid n \in \mathbb{N}\}$. Then $M \in \mathrm{V}^{set}$ and $\mathrm{S}(M)$. Suppose $e \Vdash_L z \in M$. Then $\mathrm{D}_e \neq \emptyset$ and

$$(\forall d \in \mathrm{D}_e)\exists n[\langle \jmath_1 d, n\rangle \in M \,\wedge\, \jmath_2 d \Vdash_L z = n].$$

Note that $\langle \jmath_1 d, n\rangle \in M$ and $\jmath_2 d \Vdash_L z = n$ with $n \in \mathbb{N}$ entail that $\jmath_1 d = z = n$. We then also (trivially) have $\jmath_1 d \Vdash_L N(z)$. Invoking Lemma 4.4 we have $\mathrm{D}_{\Phi(e,\lambda d.\jmath_1 d)} = \{n\}$. Thus, by Lemma 4.3, $\phi(\Phi(e, \lambda d.\jmath_1 d))\downarrow$ and $\phi(\Phi(e, \lambda d.\jmath_1 d)) \Vdash_L \mathrm{N}(z)$.

Conversely, if $e \Vdash_L \mathrm{N}(z)$, then $e = z \,\wedge\, \langle z,z\rangle \in M$, so $\theta(e) \Vdash_L z \in M$.

The upshot is that $\mathfrak{sg}(\jmath(\lambda x.\phi(\Phi(x, \lambda d.\jmath_1 d)), \lambda x.\theta(x))) \Vdash_L \forall u(u \in M \,\leftrightarrow\, \mathrm{N}(u))$. Hence

$$\mathfrak{sg}(\jmath(0, \mathfrak{sg}(\jmath(\lambda x.\phi(\Phi(x, \lambda d.\jmath_1 d)), \lambda x.\theta(x))))) \Vdash_L \exists z[\mathrm{S}(z) \,\wedge\, \forall u(u \in z \,\leftrightarrow\, \mathrm{N}(u))].$$

(**Powerset**): Let $a \in \mathrm{V}^{set}_\alpha$. It suffices to find a realizer for the formula

$$\exists y[\mathrm{S}(y) \,\wedge\, \forall x(\mathrm{S}(x) \,\wedge\, x \subseteq a \to x \in y)]$$

since realizability of the power set axiom follows then with the help of Separation. Define

$$\mathfrak{V}_\alpha := \{\langle q, b\rangle \mid b \in \mathrm{V}^{set}_\alpha \,\wedge\, q \in \mathbb{N} \,\wedge\, q \Vdash_L \forall x(x \in b \to x \in a)\}.$$

Then $\mathfrak{V}_\alpha \in \mathrm{V}^{set}$. Suppose $b \in \mathrm{V}^{set}$ and $e \Vdash_L b \subseteq a$. Then $\mathfrak{sub}(e) \Vdash_L b = b^*$ for some $b^* \in \mathrm{V}^{set}_\alpha$ by Lemma 5.1. Thus, as $\langle \jmath_1(\mathfrak{sub}(e)), b^*\rangle \in \mathfrak{V}_\alpha$, we have

$$\mathfrak{sg}(\jmath(\jmath_1(\mathfrak{sub}(e)), \mathfrak{sub}(e))) \Vdash_L b \in \mathfrak{V}_\alpha.$$

Thus $\mathfrak{sg}(\lambda f.\mathfrak{sg}(\jmath(\jmath_1(\mathfrak{sub}(\jmath_2 f)), \mathfrak{sub}(\jmath_2 f)))) \Vdash_L \forall x(\mathrm{S}(x) \wedge x \subseteq a \to x \in \mathfrak{V}_\alpha)$ and consequently

$$\mathfrak{sg}(\jmath(0, \mathfrak{sg}(\lambda f.\mathfrak{sg}(\jmath(\jmath_1(\mathfrak{sub}(\jmath_2 f)), \mathfrak{sub}(\jmath_2 f)))))) \Vdash_L \exists y[\mathrm{S}(y) \wedge \forall x(\mathrm{S}(x) \wedge x \subseteq a \to x \in y)].$$

(**Set Induction**): Suppose $\bar{e} \Vdash_L \forall x[\forall y(y \in x \to A(y)) \to A(x)]$. Then $\mathrm{D}_{\bar{e}} \neq \emptyset$ and

$$(\forall d \in \mathrm{D}_{\bar{e}})(\forall x \in \mathrm{V}(L))\, d \Vdash_L \forall y(y \in x \to A(y)) \to A(x). \tag{16}$$

Let $a \in \mathrm{V}(L)$. Suppose there is an index e^* such that for all $\langle f', b\rangle \in a$ and $d \in \mathrm{D}_{\bar{e}}$ we have $e^* \bullet d\!\downarrow$ and $e^* \bullet d \Vdash_L A(b)$. Assume $f \Vdash_L y \in a$. Then $\mathrm{D}_f \neq \emptyset$ and

$$\forall d' \in \mathrm{D}_f\ \exists b\,(\langle \jmath_1 d', b\rangle \in a \wedge \jmath_2 d' \Vdash_L y = b).$$

Using Lemma 4.8, we can explicitly engineer an index $\hat{\mathbf{i}}_A$ such that for all $d \in \mathrm{D}_{\bar{e}}$ and $d' \in \mathrm{D}_f$ we have

$$\hat{\mathbf{i}}_A(\jmath(e^* \bullet d, \jmath_2 d')) \Vdash_L A(y).$$

Fix $d \in \mathrm{D}_{\bar{e}}$. Then $(\lambda x.\hat{\mathbf{i}}_A(\jmath(e^* \bullet d, \jmath_2 x))) \bullet d' \Vdash_L A(y)$ holds for all $d' \in \mathrm{D}_f$, and hence

$$\forall h \in \mathrm{D}_{\Phi(f, \lambda x.\hat{\mathbf{i}}_A(\jmath(e^* \bullet d, \jmath_2 x)))}\ h \Vdash_L A(y)$$

using Lemma 4.4, so that with the aid of Lemma 4.6 we have

$$\chi_A(\Phi(f, \lambda x.\hat{\mathbf{i}}_A(\jmath(e^* \bullet d, \jmath_2 x)))) \Vdash_L A(y).$$

As a consequence we have

$$\lambda f.\chi_A(\Phi(f, \lambda x.\hat{\mathbf{i}}_A(\jmath(e^* \bullet d, \jmath_2 x)))) \Vdash_L y \in a \to A(y),$$

so

$$\mathfrak{l}^*(e^*, d) := \mathfrak{sg}(\lambda f.\chi_A(\Phi(f, \lambda x.\hat{\mathbf{i}}_A(\jmath(e^* \bullet d, \jmath_2 x))))) \Vdash_L \forall y[y \in a \to A(y)].$$

Consequently, in view of (16), $(\forall d \in \mathrm{D}_{\bar{e}})\, d \bullet \mathfrak{l}^*(e^*, d)\downarrow$ and

$$(\forall d \in \mathrm{D}_{\bar{e}})\, d \bullet \mathfrak{l}^*(e^*, d) \Vdash_L A(a). \tag{17}$$

With the help of the recursion theorem we can explicitly cook up an index e^* such that

$$e^* \bullet n \simeq n \bullet \mathfrak{l}^*(e^*, n)$$

for all $n \in \mathbb{N}$. In view of the foregoing, it follows by set induction on $a \in \mathrm{V}(L)$ that for all $d \in \mathrm{D}_e$, $e^* \bullet d \Vdash_L A(a)$. Hence, by Lemma 4.4, $\mathrm{D}_{\Phi(e,\lambda d.e^* \bullet d)} \neq \emptyset$ and for all $h \in \mathrm{D}_{\Phi(e,\lambda d.e^* \bullet d)}$ and all $a \in \mathrm{V}(L)$ we have $h \Vdash_L A(a)$. Thus $\Phi(e, \lambda d.e^* \bullet d) \Vdash_L \forall x A(x)$. Hence

$$\lambda e.\Phi(e, \lambda d.e^* \bullet d) \Vdash_L \forall x[\forall y(y \in x \to A(y)) \to A(x)] \to \forall x A(x).$$

(**Separation**): Given $a \in \mathrm{V}(L)$ we seek a realizer $\mathfrak{e}$ such that

$$\mathfrak{e} \Vdash_L \exists z[\mathrm{S}(z) \wedge \forall u(u \in z \to u \in a \wedge A(a)) \wedge \forall u(u \in a \wedge A(u) \to u \in z)]. \quad (18)$$

$\mathfrak{e}$ will not depend on a nor on other parameters occurring in A. Let

$$b \quad = \quad \{\langle \jmath(f,g), x\rangle \mid \langle f, x\rangle \in a \wedge g \Vdash_L A(x)\}. \quad (19)$$

Then b is a set by separation in the background universe, and also $b \in \mathrm{V}^{set}$.

Assume $e \Vdash_L u \in b$. Then $\mathrm{D}_e \neq \emptyset$ and for every $d \in \mathrm{D}_e$ there exists x such that $\langle \jmath_1 d, x\rangle \in b \wedge \jmath_2 d \Vdash_L u = x$. By definition of b, $\jmath_1 d = \jmath(f,g)$ for some $f, g \in \mathbb{N}$ such that $\langle f, x\rangle \in a$ and $g \Vdash_L A(x)$. From $\jmath_2 d \Vdash_L u = x$ and $\theta(f) \Vdash_L x \in a$ we deduce $\mathfrak{q}(d,f) := \mathbf{i}_0(\jmath(\jmath_2 d, \theta(f))) \Vdash_L u \in a$ with the help of Lemma 4.8(4). As $g \Vdash_L A(x)$ we get $\mathfrak{p}(d,g) := \mathbf{i}_{A'}(\jmath(\jmath_2 d, g)) \Vdash_L A(u)$ from Lemma 4.8, where A' is obtained from A by replacing parameters from $\mathrm{V}(L)$ with free variables. Thus, from the above we conclude that

$$\jmath(\mathfrak{q}(d,f), \mathfrak{p}(d,g)) \Vdash_L u \in a \wedge A(u). \quad (20)$$

We can write $\mathfrak{l}(d) := \jmath(\mathfrak{q}(d,f), \mathfrak{p}(d,g))$ solely as a partial recursive function of d since $f = \jmath_1(\jmath_1 d))$ and $g = \jmath_2(\jmath_1 d))$. Thus (20) yields $(\forall d \in \mathrm{D}_e)\, \mathfrak{l}(d) \Vdash_L u \in a \wedge A(u)$, whence $(\forall h \in \mathrm{D}_{\Phi(e,\lambda d.\mathfrak{l}(d))}) \Vdash_L u \in a \wedge A(u)$ by Lemma 4.4, so

$$\chi_B(\Phi(e, \lambda d.\mathfrak{l}(d))) \Vdash_L u \in a \wedge A(u) \quad (21)$$

by Lemma 4.6 for an appropriate formula B. (21) yields

$$e^* := \mathfrak{sg}(\lambda e.\chi_B(\Phi(e, \lambda d.\mathfrak{l}(d)))) \Vdash_L \forall u(u \in b \to u \in a \wedge A(u)). \quad (22)$$

Conversely, assume $e \Vdash_L u \in a \wedge A(u)$. Then $\jmath_1 e \Vdash_L u \in a$ and $\jmath_2 e \Vdash_L A(u)$. Thus, for all $d \in \mathrm{D}_{\jmath_1 e}$ there exists x such that $\langle \jmath_1 d, x\rangle \in a$ and $\jmath_2 d \Vdash_L u = x$. Then, by Lemma 4.8, $\mathfrak{l}_1(d,e) := \mathbf{i}_{A_0}(\jmath(\jmath_2 d, \jmath_2 e)) \Vdash_L A(x)$ for a suitable formula A_0. So $\langle \jmath(\jmath_1 d, \mathfrak{l}_1(d,e)), x\rangle \in b$, which together with $\jmath_2(d) \Vdash_L u = x$ yields

$$\mathfrak{l}_2(d,e) := \jmath(\jmath(\jmath_1 d, \mathfrak{l}_1(d,e)), \jmath_2 d) \Vdash_L u \in b.$$

Consequently, by Lemma 4.4,

$$(\forall h \in \mathrm{D}_{\Phi(\jmath_1 e, \lambda d.\mathfrak{l}_2(d,e))})\, h \Vdash_L u \in b,$$

thus $\chi_C(\Phi(\jmath_1 e, \lambda d.\mathfrak{l}_2(d,e))) \Vdash_L u \in b$ by Lemma 4.6, where $C \equiv x_1 \in x_2$. Hence

$$e^{**} := \mathfrak{sg}(\lambda e.\chi_C(\Phi(\jmath_1 e, \lambda d.\mathfrak{l}_2(d,e)))) \Vdash_L \forall u[u \in a \wedge A(u) \to u \in b]. \quad (23)$$

Finally, by (22) and (23), we arrive at (18) with $\mathfrak{e} := \mathfrak{sg}(\jmath(0, \jmath(e^*, e^{**})))$.

(**Collection**): Suppose

$$e \Vdash_L \forall u(u \in a \rightarrow \exists y\, B(u, y)). \tag{24}$$

Then $\mathrm{D}_e \neq \emptyset$ and

$$(\forall d \in \mathrm{D}_e)(\forall u \in \mathrm{V}(L))\, d \Vdash_L (u \in a \rightarrow \exists y\, B(u, y)). \tag{25}$$

Fix $d \in \mathrm{D}_e$. If $\langle f, x\rangle \in a$ then $\theta(f) \Vdash_L x \in a$, so $d \bullet \theta(f) \Vdash_L \exists y B(x, y)$. Consequently, $(\forall h \in \mathrm{D}_{d\bullet\theta(f)})(\exists y {\in} \mathrm{V}(L))\, h \Vdash_L B(x, y)$. Therefore, using Collection in the background universe, there exists a set $C \subseteq \mathrm{V}(L)$ such that

$$(\forall d \in \mathrm{D}_e)(\forall \langle f, x\rangle \in a)(\forall h \in \mathrm{D}_{d\bullet\theta(f)})(\exists y \in C)\, h \Vdash_L B(x, y). \tag{26}$$

Let

$$C^* \;=\; \{\langle \jmath(\jmath(d, f), h), y\rangle \mid d \in \mathrm{D}_e \wedge y \in C \wedge \exists x(\langle f, x\rangle \in a \wedge h \Vdash_L B(x, y))\}. \tag{27}$$

C^* is a set by Separation. Also $C^* \in \mathrm{V}^{set}$. Now assume that $d \in \mathrm{D}_e$ and $e' \Vdash_L u \in a$. Then, for all $d' \in \mathrm{D}_{e'}$ there exists x such that $\langle \jmath_1 d', x\rangle \in a$ and $\jmath_2 d' \Vdash_L u = x$. Moreover, by (25), for all $h \in \mathrm{D}_{d\bullet\theta(\jmath_1 d')}$ there exists $y \in C$ such that $h \Vdash_L B(x, y)$. Whence $\langle \mathfrak{l}_3(d, d', h), y\rangle \in C^*$, where $\mathfrak{l}_3(d, d', h) := \jmath(\jmath(d, \jmath_1 d'), h)$. From $\jmath_2 d' \Vdash_L u = x$ and $h \Vdash_L B(x, y)$ we also obtain $\mathbf{i}_{B'}(\jmath(\jmath_2 d', h)) \Vdash_L B(u, y)$ by Lemma 4.8 for an appropriate formula B'. Since $\theta(\mathfrak{l}_3(d, d', h)) \Vdash_L y \in C^*$, we have

$$\mathfrak{l}_4(d, d', h) := \jmath(\theta(\mathfrak{l}_3(d, d', h)), \mathbf{i}_{B'}(\jmath(\jmath_2 d', h))) \Vdash_L y \in C^* \wedge B(u, y), \tag{28}$$

so $\mathfrak{sg}(\mathfrak{l}_4(d, d', h)) \Vdash_L \exists y(y \in C^* \wedge B(u, y))$, hence, using Lemmata 4.4, 4.5 and 4.6 repeatedly with appropriate formulas D and E,

$$\begin{aligned}
&\mathfrak{l}_5(d, d') := \chi_D(\Phi(d \bullet \theta(\jmath_1 d'), \lambda h.\mathfrak{sg}(\mathfrak{l}_4(d, d', h)))) \Vdash_L \exists y(y \in C^* \wedge B(u, y)),\\
&\mathfrak{l}_6(d, e') := \chi_E(\Phi(e', \lambda d'.\mathfrak{l}_5(d, d'))) \Vdash_L \exists y(y \in C^* \wedge B(u, y)).
\end{aligned} \tag{29}$$

As we established (29) under the assumption $e' \Vdash_L u \in a$, we get

$$\lambda e'.\mathfrak{l}_6(d, e') \Vdash_L u \in a \rightarrow \exists y(y \in C^* \wedge B(u, y)).$$

Thus, by Lemmata 4.4 and 4.6, we have

$$\mathfrak{l}_7(e) := \chi_F(\Phi(e, \lambda d.\lambda e'.\mathfrak{l}_6(d, e'))) \Vdash_L u \in a \rightarrow \exists y(y \in C^* \wedge B(u, y)) \tag{30}$$

for an appropriate formula F. Finally, by repeatedly applying Lemma 4.2, we see that

$$\begin{aligned}
\mathfrak{sg}(\mathfrak{l}_7(e)) &\Vdash_L \forall u[u \in a \rightarrow \exists y(y \in C^* \wedge B(u, y))]\\
\mathfrak{sg}(\jmath(0, \mathfrak{sg}(\mathfrak{l}_7(e)))) &\Vdash_L \exists z(\mathrm{S}(z) \wedge \forall u[u \in a \rightarrow \exists y(y \in z \wedge B(u, y))])\\
\lambda e.\mathfrak{sg}(\jmath(0, \mathfrak{sg}(\mathfrak{l}_7(e)))) &\Vdash_L \forall u[u \in a \rightarrow \exists y\, B(u, y)] \rightarrow\\
&\qquad \exists z(\mathrm{S}(z) \wedge \forall u[u \in a \rightarrow \exists y(y \in C^* \wedge B(u, y))]).
\end{aligned}$$

$\square$

Since BΣ^0_2-MP is Lifschitz realizable by [19, Lemma 3.2] and $\mathrm{MP_{pr}}$ is also Lifschitz realizable as will be shown in Lemma 6.5, it follows that the soundness theorem 5.2 can be extended to **IZF**′.

Theorem: 5.3 *For every axiom A of* **IZF**′*, one can effectively construct an index e such that*

$$\mathbf{IZF}' \vdash \ (\bar{e} \Vdash_L A).$$

The first large set axiom proposed in the context of constructive set theory was the *Regular Extension Axiom*, **REA**, which Aczel introduced to accommodate inductive definitions in **CZF** (cf. [2]).

Definition: 5.4 A is inhabited if $\exists x\, x \in A$. An inhabited set A is *regular* if A is transitive, and for every $a \in A$ and set $R \subseteq a \times A$ if $\forall x \in a\, \exists y\, (\langle x, y\rangle \in R)$, then there is a set $b \in A$ such that

$$\forall x \in a\, \exists y \in b\, (\langle x, y\rangle \in R) \ \wedge\ \forall y \in b\, \exists x \in a\, (\langle x, y\rangle \in R).$$

In particular, if $R : a \to A$ is a function, then the image of R is an element of A.

The *Regular Extension Axiom*, **REA**, is as follows: *Every set is a subset of a regular set.*

A set I is said to be *inaccessible* if I is a transitive set such that the following are satisfied:

1. $\mathbb{N} \in I$;
2. $\forall a, b \in I\ \{a, b\} \in I$;
3. $\forall a \in I\ \bigcup a \in I$;
4. $\forall a \in I\ \mathcal{P}(a) \in I$, where $\mathcal{P}(a) = \{x \mid x \subseteq a\}$;
5. $\forall c\, \forall a \in I\, [\forall x \in a\, \exists y \in I\, \langle x, y\rangle \in c \to \exists z \in I\, \forall x \in a\, \exists y \in z\, \langle x, y\rangle \in c\,]$.

We will write $\mathbf{inac}(I)$ to convey that I is an inaccessible set, and **Inac** for the statement $\forall x\, \exists I\, [x \in I\ \wedge\ \mathbf{inac}(I)]$.

Theorem: 5.5 *One can add large set axioms to* **IZF**′ *such as axioms asserting the existence of regular sets, inaccessible sets, Mahlo sets and other large sets. Such largeness notions have been considered in [21] and [2]. It would then turn out that these axioms are validated in* $\mathrm{V}(L)$*, too, if they hold in the background universe. We shall be content with stating one of these examples rigorously:*

$$\mathbf{IZF}' + \mathbf{Inac} \ \textit{proves that}\ \ \mathrm{V}(L) \models \mathbf{Inac}.$$

Proof: We will demonstrate the latter result. So our background theory will be $\mathbf{IZF}' + \mathbf{Inac}$. The proof is a modification of [22, Theorem 6.2]. Let $a \in \mathrm{V}(L)$. By **Inac** there exists an inaccessible set I such that $a \in I$. Let

$$\begin{aligned} \mathfrak{A} &:= I \cap \mathrm{V}(L), \\ \mathfrak{C} &:= \{\langle 0, x\rangle \,:\, x \in \mathfrak{A}\}. \end{aligned}$$

$\mathfrak{A}$ is a set by separation, and since $\mathfrak{A} \subseteq \mathrm{V}(L)$, $\mathfrak{C}$ is a set belonging to $\mathrm{V}(L)$. Let $\kappa := \bigcup\{x \in \mathfrak{A} \mid x \text{ is an ordinal}\}$. One easily verifies that $\mathfrak{A} = \mathrm{V}_\kappa^{set} \cup \mathbb{N}$. I being inaccessible it is clear from Theorem 5.2 that $\mathfrak{A}$ realizes all theorems of **IZF**. In the main, we have to go through the proof of Theorem 5.2 to ascertain that the witnesses for set existence axioms of **IZF** can be found in $\mathfrak{C}$. We shall do a few examples.

For **Pair** suppose that $\langle 0, u\rangle, \langle 0, v\rangle \in \mathfrak{C}$. Put $a := \langle 0, \{\langle 0, u\rangle, \langle 0, u\rangle\}\rangle$. Then $a \in \mathfrak{C}$ since $\{\langle 0, u\rangle, \langle 0, u\rangle\} \in \mathfrak{A}$. As $\theta(0) \Vdash_L u \in a$, $\theta(0) \Vdash_L v \in a$, and $\theta(0) \Vdash_L a \in \mathfrak{C}$ we have $\jmath(\theta(0), \jmath(0, \jmath(\theta(0), \theta(0)))) \Vdash_L a \in \mathfrak{C} \wedge (\mathrm{S}(a) \wedge u \in a \wedge v \in a)$. Thus $\mathfrak{q} \Vdash_L \exists y \in \mathfrak{C}[\mathrm{S}(y) \wedge u \in y \wedge v \in y]$, where $\mathfrak{q} := \mathfrak{sg}(\jmath(\theta(0), \jmath(0, \jmath(\theta(0), \theta(0)))))$. Hence using 4.11(ii) twice we arrive at

$$\Upsilon_2(\lambda x.\Upsilon_2(\lambda x.\mathfrak{q})) \Vdash_L \forall u \in \mathfrak{C}\, \forall v \in \mathfrak{C}\, \exists y \in \mathfrak{C}[\mathrm{S}(y) \wedge u \in y \wedge v \in y]\,.$$

(**Union**): For each $\langle 0, u\rangle \in \mathfrak{C}$, put

$$\mathrm{Un}(u) \quad = \quad \{\langle \jmath(f, h), y\rangle \mid \exists x(\langle f, x\rangle \in u \wedge \langle h, y\rangle \in x)\}.$$

Then $\mathrm{Un}(u) \in \mathrm{V}^{set} \cap \mathfrak{A}$ and thus $\langle 0, \mathrm{Un}(u)\rangle \in \mathfrak{C}$. By the same proof as in Theorem 5.2 we find a realizer $\mathfrak{s}$ such that $\mathfrak{s} \Vdash_L \forall z(\exists x(x \in u \wedge z \in x) \to z \in \mathrm{Un}(u))$. Hence by Lemma 4.11(ii) we have $\Upsilon_2(\lambda x.q) \Vdash_L \forall u \in \mathfrak{C}\, \forall z[\exists x(x \in u \wedge z \in x) \to z \in \mathrm{Un}(u)]$. Thus $\mathfrak{sg}(\jmath(\theta(0), \Upsilon_2(\lambda x.q))) \Vdash_L \exists y \in \mathfrak{C}\, \forall z[\exists x(x \in u \wedge z \in x) \to z \in \mathrm{Un}(u)]$.

(**Powerset**): Let $\langle 0, a\rangle \in \mathfrak{C}$. Then $a \in \mathrm{V}_\alpha^{set}$ for some $\alpha \in \kappa$. Define

$$\mathfrak{V}_\alpha := \{\langle q, b\rangle \mid b \in \mathrm{V}_\alpha^{set} \wedge q \in \mathbb{N} \wedge q \Vdash_L \forall x(x \in b \to x \in a)\}.$$

Then $\mathfrak{V}_\alpha \in \mathfrak{A}$ and hence $\langle 0, \mathfrak{V}_\alpha\rangle \in \mathfrak{C}$. Then proceed as in the proof of Theorem 5.2.

The remaining axioms are also dealt with by similar adaptations of the proof of Theorem 5.2. So the upshot is that we find $\mathfrak{r}$ (not depending on a) such that $\mathfrak{r} \Vdash_L \mathbf{inac}(\mathfrak{C})$, and therefore also a realizer for **Inac**. □

6 Church's thesis in V(L)

Lemma: 6.1 ($\mathbf{IZF}'$) $\quad \mathrm{V}(L) \models \mathbf{CT}_0!$.

Proof: Note that according to Lemma 4.7 our realizability for arithmetic formulae is the same as in [15]. As a result, the same proof as in [15, Lemma 3] will do. □

Lemma: 6.2 $\mathrm{V}(L) \not\models \mathbf{CT}_0$. *More precisely, let $\bar{e}, \tilde{e} \in \mathbb{N}$ be indices of two disjoint recursively inseparable r.e. sets, i.e. $X = \{m \mid \exists m\, \mathrm{T}(\bar{e}, n, m)\}$ and $Y = \{m \mid \exists m\, \mathrm{T}(\tilde{e}, n, m)\}$ are disjoint and recursively inseparable. Let $A(n) := \forall m\, \neg \mathrm{T}(\bar{e}, n, m)$, $B(n) := \forall m\, \neg \mathrm{T}(\tilde{e}, n, m)$ and $C(n, k) := (A(n) \wedge k = 0) \vee (B(n) \wedge k = 1)$. Then*

$$\mathrm{V}(L) \not\models \forall n \exists k\, C(n, k) \to \exists d \forall n\, C(n, d \bullet n).$$

Proof: The proof is the same as in [15, section 4]. First one shows that $\mathrm{V}(L) \models \forall n \exists k\, C(n, k)$. Next one shows that from $e^* \Vdash_L \exists d \forall n\, C(n, d \bullet n)$ one would be able to engineer a recursive separation of X and Y above, which is impossible. □

The foregoing Lemmata also show that a "binary" version of number choice is not provable in **IZF**. Let $\mathbf{AC}_{\omega,2}$ be the statement that whenever $(A_i)_{i \in \mathbb{N}}$ is family of inhabited sets A_i with $A_i \subseteq \{0, 1\}$, then there exists a function $F : \mathbb{N} \to \bigcup_{i \in \mathbb{N}} A_i$ such that $\forall i\, F(i) \in A_i$.

Corollary: 6.3 $\mathrm{V}(L) \not\models \mathbf{AC}_{\omega,2}$. *In particular,* **IZF** *does not prove* $\mathbf{AC}_{\omega,2}$.

Proof: We argue in $\mathrm{V}(L)$. We have $\forall n \exists k\, C(n, k)$ with C as in the proof of Lemma 6.2. Then with $A_n := \{k \in \{0, 1\} \mid C(n, k)\}$, $A_n \subseteq \{0, 1\}$ and A_n is inhabited. Thus if $\mathbf{AC}_{\omega,2}$ were to hold in $\mathrm{V}(L)$ we would get a function $F : \mathbb{N} \to \bigcup_{n \in \mathbb{N}} A_n$ such that $\forall n\, F(n) \in A_n$. Since $\forall n \exists ! k F(n) = k$, $\mathbf{CT}_0!$ implies the existence of an index d such that $\forall n\, F(n) = d \bullet n$, and hence $\exists d \forall n\, C(n, d \bullet n)$. This contradicts Lemma 6.2. □

The *presentation axiom*, **PAx**, was considered by Aczel [1] and Blass [6]. In category theory it is also known as the *existence of enough projective sets*. More details about **PAx** can be found in [2]. Since **PAx** implies countable choice we can infer the following result:

Corollary: 6.4 $\mathrm{V}(L)$ *does not model the presentation axiom.*

Recall $\mathrm{MP}_{\mathrm{pr}}$ from Definition 4.1.

Lemma: 6.5 $(\mathbf{IZF}')$ $\quad \mathrm{V}(L) \models \mathrm{MP}_{\mathrm{pr}}$.

Proof: Assume $e \Vdash_L \neg\neg \exists n\, A(n)$ where $A(n)$ is of the form $R(n, \vec{k}\,)$ with R primitive recursive and $\vec{k} \in \mathbb{N}$. Then $\neg\neg \exists f\, f \Vdash_L \exists n\, A(n)$, and thus by Lemma 4.7, $\neg\neg \exists f \Vdash_L A(\jmath_1 f)$, thus $\neg\neg \exists f\, A(\jmath_1 f)$. Using $\mathrm{MP}_{\mathrm{pr}}$ in the background universe we have $\exists n\, A(n)$. Then, with $r := \mu n. A(n)$, we have $\mathfrak{sg}(\jmath(r, 0)) \Vdash_L \exists n\, A(n)$. Whence $\lambda e. \mathfrak{sg}(\jmath(r, 0))$ realizes this instance of $\mathrm{MP}_{\mathrm{pr}}$. □

7 More classical and non-classical principles that hold in V(L)

The next definitions lists several interesting principles that are validated in V(L).

Definition: 7.1 1. UP, the *Uniformity Principle*, is expressed by the schema:

$$\forall x\,[\mathrm{S}(x) \rightarrow \exists n\, A(x,n)] \;\rightarrow\; \exists n\, \forall x[\mathrm{S}(x) \rightarrow A(n,x)].$$

2. *Unzerlegbarkeit*, UZ, is the schema

$$\forall x[\mathrm{S}(x) \rightarrow (A(x) \vee B(x))] \;\rightarrow\; \forall x(\mathrm{S}(x) \rightarrow A(x)) \;\vee\; \forall x(\mathrm{S}(x) \rightarrow B(x))$$

for all formulas A, B.

Lemma: 7.2 **(IZF$'$)** V$(L) \models$ UP $\wedge$ UZ.

Proof: Suppose $e \Vdash_L \forall x\,[\mathrm{S}(x) \rightarrow \exists n\, A(x,n)]$. Then $\mathrm{D}_e \neq \emptyset$. Since $0 \Vdash_L \mathrm{S}(a)$ holds for all $a \in \mathrm{V}^{set}$, we have

$$(\forall d \in \mathrm{D}_e)(\forall a \in \mathrm{V}^{set})\jmath_1 d \bullet 0 \Vdash_L \exists y[\mathrm{N}(y) \;\wedge\; A(a,y)].$$

Let $d \in \mathrm{D}_e$ and $a \in \mathrm{V}^{set}$. If $f \in \mathrm{D}_{\jmath_1 d \bullet 0}$ then there exists $y \in \mathrm{V}(L)$ such that $f \Vdash_L \mathrm{N}(y) \wedge A(a,y)$, thus $\jmath_1 f = y$ and $\jmath_2 f \Vdash_L A(a, \jmath_1 f)$. Hence

$$(\forall f \in \mathrm{D}_{\jmath_1 d \bullet 0})(\forall a \in \mathrm{V}^{set})\jmath_2 f \Vdash_L A(a, \jmath_1 f),$$

and so

$$\begin{aligned}
(\forall f \in \mathrm{D}_{\jmath_1 d \bullet 0})\lambda x.\jmath_2 f &\Vdash_L \forall x[\mathrm{S}(x) \rightarrow A(x, \jmath_1 f)]\\
(\forall f \in \mathrm{D}_{\jmath_1 d \bullet 0})\jmath(\jmath_1 f, \lambda x.\jmath_2 f) &\Vdash_L \mathrm{N}(\jmath_1 f) \;\wedge\; \forall x(\mathrm{S}(x) \rightarrow A(x, \jmath_1 f)),\\
\mathfrak{l}(d) := \Phi(\jmath_1 d \bullet 0, \lambda f.\jmath(\jmath_1 f, \lambda x.\jmath_2 f)) &\Vdash_L \exists y[\mathrm{N}(y) \;\wedge\; \forall x(\mathrm{S}(x) \rightarrow A(x,y))],
\end{aligned}$$

where we used Lemma 4.4 in the last step. Finally, by applying Lemmata 4.3 and 4.5 we arrive at

$$\chi_{A'}(\Phi(e, \lambda d.\mathfrak{l}(d))) \Vdash_L \exists y[\mathrm{N}(y) \;\wedge\; \forall x(\mathrm{S}(x) \rightarrow A(x,y))]$$

for an appropriate formula A'. Hence, with $e^* := \lambda e.\chi_{A'}(\Phi(e, \lambda d.\mathfrak{l}(d)))$,

$$e^* \Vdash_L \forall x\,[\mathrm{S}(x) \rightarrow \exists n\, A(x,n)] \rightarrow \exists y[\mathrm{N}(y) \;\wedge\; \forall x(\mathrm{S}(x) \rightarrow A(x,y))].$$

As to Lifschitz realizability of UZ, note that $\forall x[\mathrm{S}(x) \rightarrow (A(x) \vee B(x))]$ implies $\forall x[\mathrm{S}(x) \rightarrow \exists n[(n = 0 \wedge A(x)) \vee (n \neq 0 \wedge B(x))]$. The latter yields

$$\exists n\, \forall x[\mathrm{S}(x) \rightarrow [(n = 0 \;\wedge\; A(x)) \vee (n \neq 0 \;\wedge\; B(x))]$$

via UP, and hence $\forall x(\mathrm{S}(x) \rightarrow A(x)) \;\vee\; \forall x(\mathrm{S}(x) \rightarrow B(x))$. Thus UZ is a consequence of UP. Therefore V$(L) \models$ UZ. □

A classically valid principle considered in connection with intuitionistic theories is the *Principle of Independence of Premisses*, IP, which is expressed by the schema

$$(\neg A \to \exists x\, B(x)) \;\to\; \exists x(\neg A \to B(x)),$$

where A has to be assumed to be a closed formula.

Lemma: 7.3 *Assuming classical logic in V,* $\mathrm{V}(L) \models \mathrm{IP}$.

Proof: (1). Assume $e \Vdash_L \neg\neg\exists n\, A(n)$ where $A(n)$ is of the form $R(n, \vec{k})$ with R primitive recursive and $\vec{k} \in \mathbb{N}$. Then $\neg\neg\exists f\, f \Vdash_L \exists n\, A(n)$, and thus by Lemma 4.7, $\neg\neg\exists f\; \Vdash_L A(\jmath_1 f)$, thus $\neg\neg\exists f\, A(\jmath_1 f)$. Using $\mathrm{MP_{pr}}$ in the background universe we have $\exists n\, A(n)$. Then, with $r := \mu n.A(n)$, we have $\mathfrak{sg}(\jmath(r,0)) \Vdash_L \exists n\, A(n)$. Whence $\lambda e.\mathfrak{sg}(\jmath(r,0))$ realizes this instance of $\mathrm{MP_{pr}}$.

(2). Assume that $e \Vdash_L \neg A \to \exists x\, B(x)$. Then, if $g \Vdash_L \neg A$, $0 \Vdash_L \neg A$ and $e \bullet 0 \Vdash_L \exists x\, B(x)$. Therefore, $\mathrm{D}_{e\bullet 0} \neq \emptyset$ and for all $d \in \mathrm{D}_{e\bullet 0}$ there is an $a \in \mathrm{V}(L)$ such that $d \Vdash_L B(a)$, and therefore $\lambda u.d \Vdash_L \neg A \to B(a)$. Hence, if A is not realized,

$$\Phi(e \bullet 0, \lambda d.\lambda u.d) \Vdash_L \exists x(\neg A \to B(x)).$$

On the other hand, should A be realized, then $\neg A$ is never realized, so $\lambda u.u$ would realize this instance of IP. $\square$

8 The reals in $\mathrm{V}(L)$

By Lemma 6.1 the Cauchy reals in $\mathrm{V}(L)$ are the recursive reals. A well-known consequence of $\mathbf{AC}_{\omega,2}$ is that the sets of Cauchy reals and Dedekind reals are isomorphic. As it turns out, the notions of Cauchy real and Dedekind real coincide in $\mathrm{V}(L)$ despite the failure of $\mathbf{AC}_{\omega,2}$.

Definition: 8.1 A *Dedekind cut* is a pair (L, U) of subsets of $\mathbb{Q}$, satisfying:

(i) $q \in L \leftrightarrow \exists r \in L\; q < r$

(ii) $q \in U \leftrightarrow \exists r \in U\; r < q$

(iii) $q \in L \;\wedge\; r \in U \to q < r$

(iv) $\forall n\, \exists q \in L\, \exists r \in U\; r - q < \frac{1}{2^n}$ (locatedness)

Call (L, U) a *strong real* if there exists $f : \mathbb{Q}^2 \to \mathbb{N}$ such that

$$\forall q, r\, (q < r \to [[f(q,r) = 0 \;\wedge\; q \in L] \;\vee\; [f(q,r) \neq 0 \;\wedge\; r \in U]]). \tag{31}$$

Van Oosten showed that in the so-called Lifschitz topos [20, IV. Proposition 2.5] the two notions of reals agree.

Theorem: 8.2 *In* $\mathrm{V}(L)$ *the set of Cauchy reals is order-isomorphic to the set of Dedekind reals.*

Proof: The proof utilizes [25, Ch.5 Proposition 5.10] saying that the collection of strong Dedekind reals is order-isomorphic to the collection of Cauchy reals. Thus it remains to show that in $\mathrm{V}(L)$ every Dedekind real is strong. So assume

$$e \Vdash_L (L,U) \text{ is a Dedekind real.} \tag{32}$$

We will assume that the rationals are coded as natural numbers, more specifically we will assume that the property of being a rational number, the ordering between rationals and their distance relation are primitive recursive. From e we have to construct (an index of) a recursive function $f : \mathbb{Q}^2 \to \mathbb{N}$ such that (31) holds. From e we can compute an index $\tilde{e}$ such that

$$\begin{array}{ll} & \forall n\ \tilde{e} \bullet n \Vdash_L \exists r \in L\ \exists r' \in U\ r' - r < 2^{-n} \\ \leftrightarrow & \forall n\,[\mathrm{D}_{\tilde{e}\bullet n} \neq \emptyset \ \wedge\ \forall d \in \mathrm{D}_{\tilde{e}\bullet n}\ \jmath_2 d \Vdash_L (\jmath_1 d \in L \ \wedge\ \exists r' \in U\ r' - r < 2^{-n})] \\ \leftrightarrow & \forall n\,[\mathrm{D}_{\tilde{e}\bullet n} \neq \emptyset \ \wedge\ \forall d \in \mathrm{D}_{\tilde{e}\bullet n}\ (\jmath_1(\jmath_2 d) \Vdash_L (\jmath_1 d \in L) \ \wedge\ \mathrm{D}_{\jmath_2(\jmath_2 d)} \neq \emptyset \\ & \qquad\qquad \wedge\ \forall h \in \mathrm{D}_{\jmath_2(\jmath_2 d)}\ \jmath_2 h \Vdash_L (\jmath_1 h \in U \ \wedge\ \jmath_1 h - \jmath_1 d < 2^{-n}))]. \end{array}$$

Now, given $p, q \in \mathbb{Q}$ with $p < q$ we can compute a natural number n_0 such that $2^{-n_0} < \frac{q-p}{2}$. In view of the above we then either have

$$\forall d \in \mathrm{D}_{\tilde{e}\bullet n_0}\ p \leq \jmath_1 d \tag{33}$$

and thus $\mathrm{V}(L) \models p \in L$, or else there exists $d_0 \in \mathrm{D}_{\tilde{e}\bullet n_0}$ such that $\jmath_1 d_0 < p$. Let us assume that the latter case obtains. Pick $h_0 \in \mathrm{D}_{\jmath_2(\jmath_2 d_0)}$. Let $d \in \mathrm{D}_{\tilde{e}\bullet n_0}$ and $h \in \mathrm{D}_{\jmath_2(\jmath_2 d)}$. We then have $\jmath_1 h_0 - \jmath_1 d_0 < 2^{-n_0}$, $\jmath_1 h - \jmath_1 d < 2^{-n_0}$. As $\mathrm{V}(L) \models \jmath_1 d \in L$ and $\mathrm{V}(L) \models \jmath_1 h_0 \in L$, we must have $\jmath_1 d < \jmath_1 h_0$, so that

$$\jmath_1 h < \jmath_1 d + 2^{-n_0} < \jmath_1 h_0 + 2^{-n_0} < \jmath_1 d_0 + 2 \cdot 2^{-n_0} < \jmath_1 d_0 + (q-p) < p + (q-p) = q,$$

and hence $\mathrm{V}(L) \models q \in U$. As a result we have

$$\forall d \in \mathrm{D}_{\tilde{e}\bullet n_0}\ \forall h \in \mathrm{D}_{\jmath_2(\jmath_2 d)}\ \jmath_1 h < q\,. \tag{34}$$

We are now lucky since the sentences in (33) and (34) are Σ^0_1 and at least one of them must be true. So we can define a recursive function f by simultaneously searching for a witness for (33) and for (34). If we find a witness for (33) before we find one for (34), let $f(p,q) = 0$, and if it is the other way round let $f(p,q) = 1$. □

Of course, in the proof of the previous Theorem we used a certain amount of classical logic beyond that available in $\mathbf{IZF}'$. But we leave it to the reader to spell out the details.

9 The lesser limited principle of omniscience

Recall the following two principles under the names given to them by Bishop:

Definition: 9.1 *The limited principle of omniscience*, **LPO**: If $f : \mathbf{N} \to \{0,1\}$, then either there exists $n \in \mathbb{N}$ such that $f(n) = 1$, or else $f(n) = 0$ for each $n \in \mathbb{N}$.

The lesser limited principle of omniscience, **LLPO**: If $f : \mathbf{N} \to \{0,1\}$ such $f(n) = 1$ holds for at most one n, then either $f(2n) = 0$ for each $n \in \mathbb{N}$, or else $f(2n+1) = 0$ for each $n \in \mathbb{N}$.

LPO is incompatible with $\mathbf{CT}_0$! (see Corollary 9.3). Albeit being incompatible with $\mathbf{CT}_0$ and thus invalidated in the usual Kleene-type realizability models, **LLPO** turns out to be compatible with $\mathbf{CT}_0$!.

Lemma: 9.2 *(i)* ($\mathbf{IZF}'$) $\mathrm{V}(L) \models$ **LLPO**.

(ii) ($\mathbf{IZF}'$) $\mathrm{V}(L) \not\models$ **LPO**.

Proof: (i) First we use the fact that the principle Σ^0_1-**LLPO** from [4] is Lifschitz realizable. This is the principle

$$\neg[\exists n\, P(n,\vec{k}\,) \wedge \exists m\, Q(m,\vec{l}\,)] \to [\forall n\, \neg P(n,\vec{k}\,) \vee \forall n\, \neg Q(n,\vec{l}\,)]$$

where P, Q are primitive recursive and $\vec{k}, \vec{l}$ are parameters from $\mathbb{N}$. It was observed in [4, Theorem 3.14] that **LLPO** is Lifschitz realizable since it is a consequence of $\mathrm{B}\Sigma^0_2$-MP and the latter is Lifschitz realizable by [19, Lemma 3.2]. One sees that Σ^0_1-**LLPO** is a consequence of $\mathrm{B}\Sigma^0_2$-MP and $\mathrm{MP_{pr}}$ (Lemma 6.5) as follows, letting $P'(n) := P(n,\vec{k}\,)$ and $Q'(n) := Q(n,\vec{l}\,)$:

$$\begin{aligned}
&\neg\exists l' \le 1\, \forall n\, [(l' = 0 \wedge \neg P'(n)) \vee (l' = 1 \wedge \neg Q'(n))] \\
&\quad \to\ \neg\forall n\, \neg P'(n) \wedge \forall n\, \neg Q'(n) \\
&\quad \to\ \neg\neg\exists n\, P'(n) \wedge \neg\neg\exists n\, Q'(n) \\
&\quad \to\ \exists n\, P'(n) \wedge \exists n\, Q'(n)
\end{aligned}$$

using $\mathrm{MP_{pr}}$ in the last step. As a result, with the aid of $\mathrm{B}\Sigma^0_2$-MP we obtain

$$\begin{aligned}
\neg[\exists n\, P'(n) \wedge \exists n\, Q'(n)] &\to \neg\neg\exists l' \le 1\, \forall n\, [(l' = 0 \wedge \neg P'(n)) \vee (l' = 1 \wedge \neg Q'(n))] \\
\neg[\exists n\, P'(n) \wedge \exists n\, Q'(n)] &\to \exists l' \le 1\, \forall n\, [(l' = 0 \wedge \neg P'(n)) \vee (l' = 1 \wedge \neg Q'(n))] \\
\neg[\exists n\, P'(n) \wedge \exists n\, Q'(n)] &\to \forall n\, \neg P'(n) \vee \forall n\, \neg Q'(n).
\end{aligned}$$

Now assume that $f : \mathbf{N} \to \{0,1\}$ such $f(n) = 1$ holds for at most one n. Using $\mathbf{CT}_0$! there exists an index e of a recursive function such that $\forall n\, e \bullet n = f(n)$. Let $P(n,e)$ and $Q(e,n)$ be the primitive recursive predicates defined by $\exists l \le n\, \exists l' \le n\, [T(e, 2l, l') \wedge U(l') = 1]$ and $\exists l \le n\, \exists l' \le n\, [T(e, 2l+1, l') \wedge U(l') = 1]$, respectively, with T being Kleene's T-predicate and U the result extracting primitive recursive function. We then have

$\neg[\exists n\, P(n,e) \wedge \exists n\, Q(n,e)]$. In view of the above we conclude with the aid of BΣ_2^0-MP that $\forall n\, \neg P(n,e) \vee \forall n\, \neg Q(n,e)$, whence $\forall n\, f(2n) = 0$ or $\forall n\, f(2n+1) = 0$. We will thus reach the desired conclusion if we can show that BΣ_2^0-MP is Lifschitz realizable. This follows from [19, Lemma 3.2] since instances of the latter scheme are BΣ_2^0-negative formulas.

(ii) Assume **LPO**. Let $f_e(n)$ be 1 if $T(e,e,n)$ holds and 0 otherwise. With **LPO** we get $\forall n\, f_e(n) = 0 \vee \exists n\, f_e(n) = 1$, whence $e \bullet e \downarrow \vee\, e \bullet e \uparrow$. Thus $\forall e\, \exists! k\, [(k = 0 \wedge e \bullet e \downarrow) \vee (k = 1 \wedge e \bullet e \uparrow)]$. In the presence of $\mathbf{CT}_0$! we would thus find a total recursive function ρ such that $\forall e\, [(\rho(e) = 0 \wedge e \bullet e \downarrow) \vee (\rho(e) = 1 \wedge e \bullet e \uparrow)]$. So ρ would solve the halting problem. As the unsolvability of the halting problem can be demonstrated in **HA** it follows that **LPO** is not Lifschitz realizable. □

Since $\mathrm{V}(L) \models$ **LLPO** it might be instructive to recall why **LLPO** and $\mathbf{CT}_0$ are incompatible: Take two disjoint recursively inseparable r.e. sets W_e and W_d. For each n define a function g_n as follows: $g_n(2k) = 1$ if $T(e,n,k) \wedge \forall k' < k\, \neg T(e,n,k') \wedge \forall k'' \leq k\, \neg T(d,n,k'')$ holds and $g_n(2k) = 0$ otherwise; $g_n(2k+1) = 1$ if $T(d,n,k) \wedge \forall k' < k\, \neg T(d,n,k') \wedge \forall k'' \leq k\, \neg T(e,n,k'')$ holds and $g_n(2k+1) = 0$ otherwise. Then, using Σ_1^0-**LLPO** (a consequence of **LLPO**), we have

$$\forall n\, [\forall k\, g_n(2k) = 0 \vee \forall k\, g_n(2k+1) = 0].$$

Thus in the presence of $\mathbf{CT}_0$ there would exist a recursive function ℓ such that

$$\forall n\, [(\ell(n) = 0 \wedge \forall k\, g_n(2k) = 0) \vee (\ell(n) = 1 \wedge \forall k\, g_n(2k+1) = 0)], \tag{35}$$

providing the recursive separation $W_d \subseteq \{n \mid \ell(n) = 0\}$ and $W_e \subseteq \{n \mid \ell(n) = 1\}$.

Corollary: 9.3 $\mathbf{CT}_0$ *refutes* **LLPO**.

Corollary: 9.4 *The combination* **LLPO** *and* $\mathbf{AC}_{\omega,2}$ *yields the existence of non-computable functions.*

Proof: The function ℓ in (35) exists with the help of **LLPO** and $\mathbf{AC}_{\omega,2}$, and ℓ is non-computable. □

On account of $\mathrm{V}(L)$ being a model **LLPO**, there are several well-known mathematical principle that also hold in $\mathrm{V}(L)$. Perhaps, the best known consequence of **LLPO** is the "weak" linearity of the reals, i.e., $\forall x, y \in \mathbb{R}(x \leq y \vee y \leq x)$. The latter can be proved with the help of **LLPO** for Cauchy reals with a modulus of continuity (cf. [25, 5.2.2]) and (crucially) without appealing to any choice principles. Therefore it holds in $\mathrm{V}(L)$ for the Dedekind reals, too.

In a paper by Ishihara [11] it is shown that the following are "constructively" equivalent: **LLPO**, König's lemma, the fan theorem, the Hahn-Banach Theorem and the minimum principle, i.e., every real valued uniformly continuous function on a compact metric space attains its minimum. However, the notion of constructivism in [11] assumes the axiom of countable choice (and perhaps dependent choices) as a basic principle. Indeed, neither König's lemma nor the (decidable) fan theorem (see [25, 4.7.2] for a definition) hold in

V(L) as follows from the incompatibility of both principles with **CT**!. For a proof of the latter fact just take Kleene's primitive recursive 01-tree from [13, Lemma 9.8] (alternatively consult [25, 4.7.6]) which has arbitrarily long paths but has no infinite recursive path.

Corollary: 9.5 V(L) $\not\models$ *Decidable Fan Theorem.*

Funding

Research [7] of the first author (see [7]) was partially supported by the Marie Curie Early Stage Training Network MATHLOGAPS (MEST-CT-2004- 504029). The second author was supported by Royal Society International Joint Projects award 2007/R1 and the EPSRC of the UK through Grant No. EP/G029520/1 (the latter in connection with Theorem 5.5 and sections 8 and 9).

References

[1] P. Aczel: *The type theoretic interpretation of constructive set theory.* In: MacIntyre, A. and Pacholski, L. and Paris, J, editor, *Logic Colloquium '77* (North Holland, Amsterdam 1978) 55–66.

[2] P. Aczel, M. Rathjen: *Notes on constructive set theory*, Technical Report 40, Institut Mittag-Leffler (The Royal Swedish Academy of Sciences, 2001). *http://www.ml.kva.se/preprints/archive2000-2001.php*

[3] M. Beeson: *Continuity in intuitionistic set theories*, in: M Boffa, D. van Dalen, K. McAloon (eds.): *Logic Colloquium '78* (North-Holland, Amsterdam, 1979).

[4] Y. Akama, S. Berardi, S. Hayashi, U. Kohlenbach: *An arithmetical hierarchy of the law of excluded middle and related principles.* In: Proc. of the 19th Annual IEEE Symposium on Logic in Computer Science (LICS'04) (IEEE Press, 2004) 192-201.

[5] M. Beeson: *Foundations of Constructive Mathematics* (Springer Verlag, Berlin, 1980).

[6] A. Blass: *Injectivity, projectivity, and the axiom of choice.* Transactions of the AMS 255 (1979) 31–59.

[7] R.-M. Chen: *Independence and conservativity results for intuitionistic set theory*, Ph.D. Thesis (University of Leeds, 2010).

[8] A.G. Dragalin: *Mathematical Intuitionism-Introduction to Proof Theory* (American Mathematical Society, 1988).

[9] H. Friedman: *Some applications of Kleene's method for intuitionistic systems.* In: A. Mathias and H. Rogers (eds.): *Cambridge Summer School in Mathematical Logic*, volume 337 of *Lectures Notes in Mathematics* (Springer, Berlin, 1973) 113–170.

[10] H. Friedman: *Set Theoretic Foundations for Constructive Analysis*, The Annals of Mathematics 105 (1977) 1–28.

[11] H. Ishihara: *An omniscience principle, the König lemma and the Hahn-Banch theorem.* Zeitschrift für mathematische Logik und grundlagen der Mathematik 36 (1990) 237–240.

[12] S.C. Kleene: *On the interpretation of intuitionistic number theory.* Journal of Symbolic Logic 10 (1945) 109–124.

[13] S.C. Kleene, R.E. Vesley: *The foundations of intuitionistic mathematics* (North-Holland,Amsterdam,1965).

[14] G. Kreisel and A.S. Troelstra: *Formal systems for some branches of intuitionistic analysis.* Annals of Mathematical Logic 1 (1970) 229–387.

[15] V. Lifschitz: CT_0 *is stronger than* $CT_0!$, Proceedings of the American Mathematical Society 73 (1979) 101-106.

[16] D.C. McCarty: *Realizability and recursive mathematics*, PhD thesis, Oxford University (1984), 281 pages.

[17] D.C. McCarty: *Realizability and recursive set theory*, Annals of Pure and Applied Logic 32 (1986) 153–183.

[18] J. Myhill: *Some properties of Intuitionistic Zermelo-Fraenkel set theory.* In: A. Mathias and H. Rogers (eds.): *Cambridge Summer School in Mathematical Logic*, volume 337 of *Lectures Notes in Mathematics* (Springer, Berlin, 1973) 206–231.

[19] J. van Oosten: *Lifschitz's Realizability*, The Journal of Symbolic Logic 55 (1990) 805-821.

[20] J. van Oosten: *Exercises in realizability*, PhD thesis (University of Amsterdam, 1991) 103 pages.

[21] M. Rathjen: *The higher infinite in proof theory.* In: J.A. Makowsky and E.V. Ravve (eds.): *Logic Colloquium '95.* Lecture Notes in Logic, vol. 11 (Springer, New York, Berlin, 1998) 275–304.

[22] M. Rathjen: *Realizability for constructive Zermelo-Fraenkel set theory.* In: J. Väänänen, V. Stoltenberg-Hansen (eds.): *Logic Colloquium '03.* Lecture Notes in Logic 24 (A.K. Peters, Wellesley, Massachusetts, 2006) 282–314.

[23] M. Rathjen: *The disjunction and other properties for constructive Zermelo-Fraenkel set theory.* Journal of Symbolic Logic 70 (2005) 1233–1254.

[24] M. Rathjen: *Metamathematical Properties of Intuitionistic Set Theories with Choice Principles.* In: S. B. Cooper, B. Löwe, A. Sorbi (eds.): *New Computational Paradigms: Changing Conceptions of What is Computable* (Springer, New York, 2008) 287–312.

[25] A.S. Troelstra and D. van Dalen: *Constructivism in Mathematics, Volumes I, II.* (North Holland, Amsterdam, 1988).