PROOF THEORY OF REFLECTION

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Abstract

The paper contains proof-theoretic investigations on extensions of Kripke-Platek set theory, $KP$, which accommodate first order reflection. Ordinal analyses for such theories are obtained by devising cut elimination procedures for infinitary calculi of ramified set theory with $\Pi_n$ reflection rules. This leads to consistency proofs for the theories $KP + \Pi_n$-reflection using a small amount of arithmetic (PRA) and the well-foundedness of a certain ordinal notation system with respect to primitive recursive descending sequences.

Regarding future work, we intend to avail ourselves of these new cut elimination techniques to attain an ordinal analysis of $\Pi_1^1$ comprehension by approaching $\Pi_1^1$ comprehension through transfinite levels of reflection.

1 Introduction

Since 1967, when Takeuti obtained a consistency proof for the subsystem of analysis based on impredicative $\Pi_1^1$ comprehension, great progress has been made in the proof theory of impredicative systems, culminating in the “Admissible Proof Theory” originating with Jäger and Pohlers in the early 80’s. In essence, admissible proof theory is a gathering of cut elimination techniques for infinitary calculi of ramified set theory with $\Sigma$ and/or $\Pi_2$ reflection rules that lends itself to ordinal analyses of theories of the form $KP + \text{“there are } x \text{ many admissibles”}$ or $KP + \text{“there are many admissibles”}$. By way of illustration, the subsystem of analysis with $\Delta_1^0$ comprehension and Bar induction can be couched in such terms, for it is naturally interpretable in the set theory $KPi := KP + \forall y \exists z (y \in z \land z \text{ is admissible})$ (cf. Jäger and Pohlers [1982]). Nonetheless, the advanced techniques of admissible proof theory are way too weak for dealing with significantly stronger theories like $\Pi_1^1$ analysis, let alone full analysis. An ordinal analysis of $\Pi_1^1$ comprehension would inherently involve one for all the theories $KP + \Pi_n$-reflection, and, therefore, a first step to be taken towards this end consists in devising ordinal notation systems that give rise to cut elimination procedures for infinitary calculi with $\Pi_n$ reflection rules.

In this paper we focus on the ordinal analysis of $\Pi_3$ reflection. This means no genuine loss of generality, as the removal of $\Pi_3$ reflection rules in derivations already exhibits the pattern of cut elimination that applies for arbitrary $\Pi_n$ reflection rules as well.

As regards the advance achieved in this paper, it should be pointed out that we cherish much higher expectations than just moving a tiny step towards $\Pi_1^1$ comprehension. The idea is that $\Pi_1^2$ comprehension can be fathomed by going through transfinite levels of

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2 Recall that the salient feature of admissible sets is that they are models of $\Delta_0$ collection and that $\Delta_0$ collection is equivalent to $\Sigma$ reflection on the basis of the other axioms of $KP$ (see Barwise [1975]). Furthermore, admissible sets of the form $L_\alpha$ also satisfy $\Pi_2$ reflection.
reflection; and thus an ordinal analysis for it should be attainable via an, admittedly, considerable extension of the machinery laid out in this paper.

The paper is organized as follows: Section 2 introduces set-theoretic reflection and situates it with regard to non-monotone inductive definitions, subsystems of analysis with \( \beta \)-model reflection and \( \Pi_1 \) comprehension. Section 3 provides a formalization of \( KP \) as sequent calculus. In Section 4, so-called collapsing functions are developed which give rise to a strong ordinal notation system \( T(K) \). \( T(K) \) is introduced in Section 5. In Section 6, we define an infinitary calculus \( RS(K) \) with \( \Pi_3 \) and \( \Pi_2 \) reflection. Here we draw on Buchholz’s [1993] approach to local predicativity, in particular, the notion of operator controlled derivations. Section 7 deals with the elimination of uncritical cuts in \( RS(K) \) derivations, i.e. cuts whose cut formulae have not been introduced by reflection rules. Section 8 is devoted to interpreting \( KP + \Pi_3-Ref \) in \( RS(K) \). Section 9 and 10 are concerned with the removal of critical cuts in \( RS(K) \) derivation. Finally, in Section 11, we indicate how ordinal analyses for arbitrary \( \Pi_n \) reflections can be obtained. This Section also contains some remarks on consistency proofs.

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2 Set-theoretic reflection and related principles

This Section provides some background information and contains (almost) no proofs. Its theorems will not be used in later Sections.

We shall consider set-theoretic reflection on the basis of Kripke-Platek set theory, \( KP \), which arises from \( ZF \) by omitting\(^4\) the power set axiom and restricting the axiom schemes of comprehension and collection to absolute predicates, i.e. \( \Delta_0 \) predicates.

Definition 2.1 A set-theoretic formula is said to be \( \Pi_n \) (respectively \( \Sigma_n \)) if it consists of a string of \( n \) alternating quantifiers beginning with a universal one (respectively existential one), followed by a \( \Delta_0 \) formula. By \( \Pi_n \) reflection we mean the scheme

\[
F \rightarrow \exists z[Tran(z) \land z \neq \emptyset \land F^z],
\]

\(^3\)Meanwhile, Kurt Schütte has given another presentation of the ordinal analysis of \( KP + \Pi_3-Ref \) using the calculus of positive and negative forms (cf. Schütte [1993]).

\(^4\)This contrasts with Barwise [1975], where the infinity axiom is not included in \( KP \).
where $F$ is $\Pi_n$ and $Tran(z)$ expresses that $z$ is a transitive set; $F^z$ denotes the formula that arises from $F$ by restricting the unbounded quantifiers to $z$, i.e. $\forall x$ gets replaced with $(\forall x \in z)$ and $\exists x$ with $(\exists x \in z)$.

An ordinal $\alpha > 0$ is said to be $\Pi_n$–reflecting if $L_\alpha \models \Pi_n \text{reflection}$. $\Sigma_n$ reflection and $\Sigma_n$–reflecting are defined analogously.

Note that if $\kappa$ is $\Pi_n$–reflecting and $n \geq 2$, then $\kappa$ must be a limit ordinal $> \omega$. Therefore $L_\kappa$ is a model of all the axioms of $KP$ other than $\Delta_0$ collection. But $\Delta_0$ collection issues from $\Pi_n$ reflection, and hence $L_\kappa \models KP + \Pi_n \text{reflection}$.

$\Pi_n$–reflecting ordinals have interesting points of contact with non–monotone inductive definitions.

**Definition 2.2** A function $\Gamma$ from the power set of $\mathbb{N}$ into itself is called an operator on $\mathbb{N}$. $\Gamma$ determines a transfinite sequence $\langle \Gamma^\xi : \xi \in ON \rangle$ of subsets of $\mathbb{N}$,

$$\Gamma^\lambda = \bigcup_{\xi < \lambda} \Gamma^\xi \cup \Gamma(\Gamma^\xi),$$

where $\Gamma^{< \lambda} = \bigcup_{\xi < \lambda} \Gamma^\xi$.

The *closure ordinal* $|\Gamma|$ of $\Gamma$ is the least ordinal $\rho$ such that $\Gamma^{\rho+1} = \Gamma^\rho$. $\Gamma$ is said to be $\Pi_k^0$ when there is an arithmetic $\Pi_k^0$ formula $F(U,u)$ with second order variable $U$ such that, for all $X \subseteq \mathbb{N}$,

$$\Gamma(X) = \{ n \in \mathbb{N} : F(X,n) \}.$$

Let $| \Pi_k^0 | := \sup\{ |\Gamma| : \Gamma \text{ is } \Pi_k^0 \}$.

Owing to Aczel and Richter [1974], we have the following characterization.

**Theorem 2.3** For $k > 0$,

$$| \Pi_k^0 | = \text{first } \Pi_{k+1} \text{–reflecting ordinal}.$$

Several notions of recursively large ordinals are modelled upon notions of large cardinals. This is especially true of notions like “recursively inaccessible ordinal” and “recursively Mahlo ordinal”. It turns out that the least $\Pi_3$–reflecting ordinal is greater than the least recursively Mahlo ordinal, indeed much greater than any transfinite iteration of recursive “Mahloness” from below. For instance, every $\Pi_3$–reflecting ordinal $\kappa$ is recursively $\kappa$–Mahlo.

**Definition 2.4** An ordinal $\kappa$ is *recursively Mahlo* if for every $\kappa$–recursive function $f : \kappa \rightarrow \kappa$ there exists an admissible ordinal $\rho < \kappa$ that is closed under $f$.

A recursively Mahlo ordinal $\kappa$ is *recursively $\alpha$–Mahlo* if for every $\kappa$–recursive function $f : \kappa \rightarrow \kappa$ there exists an admissible ordinal $\rho < \kappa$ closed under $f$ such that $\rho$ is recursively $\beta$–Mahlo for all $\beta < \alpha$. 2
Regarding a notion of large cardinal to which \( \Pi_3 \)-reflecting ordinals provide the recursive counterparts, Aczel and Richter [1974] have convincingly argued that this should be the weakly compact (or \( \Pi_1^1 \) indescribable) cardinals. By the same token, for \( n > 1 \), \( \Pi_{n+2} \)-reflecting ordinals should be regarded as the recursive analogues of \( \Pi_n^1 \) indescribable cardinals.

Since subsystems of analysis appear to be the most common measure for the calibration of proof-theoretic strength of theories, we shall also give a characterization of \( \text{KP} + \Pi_n \text{reflection} \) (for \( n > 2 \)) in terms of subsystems of analysis. However, \( \Pi_n \text{reflection} \) does not simply translate into familiar levels of comprehension of the projective hierarchy. In proof-theoretic strength, the theories \( \text{KP} + \Pi_n \text{reflection} \) \((n > 2)\) are strictly between \( \Delta^1_2 \) comprehension plus Bar-induction and \( \Pi_2^1 \) comprehension. It turns out that set-theoretic reflection by transitive sets is related to \( \beta \)-model reflection.

Via coding, any set of natural numbers \( X \) gives rise to a countable collection of subsets of \( \mathbb{N} \), \( \{(X)_k : k \in \mathbb{N}\} \), where \( (X)_k = \{m : 2^k 3^m \in X\} \). The structure

\[
B_X = \langle \mathbb{N}, \{(X)_k : k \in \mathbb{N}\}, 0, 1, +, \ldots, =, \in \rangle
\]

(where the first order part is standard) is a \( \beta \)-model if, for any \( \Pi_1^1 \) sentence \( A \) with parameters from \( B_X \), \( A \) holds in \( B_X \) iff \( A \) is true (or, equivalently, the notion of well-foundedness is absolute with regard to \( B_X \)). We shall refer to \( B_X \) as the the model coded by \( X \). The notion of \textit{countably coded} \( \beta \)-model can be formalized in analysis. Hereditarily countable sets can be identified with certain well-founded trees on \( \mathbb{N} \) and thus can be modelled in second order arithmetic (see Apt and Marek [1974]). Let \( \text{ACA} \) denote the subsystem of second order arithmetic with comprehension restricted to arithmetic predicates. We use \( Z \in X \) as an abbreviation for \( \exists k[Z = (X)_k] \). The following characterization can be obtained (Rathjen [1991b]).

\textbf{Theorem 2.5} For \( n > 2 \), \( \text{KP} + \Pi_n \text{reflection} \) proves the same \( \Pi_1^1 \) sentences of second order arithmetic as \( \text{ACA} \) plus Bar-induction augmented by the scheme

\[
\forall Z_1, \ldots, Z_k [A(Z_1, \ldots, Z_k) \rightarrow \exists X[Z_1 \in X \land \ldots Z_k \in X \land B_X \models A(Z_1, \ldots, Z_k)]],
\]

where \( A \) ranges over the \( \Pi_{n+1}^1 \) formulae of second order arithmetic and the free second order variables of \( A \) are among the ones shown. It is readily shown that \( \Delta_1^2 \) comprehension is derivable in the latter theory.

Next, we are going to explain why an ordinal analysis of \( \Pi_2^1 \) comprehension, unlike \( \Delta_1^1 \) comprehension, has to exceed the methods of admissible proof theory. On the set-theoretic side, \( \Pi_2^1 \) comprehension corresponds to \( \Sigma_1 \text{ separation} \), i.e. the scheme

\[
\exists z(z = \{x \in a : F(x)\})
\]

for all \( \Sigma_1 \) formulae \( F(x) \) in which \( z \) does not occur free. The precise relationship reads as follows.
Theorem 2.6 $KP + \Sigma_1$ separation and $(\Pi^1_2 - CA) + BI$ prove the same theorems of second order arithmetic.\textsuperscript{5}

The ordinals $\kappa$ such that $L_\kappa \models KP + \Sigma_1$ separation are familiar from ordinal recursion theory (see Barwise [1975], Hinman [1978]). An admissible ordinal $\kappa$ is said to be non-projectible if there is no (total) $\kappa$–recursive function mapping $\kappa$ one–one into some $\beta < \kappa$.

The key to the “largeness” properties of nonprojectible ordinals is the following.

Theorem 2.7 For any nonprojectible ordinal $\kappa$, $L_\kappa$ is a limit of $\Sigma_1$–elementary substruc-

Ordinals $\rho$ satisfying $L_\rho \prec L_\kappa$ for some $\kappa > \rho$ have strong reflecting properties. For instance, if $L_\rho \models F$ for some set–theoretic sentence $F$ (possibly containing parameters from $L_\rho$), then there exists a $\gamma < \rho$ such that $L_\gamma \models F$ because from $L_\rho \models F$ we can infer $L_\kappa \models \exists \gamma F^{L_\gamma}$ which yields $L_\rho \models \exists \gamma F^{L_\gamma}$ using $L_\rho \prec L_\kappa$.

The last remark makes it clear that an ordinal analysis of $\Pi^1_2$ comprehension would necessarily involve a proof–theoretic treatment of reflections.

3 A sequent calculus for $KP$

Since later on we are going to interpret $KP$ in an infinitary sequent calculus $RS(\kappa)$, we will furnish $KP$ in sequent calculus style. For technical reasons we shall treat equality as a defined symbol and assume that formulae are in negation normal form. Also bounded quantifiers will be treated syntactically as quantifiers in their own right.

The language of $KP$, $L$, consists of: free variables $a_1, a_2, a_3, \ldots$; bound variables $x_1, x_2, x_3, \ldots$; the predicate symbol $\in$; the logical symbols $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$; and parenthesis.

The atomic formulae are those of the form $(a \in b)$ with free variables $a, b$. Formulae are built from atomic and negated atomic formulae by means of the connectives $\wedge$, $\vee$ and the following construction step: If $b$ is a free variable and $F(a)$ is a formula in which the bound variable $x$ does not occur, then $(\forall x \in b)F(x), (\exists x \in b)F(x), \forall x F(x), \exists x F(x)$ are formulae.

A formula which contains only bounded quantifiers, i.e. quantifiers of the form $(\forall x \in b), (\exists x \in b)$, is said to be a $\Delta_0$–formula. The negation, $\neg A$, of a non–atomic formula $A$ is defined to be the formula obtained from $A$ by (i) putting $\neg$ in front any atomic subformula, (ii) replacing $\wedge$, $\vee$, $(\forall x \in b), (\exists x \in b)$, $\forall x$, $\exists x$ by $\forall$, $\neg$, $(\exists x \in b), (\forall x \in b), \exists x, \forall x$, respectively, and (iii) dropping double negations.

\textsuperscript{5}For this result to hold it is crucial that Infinity is among the axioms of $KP$.

\textsuperscript{6}$L_\rho$ is a $\Sigma_1$–elementary substructure of $L_\kappa$ if every $\Sigma_1$ sentence with parameters from $L_\rho$ that holds in $L_\kappa$ holds in $L_\rho$ as well.
Equality is defined by \( a = b : \iff (\forall x \in a)(x \in b) \land (\forall x \in b)(x \in a) \). As a result of this, we will have to state the Axiom of Extensionality in a different way than usually.

We use \( A, B, C, \ldots, F(a), G(a) \) as meta–variables for formulae. Upper case Greek letters \( \Delta, \Gamma, \Lambda, \ldots \) range over finite sets of formulae. The meaning of \( \{A_1, \ldots, A_n\} \) is the disjunction \( A_1 \lor \cdots \lor A_n \). \( \Gamma, A \) stands for \( \Gamma \cup \{A\} \). As usual, \( A \to B \) abbreviates \( \lnot A \lor B \).

For any \( \Gamma \) and formula \( A \),

\[
\Gamma, A, \lnot A
\]

is a logical axiom of \( KP \).

The set–theoretic axioms of \( KP \) are:

- **Extensionality:** \( \Gamma, a = b \to [F(a) \leftrightarrow F(b)] \) for all formulae \( F(a) \).
- **Foundation:** \( \Gamma, \exists x G(x) \to \exists x[G(x) \land (\forall y \in x)\lnot G(y)] \) for all formulae \( G(b) \).
- **Pairing:** \( \Gamma, \exists x (x = \{a, b\}) \).
- **Union:** \( \Gamma, \exists x (x = \bigcup a) \).
- **Infinity:** \( \Gamma, \exists x [x \neq \emptyset \land (\forall y \in x)(\exists z \in x)(y \in z)] \).
- **\( \Delta_0 \)–Separation:** \( \Gamma, \exists x (x = \{y \in a : F(y)\}) \) for all \( \Delta_0 \)–formulae \( F(b) \)
- **\( \Delta_0 \)–Collection:** \( \Gamma, (\forall x \in a)\exists y G(x, y) \to \exists z(\forall x \in a)(\exists y \in z)G(x, y) \) for all \( \Delta_0 \)–formulae \( G(b) \).

The logical rules of inference are:

\[
\begin{align*}
(\land) & \quad \frac{\Gamma, A}{\Gamma, A \land A'} & (\lor) & \quad \frac{\Gamma, A}{\Gamma, A \lor A} \quad \text{if } i \in \{0, 1\} \\
(b\forall) & \quad \frac{\Gamma, a \in b \to F(a)}{\Gamma, (\forall x \in b)F(x)} & (\forall) & \quad \frac{\Gamma, F(a)}{\Gamma, \forall x F(x)} \\
(b\exists) & \quad \frac{\Gamma, a \in b \land F(a)}{\Gamma, (\exists x \in b)F(x)} & (\exists) & \quad \frac{\Gamma, F(a)}{\Gamma, \exists x F(x)} \\
(Cut) & \quad \frac{\Gamma, A}{\Gamma, \lnot A}
\end{align*}
\]

where in \( \lor \) and \( b\forall \) the free variable \( a \) is not to occur in the conclusion of the inference.

We formalize \( \Pi_n \)–reflection as an inference rule.

**Definition 3.1** The sequent calculus \( KP + \Pi_n–Ref \) arises from \( KP \) by adjoining the \( \Pi_n \)–reflection rule of inference

\[
(\Pi_n–Ref) \quad \frac{\Gamma, A}{\Gamma, \exists \overline{z}[Tran(z) \land z \neq \emptyset \land A^\overline{z}]} \]

\[5\]
for all Πₙ-formulae A.

4 Collapsing functions

We are going to develop so-called collapsing functions which give rise to a strong ordinal notation system \( T(K) \). Rather than developing such functions on the basis of \( \Pi_3 \) reflecting ordinals, we build them by employing a weakly compact cardinal. This is not a far-fetched assumption since \( \Pi_3 \) reflecting ordinals are the recursive analogues of weakly compact cardinals (see Aczel and Richter [1974]). Proceeding this way, allows us to develop the right intuitions about these functions and to side-step fiddly and delicate ordinal recursion theory (cf. Rathjen [1993a] and [1993c]). Of course, another option would be to abstain completely from set theory by directly defining the primitive recursive notation system. However, nude ordinal notation systems without any set-theoretic interpretation tend to be hard to grasp.

Firstly, we remind the reader of some set-theoretical notions and take this as an opportunity to fix some notations.

**Definition 4.1** Let \( \text{On} \) denote the class of ordinals and let \( \text{Lim} \) be the class of limit ordinals. The cumulative hierarchy, \( V = \bigcup \{ V_\alpha : \alpha \in \text{On} \} \), is defined by: \( V_0 = \emptyset \), \( V_{\alpha+1} = \{ X : X \subseteq V_\alpha \} \), \( V_\lambda = \bigcup \{ V_\xi : \xi < \lambda \} \) for \( \lambda \in \text{Lim} \).

Let \( \mathbb{A} = \langle A, U_1, \ldots, f_1, \ldots, c_1, \ldots \rangle \) be a structure for a language. The extension of \( \mathcal{L} \) to second order, denoted \( \mathcal{L}_2 \), is given as follows. Besides symbols of \( \mathcal{L} \), a formula of \( \mathcal{L}_2 \) may contain second order quantifiers \( \forall X, \exists X \), and atomic formulae \( X(t) \), where \( X \) is a second order variable and \( t \) is a term of \( \mathcal{L} \).

Satisfaction of sentences of \( \mathcal{L}_2 \) in \( \mathbb{A} \) is defined as follows. Variables of first order range over elements of \( A \). Variables of second order range over the full power set of \( A \). A formula \( X(t) \) is interpreted as \( t \in X \).

A formula of \( \mathcal{L}_2 \) is \( \Pi^1_n \) if it is of the form

\[
\forall X_1 \exists X_2 \cdots QX_n F(X_1, \cdots, X_n),
\]

where \( F(X_1, \cdots, X_n) \) does not contain second order quantifiers and the \( n \) second order quantifiers in \( \forall X_1 \exists X_2 \cdots QX_n \) are alternating.

**Definition 4.2** A cardinal \( \kappa \) is \( \Pi^1_n \)-indestructible, if whenever \( U_1, \ldots, U_m \subseteq V_\kappa \) and \( F \) is a \( \Pi^1_n \) sentence of the language of \( \langle V_\kappa, \in, U_1, \ldots, U_m \rangle \) such that

\[
\langle V_\kappa, \in, U_1, \ldots, U_m \rangle \models F
\]

then, for some \( 0 < \alpha < \kappa \),

\[
\langle V_\alpha, \in, U_1 \cap V_\alpha, \ldots, U_m \cap V_\alpha \rangle \models F.
\]
Definition 4.3 A class of ordinals $C$ is unbounded in $\alpha \in \text{Lim}$ if $(\forall \xi < \alpha)(\exists \delta \in C)(\xi < \delta \land \delta < \alpha)$.

Let $\kappa$ be a regular cardinal $> \omega$. A class $C$ of ordinals is closed in $\kappa$ if whenever $\lambda$ is a limit ordinal $< \kappa$ such that $C$ is unbounded in $\lambda$, then $\lambda \in C$.

A class of ordinals $S$ is stationary in $\kappa$ if, for all $C$ which are closed and unbounded in $\kappa$, $S \cap C \neq \emptyset$.

$\kappa$ is Mahlo on $X \subseteq \text{On}$ if $\kappa \in X$ and $X$ is stationary in $\kappa$. The Mahlo thinning–operation $M$ is defined as follows

$$M(X) = \{\alpha \in X : X \text{ ist stationary in } \alpha\}.$$ 

The $\Pi^1_1$ indescribable cardinals are also called (or proved to be the same as) the weakly compact cardinals (see Jech [1979]). To give an inkling as to the strength of weakly compact cardinals, we introduce the notion of Mahlo cardinal. A cardinal is called Mahlo cardinal (respectively, weakly Mahlo cardinal) if, for every function $f : \kappa \rightarrow \kappa$, there exists an inaccessible cardinal (respectively, weakly inaccessible cardinal) $\rho < \kappa$ such that $\rho$ is closed under $f$. Equivalently, $\kappa$ is Mahlo (respectively, weakly Mahlo) iff the inaccessible cardinals (respectively, weakly inaccessible cardinals) are stationary in $\kappa$.

Remark 4.4 If $\kappa$ is weakly compact, then $\kappa$ is Mahlo and the Mahlo cardinals are stationary in $\kappa$.

The Veblen–function figures prominently in predicative proof theory (cf. Feferman [1968], Schütte [1977], Sec.13 and Pohlers [1989].) We are going to incorporate this function in our notation system.

Definition 4.5 The Veblen–function $\varphi_{\alpha \beta} := \varphi_{\alpha}^{\beta}$ is defined by transfinite recursion on $\alpha$ by letting $\varphi_{\alpha}$ be the function that enumerates the class of ordinals

$$\{\omega^\gamma : \gamma \in \text{On} \land (\forall \xi < \alpha)[\varphi_{\xi}(\omega^\gamma) = \omega^\gamma]\}.$$ 

Corollary 4.6 (i) $\varphi_{0}^{\beta} = \omega^{\beta}$.

(ii) $\xi, \eta < \varphi_{\alpha \beta} \Rightarrow \xi + \eta < \varphi_{\alpha \beta}$.

(iii) $\xi < \zeta \Rightarrow \varphi_{\alpha \xi} < \varphi_{\alpha \zeta}$.

(iv) $\alpha < \beta \Rightarrow \varphi_{\alpha \varphi_{\beta \xi}} = \varphi_{\beta \xi}$.

Definition 4.7 To save space, we introduce some abbreviations. $\text{fun}(g)$ abbreviates that $g$ is a function. $\text{dom}(g)$ and $\text{ran}(g)$ denote the domain and the range of $g$, respectively. $g^{\prime}x$ stands for the set $\{g(u) : u \in x \cap \text{dom}(g)\}$. Let $\text{pow}(a) := \{x : x \subseteq a\}$. For $U$ a second order variable, let $\text{club}(U)$ be the formula expressing that $U$ is closed and unbounded in $\text{On}$, i.e. $\forall \alpha(\exists \beta \in U)(\alpha < \beta) \land (\forall \lambda \in \text{Lim})[(\forall \xi < \lambda)(\exists \delta \in U)(\xi < \delta < \lambda) \rightarrow \lambda \in U]$. 

7
For classes $G$, one defines $\text{fun}(G)$, $\text{ran}(G)$ and $\text{dom}(G)$ analogously.

Let

$$\Omega_\xi = \begin{cases} \aleph_\xi & \text{if } \xi > 0 \\ 0 & \text{otherwise.} \end{cases}$$

**General assumption:** From now on, we assume that there exists a weakly compact cardinal, denoted $K$.

$\text{Reg}$ denotes the set of uncountable regular cardinals $< K$. We shall use the variables $\kappa, \pi, \tau, \kappa', \pi'$ exclusively for elements of $\text{Reg}$.

**Definition 4.8** By recursion on $\alpha$, we define sets $C(\alpha, \beta)$ and $M^\alpha$, and ordinals $\Xi_\kappa$ and $\Psi_\pi^\xi(\alpha)$ as follows:\footnote{Closure of $C(\alpha, \beta)$ under $(\xi \mapsto \Omega_\xi)_{\xi < K}$ is only demanded for technical convenience. This closure property does not contribute to the strength of the intended ordinal notation system. Likewise, it would suffice to demand only closure under $\xi \mapsto \omega^\xi$ instead of $\varphi$.}

$$C(\alpha, \beta) = \begin{cases} \text{closure of } \beta \cup \{0, K\} \\ \text{under } +, \\ (\xi \mapsto \varphi \xi \eta), \\ (\xi \mapsto \Omega_\xi)_{\xi < K}, \\ (\xi \mapsto \Xi(\xi))_{\xi < \alpha}, \\ (\xi \pi \delta \mapsto \Psi(\xi)_{\pi}(\delta))_{\xi \leq \delta < \alpha}. \end{cases}$$

$M^0 = K \cap \text{Lim}$, and, for $\alpha > 0$,

$$M^\alpha = \left\{ \pi < K : C(\alpha, \pi) \cap K = \pi \land (\forall \xi \in C(\alpha, \pi) \cap \alpha)[M^\xi \text{ stationary in } \pi] \land \alpha \in C(\alpha, \pi) \right\}$$

$$\Xi(\alpha) = \min(M^\alpha \cup \{K\}).$$

For $\xi \leq \alpha$,

$$\Psi_\pi^\xi(\alpha) = \min(\{\rho \in M^\xi \cap \pi : C(\alpha, \rho) \cap \pi = \rho \land \pi, \alpha \in C(\alpha, \rho)\} \cup \{\pi\}).$$

*Note that in the above definition, we tacitly assume, in keeping with our convention, that $\pi$ ranges over regular cardinals.*

**Remark 4.9** To gain a better picture of the sets $M^\alpha$, it is instructive to study some initial cases. It is readily verified that any $\kappa \in M^1$ is weakly inaccessible since $\kappa$ is regular and closed under $\Omega$. Therefore, $M^1$ consists of the weakly inaccessible cardinals below $K$. Subsequently, we come to see that, for any $\pi \in M^2$, $M^1$ is stationary in $\pi$ and hence $\pi$ is weakly Mahlo. This pattern continues for quite a while, i.e., $M^3$ consists of the weakly hyper–Mahlo cardinals below $K$, $M^4$ consists of the weakly hyper–hyper–Mahlo cardinals below $K$ and so forth. However, only for weakly $\alpha < K$, $M^\alpha$ can be couched in terms of $\alpha$–hyper–Mahloness. By way of contrast, $M^K$ is obtained by diagonalizing over the sequence $(M^\alpha)_{\alpha < K}$. 
**Remark 4.10** The inductive generation of $C(\alpha, \beta)$ is completed after $\omega$ stages. Therefore $C(\alpha, \beta)$ can be depicted as $C(\alpha, \beta) = \bigcup_{n<\omega} C_n(\alpha, \beta)$, where $C_n(\alpha, \beta)$ consists of the elements constructed up to stage $n$. We emphasize this build-up of $C(\alpha, \beta)$. Since we will be proving properties of the elements of this set by induction on stages $C_n(\alpha, \beta)$.

**Lemma 4.11** (i) $\alpha \leq \alpha' \land \beta \leq \beta' \implies C(\alpha, \beta) \subseteq C(\alpha', \beta')$.

(ii) $\beta < \pi \implies |C(\alpha, \beta)| < \pi$.

(iii) $\lambda \in \text{Lim} \implies C(\alpha, \lambda) = \bigcup_{\eta < \lambda} C(\alpha, \eta) \land C(\lambda, \alpha) = \bigcup_{\eta < \lambda} C(\eta, \alpha)$.

(iv) $C(\alpha, \Xi(\alpha)) \cap \mathcal{K} = \Xi(\alpha)$.

(v) $C(\alpha, \Psi_{\zeta}(\alpha)) \cap \pi = \Psi_{\zeta}(\alpha)$.

(vi) If $\pi \in M^\alpha$ and $\zeta \in C(\alpha, \pi) \cap \alpha$, then $\pi \in M^\zeta$.

(vii) If $M^\xi$ is stationary in $\pi$, then $\pi \in M^\xi$.

**Proof.** (i)–(v) are obvious.

(vi): The assumptions imply $C(\alpha, \pi) \cap \mathcal{K} = \pi$ and $(\forall \xi \in C(\alpha, \pi) \cap \alpha)[M^\xi \text{ stationary in } \pi]$; hence, a fortiori, $C(\zeta, \pi) \cap \mathcal{K} = \pi$ and $(\forall \xi \in C(\zeta, \pi) \cap \zeta)[M^\xi \text{ stationary in } \pi]$. Since $M^\xi$ is also stationary in $\pi$, we get $\zeta \in C(\zeta, \pi)$. Therefore, $\pi \in M^\zeta$.

(vii): Let $\rho \in M^\xi \cap \pi$. Then $\xi \in C(\xi, \rho)$, whence $\xi \in C(\xi, \pi)$. Since $M^\xi$ is unbounded in $\pi$ it follows $C(\xi, \pi) \cap \mathcal{K} = (\bigcup \{C(\xi, \rho) : \rho \in M^\xi \cap \pi\}) \cap \mathcal{K} = \bigcup \{C(\xi, \rho) \cap \mathcal{K} : \rho \in M^\xi \cap \pi\} = \bigcup \{\rho : \rho \in M^\xi \cap \pi\} = \pi$.

Now suppose that $\eta \in C(\xi, \pi) \cap \xi$, and let $U \subseteq \pi$ be closed and unbounded in $\pi$. Since $M^\xi$ is stationary in $\pi$, we may select a $\rho \in M^\xi \cap \pi$ so that $\eta \in C(\xi, \rho)$ and $U$ is already closed and unbounded in $\rho$. $M^\eta$ being stationary in $\rho$ implies $U \cap M^\eta \cap \rho \neq \emptyset$; thus $U \cap M^\eta \cap \pi \neq \emptyset$. Thence, $M^\eta$ is stationary in $\pi$. □

Let $\mathcal{K}^\alpha$ denote the least ordinal $\alpha > \mathcal{K}$ satisfying $(\forall \xi, \eta < \alpha)(\varphi \xi \eta < \alpha)$.

**Theorem 4.12** For all $\alpha < \mathcal{K}^\alpha$, $M^\alpha$ is stationary in $\mathcal{K}$ and hence $\Xi(\alpha) < \mathcal{K}$.

**Proof.** Each ordinal $\mathcal{K} < \beta < \mathcal{K}^\alpha$ has a unique representation of either form $\beta = \omega^\beta_1 + \cdots + \omega^\beta_n$ with $\beta > \beta_1 \geq \cdots \geq \beta_n$ and $n > 0$, or $\beta = \varphi \beta_1 \beta_2$ with $\beta > \beta_1, \beta_2$, denoted $\beta =_{NF} \omega^\beta_1 + \cdots + \omega^\beta_n$ and $\beta =_{NF} \varphi \beta_1 \beta_2$, respectively. Due to uniqueness, we can define an injective mapping

$$f : \mathcal{K}^\alpha \longrightarrow L_{\mathcal{K}}$$
by letting
$$f(\beta) = \begin{cases} 
\beta & \text{if } \beta < K \\
\{1\} & \text{if } \beta = K \\
\langle 2, f(\beta_1), \ldots, f(\beta_n) \rangle & \text{if } \beta = NF \omega^{\beta_1} + \cdots + \omega^{\beta_n} \text{ and } K < \beta \\
\langle 3, f(\beta_1), f(\beta_2) \rangle & \text{if } \beta = NF \varphi \beta_1 \beta_2 \text{ and } K < \beta.
\end{cases}$$

Putting
$$f(\alpha) \triangleleft f(\beta) : \iff \alpha < \beta,$$
\(\triangleleft\) defines a well-ordering on a subset of \(L_K\) of order type \(K^\Gamma\).

To show the Theorem, we proceed by induction on \(\alpha\), or, equivalently, by induction on \(\triangleleft\).

Assume that \(E\) is closed and unbounded in \(K\). We have to verify that \(M^\alpha \cap E \neq \emptyset\).

Since \(\alpha < K^\Gamma\), we may utilize the above representations to see that there are finitely many ordinals \(\alpha_1, \ldots, \alpha_n < K\) such that \(\alpha\) is in the closure of \(\{\alpha_1, \ldots, \alpha_n, K\}\) under + and \(\varphi\). Therefore we can pick a \(\rho_0 < K\) with \(\alpha \in C(\alpha, \rho_0)\). Since \(E \setminus \rho_0\) is also closed and unbounded in \(K\), we may assume that \(E \cap \rho_0 = \emptyset\). Using the induction hypothesis, for all \(\beta < \alpha\), \(M^\beta\) is stationary in \(K\). Define
$$U_1 := \{f(\alpha)\}, \ U_2 := \{\langle x, y \rangle : x < y\}, \ \text{and} \ U_3 := \bigcup_{\beta < \alpha} (M^\beta \times \{f(\beta)\}).$$

The following sentences are satisfied in the structure \(\langle V_K, \in, U_1, U_2, U_3, E \rangle\):

(1) \(\forall G \forall \delta [\text{fun}(G) \land \text{dom}(G) = \delta \land \text{ran}(G) \subseteq On \rightarrow \exists \gamma (G'' \delta \subseteq \gamma)]\)

(2) \(\forall a \exists b \exists \exists \exists g [b = \text{pow}(a) \land \text{fun}(g) \land \text{dom}(g) = b \land \text{ran}(g) = \beta \land \text{g injective}]\)

(3) \(U_1 \neq \emptyset \land \forall \gamma \exists \delta (\gamma < \delta \land E(\delta))\)

(4) \(\forall X \forall s \forall t [U_1(t) \land U_2((s, t)) \land \text{club}(X) \rightarrow \{y : U_3((y, s))\} \cap X \neq \emptyset]\)

Employing the \(\Pi^1_1\)-indescribability of \(K\), there exists \(\pi < K\) such that the structure
$$\langle V_\pi, \in, U_1 \cap \pi, U_2 \cap \pi, U_3 \cap \pi, E \cap \pi\rangle$$
satisfies:

(a) \(\forall G \forall \delta [\text{fun}(G) \land \text{dom}(G) = \delta \land \text{ran}(G) \subseteq On \rightarrow \exists \gamma (G'' \delta \subseteq \gamma)]\)

(b) \(\forall a \exists b \exists \exists \exists g [b = \text{pow}(a) \land \text{fun}(g) \land \text{dom}(g) = b \land \text{ran}(g) = \beta \land \text{g injective}]\)

(c) \(U_1 \cap \pi \neq \emptyset \land \forall \gamma \exists \delta (\gamma < \delta \land \delta \in E \cap \pi)\)

\(^8\langle x, y \rangle := \{\{x\}, \{x, y\}\}; \{x_1, \ldots, x_n\} := \langle (x_1, \ldots, x_n), x_{n+1} \rangle\) for \(n > 2\).
Then, \(\forall\beta < \alpha\) \(\exists\) a \(\alpha\). If, however, \(\beta <\alpha\), then Proposition 4.16 shows that \(\beta < \alpha\) must be inaccessible. Due to (c), \(f(\alpha) \in V_\pi\) and \(E\) is unbounded in \(\pi\); whence \(\pi \in E\). (d) forces that

\[
(\ast) \quad (\forall \beta < \alpha) [f(\beta) \in V_\pi \to M^\beta \text{ stationary in } \pi].
\]

Next, we want to verify

\[
(+) \quad (\forall \eta \in C(\alpha, \pi)) [f(\eta) \in V_\pi].
\]

Set \(X := \{\eta \in C(\alpha, \pi) : f(\eta) \in V_\pi\}\). Clearly, \(\pi \cup \{0, K\} \subseteq X\). If \(\eta = \omega \eta_1 + \cdots + \omega \eta_n\) and \(\eta_1, \ldots, \eta_n \in X\), then \(\eta \in X\) since \(\pi\) is closed under + and \(\zeta \mapsto \omega \zeta\) and \(V_\pi\) is closed under \(\langle \cdot, \cdot \rangle\). Likewise, \(\pi\) being closed under \(\varphi\) implies that \(X\) is closed under \(\varphi\).

For \(\sigma \in X \cap \mathcal{K}\), \(f(\sigma) = \sigma \in V_\pi\): thus \(\sigma < \pi\) and hence \(\Omega_\sigma < \pi\) because \(\pi\) is inaccessible.

If \(\beta \in X \cap \alpha\), then, according to (\(\ast\)), \(M^\beta\) is stationary in \(\pi\), yielding \(\Xi(\beta) = f(\Xi(\beta)) < \pi\).

If \(\kappa, \xi, \delta \in X\) and \(\xi \leq \delta < \alpha\), then \(f(\kappa) = \kappa < \pi\) and therefore \(\Psi_\kappa(\delta) < \pi\). So it turns out that \(X\) enjoys all the closure properties defining \(C(\alpha, \pi)\). This verifies (\(+)\).

From \(\pi \in E\) it follows \(\alpha \in C(\alpha, \pi)\). Using (\(\ast\)) and (\(+)\), we obtain

\[
(\forall \beta \in C(\alpha, \pi) \cap \alpha) [M^\beta \text{ is stationary in } \pi].
\]

Whence, \(\pi \in M^\alpha \cap E\). \(\square\)

**Corollary 4.13** When \(\alpha < \mathcal{K}^\pi\), then \(\alpha \in C(\alpha, \Xi(\alpha))\) and \(\Xi(\alpha) < \mathcal{K}\).

**Agreement:** For the remainder of this Section, we shall only consider ordinals \(< \mathcal{K}^\pi\).

**Lemma 4.14** \(\Xi(\alpha) < \Xi(\beta)\) iff either

\[
(1) \quad \alpha < \beta \land \alpha \in C(\beta, \Xi(\beta))
\]

or

\[
(2) \quad \beta < \alpha \land \beta \notin C(\alpha, \Xi(\alpha)).
\]

**Proof.** First, let \(\Xi(\alpha) < \Xi(\beta)\) be the case. If \(\alpha < \beta\), then \(\alpha \in C(\alpha, \Xi(\alpha)) \subseteq C(\beta, \Xi(\beta))\); thus (1). If, however, \(\beta < \alpha\), then \(\beta \in C(\alpha, \Xi(\alpha))\) is impossible since this would entail \(\Xi(\beta) \in C(\alpha, \Xi(\alpha))\) and consequently, \(\Xi(\beta) < \Xi(\alpha)\); thence in this case (2) is satisfied.

For the reverse implication, note that (1) yields \(\Xi(\alpha) \in C(\beta, \Xi(\beta))\) and hence \(\Xi(\alpha) < \Xi(\beta)\). (2) entails \(\beta \notin C(\beta, \Xi(\alpha))\) and therefore, utilizing \(\beta \in C(\beta, \Xi(\beta))\), \(\Xi(\alpha) < \Xi(\beta)\). \(\square\)

**Corollary 4.15** \(\alpha \neq \beta \implies \Xi(\alpha) \neq \Xi(\beta)\).

**Proposition 4.16** Let \(M^\xi\) be stationary in \(\pi\). Assume that \(\xi \leq \alpha\) and \(\xi, \pi, \alpha \in C(\alpha, \pi)\). Then,

\[
\Psi_\pi^\xi(\alpha) \in M^\xi \cap \pi.
\]

Moreover, if \(\xi > 0\), then \(M^\xi\) is not stationary in \(\Psi_\pi^\xi(\alpha)\) and, for all \(\beta > \xi\), \(\Psi_\pi^\xi(\alpha) \notin M^\beta\).
Proof. Since $\xi, \pi, \alpha \in C(\alpha, \pi)$ and $\pi \in \text{Lim}$, we may select a $\mu_0 < \pi$ so that already $\xi, \pi, \alpha \in C(\alpha, \mu_0)$.

Letting $E := \{\rho < \pi : \mu_0 \leq \rho \land C(\alpha, \rho) \cap \pi = \rho\}$, we claim that $E$ is closed and unbounded in $\pi$.

Unboundedness: Fix $\delta$ such that $\mu_0 \leq \delta < \pi$. For $\delta_0 := \delta + 1$ and $\delta_{n+1} := \sup(\{\alpha \in C(\alpha, \delta) \cap \pi : \delta < \delta \})$, one obtains, by Lemma 4.11(ii) and the regularity of $\pi$, $\delta < \delta \leq \delta_{n+1} < \pi$. The regularity of $\pi$ also ensures $\delta^* := \sup_{n<\omega} \delta_n < \pi$. Since $C(\alpha, \delta_n) \cap \pi \subseteq \delta_{n+1} \subseteq C(\alpha, \delta_{n+1}) \cap \pi$, it follows

$$C(\alpha, \delta^*) \cap \pi = \bigcup_{n<\omega} (C(\alpha, \delta_n) \cap \pi) = \delta^*.$$ 

Therefore, $\delta < \delta^* \in E$.

Closedness: Let $\lambda \in \text{Lim} \cap \pi$ and suppose that $E$ is unbounded in $\lambda$. Then $C(\alpha, \lambda) = \bigcup_{\eta \in E \cap \lambda} C(\alpha, \eta)$, and consequently $\lambda \in E$ follows from

$$C(\alpha, \lambda) \cap \pi = \bigcup_{\eta \in E \cap \lambda} (C(\alpha, \eta) \cap \pi) = \sup(E \cap \lambda) = \lambda.$$ 

By assumption, $M^\xi$ is stationary in $\pi$, so there exists a $\nu \in E \cap M^\xi$. This involves $C(\alpha, \nu) \cap \pi = \nu$. Because of $\mu_0 \leq \nu$, we get $\xi, \pi, \alpha \in C(\alpha, \nu)$. Due to the definition of $\Psi^\xi_\pi(\alpha)$, this implies $\Psi^\xi_\pi(\alpha) \leq \nu < \pi$.

Now assume $\xi > 0$. Then $\Psi^\xi_\pi(\alpha)$ is regular. We want to show that $M^\xi$ is not stationary in $\Psi^\xi_\pi(\alpha)$. Observe that $\xi, \pi, \alpha \in C(\alpha, \Psi^\xi_\pi(\alpha))$. So, if $M^\xi$ were stationary in $\Psi^\xi_\pi(\alpha)$, by applying the same arguments as in the first part of the proof, we could verify the existence of a $\rho \in M^\xi \cap \Psi^\xi_\pi(\alpha)$ with $\xi, \pi, \alpha \in C(\alpha, \rho)$ and $C(\alpha, \rho) \cap \pi = \rho$, which would collide with the definition of $\Psi^\xi_\pi(\alpha)$.

Finally, if we had $\Psi^\xi_\pi(\alpha) \in M^\beta$ for some $\beta > \xi$, then, since $\xi \in C(\xi, \Psi^\xi_\pi(\alpha))$, we would get $\xi \in C(\beta, \Psi^\xi_\pi(\alpha)) \cap \beta$, leading to the contradiction that $M^\xi$ is stationary in $\Psi^\xi_\pi(\alpha)$.

\begin{proposition}
\begin{enumerate}[label=(\roman*)]
\item $\Psi^\xi_\pi(\alpha) < \pi \implies \Psi^\xi_\pi(\alpha) \neq \Xi(\beta)$.
\item $\Psi^\xi_\pi(\alpha) < \pi \land \Psi^\xi_\pi(\beta) < \kappa \land \Psi^\xi_\pi(\alpha) = \Psi^\xi_\pi(\beta) \implies \alpha = \beta \land \pi = \kappa \land \xi = \sigma$.
\end{enumerate}
\end{proposition}

\begin{proof}
(i): By way of a contradiction, suppose $\Psi^\xi_\pi(\alpha) = \Xi(\beta)$. $\Psi^\xi_\pi(\alpha) < \pi$ implies $\pi \in C(\alpha, \Psi^\xi_\pi(\alpha))$. From $\alpha \leq \beta$ we could deduce $\pi \in C(\beta, \Xi(\beta))$ and therefore the contradiction $\pi < \Xi(\beta)$. From $\beta < \alpha$ we would get $\beta \in C(\beta, \Xi(\beta)) \subseteq C(\alpha, \Psi^\xi_\pi(\alpha))$ and consequently $\Xi(\beta) \in C(\alpha, \Psi^\xi_\pi(\alpha))$, contradicting $\Psi^\xi_\pi(\alpha) \notin C(\alpha, \Psi^\xi_\pi(\alpha))$.

(ii): The hypotheses imply

(a) $\xi, \alpha, \pi \in C(\alpha, \Psi^\xi_\pi(\beta))$

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and

\[(b) \quad \sigma, \kappa, \beta \in C(\beta, \Psi^\xi_\pi(\alpha)).\]

From \(\alpha < \beta\), using (a), we would get \(\xi, \alpha, \pi \in C(\beta, \Psi^\sigma_\pi(\beta))\) and hence \(\Psi^\xi_\pi(\alpha) \in C(\beta, \Psi^\sigma_\pi(\beta))\), contradicting \(\Psi^\sigma_\pi(\beta) \notin C(\beta, \Psi^\sigma_\pi(\beta))\). Similarly, using (b), the assumption \(\beta < \alpha\) leads to a contradiction. Therefore, \(\alpha = \beta\).

From \(\pi < \kappa\) we would get \(\pi \in C(\beta, \Psi^\sigma_\pi(\beta)) \cap \kappa\) by (a); but this is impossible since \(C(\beta, \Psi^\sigma_\pi(\beta)) \cap \kappa = \Psi^\sigma_\pi(\beta) = \Psi^\xi_\pi(\alpha) < \pi\). Using (b), we can also exclude that \(\kappa < \pi\). Consequently, \(\pi = \kappa\).

Finally, we have to show \(\xi = \sigma\). For a contradiction, assume \(\xi < \sigma\). \(\Psi^\xi_\pi(\alpha) < \pi\) yields \(\Psi^\xi_\pi(\alpha) \in M^\xi\) and thus \(\xi \in C(\xi, \Psi^\xi_\pi(\alpha))\). Therefore, \(\xi \in C(\sigma, \Psi^\sigma_\pi(\beta))\). Utilizing the definition of \(\Psi^\sigma_\pi(\beta)\), the latter implies that \(M^\xi\) is stationary in \(\Psi^\sigma_\pi(\beta)\). Letting

\[Y := \{\eta < \Psi^\sigma_\pi(\beta) : C(\alpha, \eta) \cap \Psi^\sigma_\pi(\beta) = \eta \land \alpha, \pi \in C(\alpha, \eta)\},\]

we obtain a set that is unbounded and closed in \(\Psi^\sigma_\pi(\beta)\). But then \(M^\xi \cap Y \neq \emptyset\) and, as a consequence, \(\Psi^\xi_\pi(\alpha) = \min(M^\xi \cap Y) < \Psi^\xi_\pi(\alpha)\), contradicting \(\Psi^\xi_\pi(\alpha) = \Psi^\sigma_\pi(\beta)\). Interchanging the roles of \(\sigma\) and \(\xi\) in the preceding argument, one also excludes \(\sigma < \xi\). \(\square\)

**Lemma 4.18**

(i) \(\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \implies [\alpha \in C(\zeta, \rho) \iff \alpha_1, \ldots, \alpha_n \in C(\zeta, \rho)]\).

(ii) \(\alpha =_{NF} \varphi \alpha_1 \alpha_2 \implies [\alpha \in C(\zeta, \rho) \iff \alpha_1, \alpha_2 \in C(\zeta, \rho)]\).

(iii) \(\sigma < \kappa \implies [\sigma \in C(\zeta, \rho) \iff \Omega_\sigma \in C(\zeta, \rho)]\).

**Proof.** (i) Using induction on \(n\), one easily shows that \(\alpha \in C_n(\zeta, \rho)\) implies \(\alpha_1, \ldots, \alpha_n \in C_n(\zeta, \rho)\). Similarly one proves (ii) and (iii). \(\square\)

**Lemma 4.19**

(i) \(0 < \alpha \land \pi \in M^\alpha \implies \Omega_\pi = \pi\).

(ii) \(\pi \in M^1 \implies \Omega_{\Psi^0_\pi(\alpha)} = \Psi^0_\pi(\alpha)\).

(iii) \(\pi = \Omega_{\xi+1} \land \alpha \in C(\alpha, \pi) \implies \Omega^{\xi}_\zeta < \Psi^0_\pi(\alpha) < \Omega^{\xi+1}_\zeta\).

(iv) \(\Psi^0_\pi(\alpha) < \pi \implies \Psi^0_\pi(\alpha) \notin \text{Reg.}\)

**Proof.** (i): The hypotheses imply \(C(\alpha, \pi) \cap K = \pi\). Therefore \(\pi\) is closed under \(\sigma \mapsto \Omega_\sigma\); whence \(\Omega_\pi = \pi\).

(ii) follows from (i), noting that \(C(\alpha, \Psi^0_\pi(\alpha)) \cap \pi = \Psi^0_\pi(\alpha)\).

(iii): As \(\zeta < \pi\) and \(\alpha \in C(\alpha, \pi)\), there is an \(\eta < \pi\) with \(\alpha, \pi \in C(\alpha, \eta)\). Utilizing the regularity of \(\pi\), we can find a \(\rho < \pi\) so that simultaneously \(\alpha, \pi \in C(\alpha, \rho)\) and
\(C(\alpha, \rho) \cap \pi = \rho\). This shows \(\Psi^0_\pi(\alpha) < \Omega_{\zeta+1}\). Therefore \(\pi \in C(\alpha, \Psi^0_\pi(\alpha))\), and hence, by 4.18, \(\zeta \in C(\alpha, \Psi^0_\pi(\alpha))\). Consequently, \(\Omega_\zeta \in C(\alpha, \Psi^0_\pi(\alpha)) \cap \pi = \Psi^0_\pi(\alpha)\).

(iv): \(\Psi^0_\pi(\alpha) < \pi\) implies \(\alpha, \pi \in C(\alpha, \Psi^0_\pi(\alpha))\). Let \(\sigma_0\) be minimal with the property \(\alpha, \pi \in C(\alpha, \sigma_0)\). In view of Lemma 4.11(iii), \(\sigma_0\) is not a limit; hence \(\sigma_0 < \Psi^0_\pi(\alpha)\).

Put \(\sigma_{n+1} := \sup(C(\alpha, \sigma_n) \cap \pi)\) and \(\sigma^* := \sup_{n<\omega}\sigma_n\). Then \(\sigma_n \leq \sigma_{n+1} \leq \sigma^* < \pi\). Using induction on \(n\), we see that \(\sigma_n \leq \Psi^0_\pi(\alpha)\). Since \(C(\alpha, \sigma_n) \cap \pi \subseteq \sigma_{n+1}\) and \(\bigcup_{n<\omega} C(\alpha, \sigma_n) = C(\alpha, \sigma^*)\), we get \(C(\alpha, \sigma^*) \cap \pi = \sigma^*\). Further, \(\alpha, \pi \in C(\alpha, \sigma^*)\). Therefore, \(\Psi^0_\pi(\alpha) \leq \sigma^*\). This verifies \(\Psi^0_\pi(\alpha) = \sigma^*\).

Regarding the sequence of \(\sigma_n\)'s, there are two possible outcomes. In the first case, this sequence is strictly increasing and therefore \(\Psi^0_\pi(\alpha)\) has cofinality \(\omega\), yielding that \(\Psi^0_\pi(\alpha)\) is singular.

In the second case, there exists an \(n_0\) such that \(\sigma_{n_0} < \sigma_{n_0+1} = \sigma_{n_0+2}\). To see this, note that \(\sigma_0\) is not a limit whereas \(\sigma_n = \text{Lim}\) for \(n > 0\). In this case we also have \(\sigma_{n_0+1} = \sigma^* = \Psi^0_\pi(\alpha)\). Further, \(|C(\alpha, \sigma_{n_0}) \cap \pi| = \text{max}(\omega, |\sigma_{n_0}|) < \sigma_{n_0+1}\). On the other hand, \(\sigma_{n_0+1} = \sup(C(\alpha, \sigma_{n_0}) \cap \pi)\), so \(\sigma_{n_0+1}\) must be singular. Whence, \(\Psi^0_\pi(\alpha) \notin \text{Reg}\). □

In the rest of this Section, we provide “recursive” \(<\sim\)-comparisons for ordinals which are presented in terms of \(\Psi\) and \(\Xi\).

**Proposition 4.20** Suppose that \(\Psi^0_\pi(\alpha) < \pi, \Psi^0_\pi(\beta) < \kappa, \text{ and } \Psi^0_\pi(\beta) < \pi\). Then
\[\Psi^\xi_\pi(\alpha) < \Psi^\xi_\pi(\beta)\]
iff one of the following cases holds:

1. \(\alpha < \beta \land \alpha, \pi \in C(\beta, \Psi^\sigma_\pi(\beta)) \land \Psi^\xi_\pi(\alpha) < \kappa\).
2. \(\beta \leq \alpha \land \{\beta, \sigma, \kappa\} \notin C(\alpha, \Psi^\xi_\pi(\alpha))\).
3. \(\alpha = \beta \land \kappa = \pi \land \xi < \sigma \land \xi \in C(\sigma, \Psi^\sigma_\pi(\beta))\).
4. \(\sigma < \xi \land \sigma \notin C(\xi, \Psi^\xi_\pi(\alpha))\).

**Proof.** From (1) it follows \(\Psi^\xi_\pi(\alpha) \in C(\beta, \Psi^\sigma_\pi(\beta)) \cap \kappa\), whence \(\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\pi(\beta)\).

(2) yields \(\{\beta, \sigma, \kappa\} \notin C(\beta, \Psi^\xi_\pi(\alpha))\); so, because of \(\{\beta, \sigma, \kappa\} \subseteq C(\beta, \Psi^\sigma_\pi(\beta))\), this becomes \(\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\pi(\beta)\).

(3) implies that \(M^\xi\) is stationary in \(\Psi^\sigma_\pi(\beta)\). As \(\alpha, \pi, \xi, \in C(\beta, \Psi^\sigma_\pi(\beta))\), \(\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\pi(\beta)\) follows from 4.16.

(4) yields \(\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\pi(\beta)\) since \(\sigma \in C(\sigma, \Psi^\sigma_\pi(\beta))\).

Next, assume \(\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\pi(\beta)\). Then \(\Psi^\xi_\pi(\alpha) < \kappa\). We have to show that one of (1)–(4) holds.

First, assume \(\alpha < \beta\). From \(\{\alpha, \xi, \pi\} \notin C(\beta, \Psi^\sigma_\pi(\beta))\) we would get \(\{\alpha, \xi, \pi\} \notin C(\alpha, \Psi^\xi_\pi(\alpha))\), contradicting \(\Psi^\xi_\pi(\alpha) < \pi\). So (1) must be the case.
If $\beta < \alpha$, then $\{\beta, \sigma, \kappa\} \subseteq C(\alpha, \Psi^\xi(\alpha))$ cannot hold since this would imply $\Psi^\sigma(\beta) \in C(\alpha, \Psi^\xi(\alpha)) \cap \pi$ and therefore $\Psi^\xi(\alpha) < \Psi^\sigma(\beta)$. This shows that $\beta < \alpha$ implies (2).

Finally, suppose $\alpha = \beta$. If $\kappa < \pi$, then $\kappa \notin C(\alpha, \Psi^\xi(\alpha))$; whence (2). $\pi < \kappa$ would force $\pi \in C(\alpha, \Psi^\sigma(\beta)) \cap \kappa = \Psi^\sigma(\beta)$, contradicting $\Psi^\sigma(\beta) < \pi$.

So it remains to prove the assertion when $\alpha = \beta$ and $\pi = \kappa$. If $\sigma \notin C(\alpha, \Psi^\xi(\alpha))$, then (2) is satisfied. So assume $\sigma \in C(\alpha, \Psi^\xi(\alpha))$. From $\Psi^\xi(\alpha) < \pi$ we get $\Psi^\xi(\alpha) \in M^\xi$, in particular, $\xi \in C(\xi, \Psi^\xi(\alpha))$. Also, by assumption, we have $\Psi^\xi(\alpha) < \Psi^\sigma(\beta)$. Consequently, if $\xi < \sigma$, then $\xi \in C(\sigma, \Psi^\sigma(\beta))$, so (3) holds. 4.17 excludes that $\xi = \sigma$. Furthermore, $\sigma < \xi \land \sigma \in C(\xi, \Psi^\xi(\alpha))$ can be excluded since this would lead to the contradiction $\Psi^\sigma(\beta) < \Psi^\xi(\alpha)$ by 4.16. Therefore $\sigma < \xi$ yields (4).

**Proposition 4.21**

\[ \Psi^\xi(\alpha) < \Xi(\beta) \iff [\pi \leq \Xi(\beta) \lor (\beta < \alpha \land \beta \notin C(\alpha, \Psi^\xi(\alpha))] \]

**Proof.** “$\Leftarrow$” is immediate.

To verify “$\Rightarrow$”, we assume $\Psi^\xi(\alpha) < \Xi(\beta)$ and $\Xi(\beta) < \pi$. We have to verify $\beta < \alpha \land \beta \notin C(\alpha, \Psi^\xi(\alpha))$.

$\alpha \leq \beta$ would imply $\alpha, \xi, \pi \in C(\alpha, \Psi^\xi(\alpha)) \subseteq C(\beta, \Xi(\beta))$, and hence the contradiction $\pi < \Xi(\beta)$. So we must have $\beta < \alpha$. If $\beta \notin C(\alpha, \Psi^\xi(\alpha))$, then $\Xi(\beta) \in C(\alpha, \Psi^\xi(\alpha)) \cap \pi$, yielding the contradiction $\Xi(\beta) < \Psi^\xi(\alpha)$.

\[ \square \]

5 The ordinal notation system $T(\mathcal{K})$

We are going to define a set of ordinals $T(\mathcal{K}) \subseteq C(\mathcal{K}^T, \tau)$ in conjunction with a function $m$ which assigns to inaccessibles $\pi \in T(\mathcal{K}) \cap \mathcal{K}$ the maximal $\alpha$ with $\pi \in M^\alpha$. However, $m(\pi)$ will be defined “constructively” from a normal form representation of $\pi$, and only later we shall verify the identity

\[(*) \quad m(\pi) = \sup\{\beta : \pi \in M^\beta\}.\]

We shall demand closure of $T(\mathcal{K})$ under $\Psi^\xi$ only when $M^\xi$ is stationary in $\pi$ (and $\xi, \pi \in T(\mathcal{K})$). It will transpire that, for $\pi \in T(\mathcal{K})$, stationarity of $M^\xi$ in $\pi$ is equivalent to $\xi \in C(m(\pi), \pi) \cap m(\pi)$.

Finally, by utilizing normal forms and the $\prec$-comparisons of the previous Section, we will come to see that $\langle T(\mathcal{K}), \prec \rangle$ gives rise to a primitive recursive ordinal notation system.

**Definition 5.1** The set of ordinals $T(\mathcal{K})$ and a function

\[ m : T(\mathcal{K}) \cap \mathcal{R} \} \rightarrow T(\mathcal{K}) \]

are inductively defined by the following clauses.
\[(T1)\] \(0, K \in \mathcal{T}(K).\)

\[(T2)\] If \(\alpha =_{NF} \alpha_1 + \cdots + \alpha_n\) and \(\alpha_1, \ldots, \alpha_n \in \mathcal{T}(K),\) then \(\alpha \in \mathcal{T}(K).\)

\[(T3)\] If \(\alpha =_{NF} \varphi \alpha_1 \alpha_2\) with \(\alpha_1, \alpha_2 \in \mathcal{T}(K),\) then \(\alpha \in \mathcal{T}(K).\)

\[(T4)\] If \(\xi \in \mathcal{T}(K) \cap K\) and \(0 < \xi < \Omega_\xi,\) then \(\Omega_\xi \in \mathcal{T}(K).\)

\[(T5)\] If \(\alpha \in \mathcal{T}(K)\) and \(0 < \alpha,\) then \(\Xi(\alpha) \in \mathcal{T}(K)\) and \(m(\Xi(\alpha)) = \alpha.\)

\[(T6)\] If \(\alpha, \xi, \pi \in \mathcal{T}(K)\) and \(\alpha, \xi, \pi \in C(\alpha, \pi)\) and \(\xi \leq \alpha\) and \(\xi \in C(m(\pi), \pi) \cap m(\pi),\) then \(\Psi_\pi^\xi(\alpha) \in \mathcal{T}(K).\)

\[m(\Psi_\pi^\xi(\alpha)) = \xi,\] providing that \(\xi > 0.\)

We shall write \(\delta =_{NF} \Psi_\pi^\xi(\alpha)\) if \(\delta = \Psi_\pi^\xi(\alpha)\) and the requirements of \((T6)\) are fulfilled.

The meaning of the function \(m\) and the condition \(\xi \in C(m(\pi), \pi) \cap m(\pi)\) in \((T6)\) are elucidated in the following Lemma.

**Lemma 5.2** Let \(\delta \in \mathcal{T}(K).\) Then:

(i) \(\delta \in C(\mathcal{K}_\Gamma, 0).\)

(ii) When \(\delta\) is weakly inaccessible and \(\delta < K,\) then \(\delta \in M^{m(\delta)};\) moreover, \(M^{m(\delta)}\) is not stationary in \(\delta\) and \(m(\delta) = \sup\{\beta : \beta \in M^\beta\}.\)

(iii) If \(\pi, \xi \in \mathcal{T}(K),\) then \(M^\xi\) is stationary in \(\pi\) iff \(\xi \in C(m(\pi), \pi) \cap m(\pi).\)

(iv) The clauses defining \(\mathcal{T}(K)\) are deterministic, i.e., for each \(\beta \in \mathcal{T}(K),\) there is only one way to get into \(\mathcal{T}(K).\) Whence, each ordinal in \(\mathcal{T}(K)\) can be denoted uniquely using only the symbols \(0, K, \ldots, \varphi, \Omega, \Xi, \Psi.\)

**Proof.** (i): We prove \((a), (b)\) simultaneously by induction on the definition of \(\delta \in \mathcal{T}(K).\) During the proof, we frequently use the fact that \(C(\mathcal{K}_\Gamma, 0) \subseteq \mathcal{K}_\Gamma,\) which easily follows from the definition of \(C(\mathcal{K}_\Gamma, 0).\)

Suppose \(\delta = \Xi(\alpha)\) with \(\alpha \in \mathcal{T}(K)\). The induction hypothesis yields \(\alpha \in C(\mathcal{K}_\Gamma, 0) \cap K.\Gamma.\)

Therefore, \(\delta \in C(\mathcal{K}^\Gamma, 0)\) and \(m(\delta) = \alpha\) and, according to 4.12, \(\delta \in M^\pi(\delta)\). If \(\delta \in M^\beta\) for some \(\beta > \alpha\), then, as \(\alpha \in C(\alpha, \delta),\) we would get \(\alpha \in C(\beta, \delta)\) and thus the contradiction that \(M^\alpha\) is stationary in \(\Xi(\alpha)\). Hence, \(m(\delta) =_{NF} \sup\{\beta : \beta \in M^\beta\}.\)

Assume further that \(\delta\) is weakly inaccessible. Then, by 4.19(iii), \(\pi\) must be weakly inaccessible, too, and \(\xi > 0.\) The induction hypothesis yields \(\pi \in M^{m(\pi)}\). Hence, from \(\xi \in C(m(\pi), \pi) \cap m(\pi),\) it follows that \(M^\xi\) is stationary in \(\pi.\) So, using 4.16, we can infer that \(\delta \in M^\xi,\) \(M^\xi\) is not stationary in \(\delta\) and \(\xi = \sup\{\beta : \beta \in M^\beta\}.\) This gives the assertion since \(m(\delta) = \xi.\)
Finally, if $\delta$ enters $T(\mathcal{K})$ by one of the clauses (T1),(T2),(T3),(T4), then (a) is immediate by the inductive assumption.

(ii): First, assume that $M^\xi$ is stationary in $\pi$. Observe that (ii) is trivial for successor cardinals. So let $\pi$ be weakly inaccessible. Then, using 4.11(vii), $\pi \in M^\xi$; thus $\xi < m(\pi)$ by (i)(b). Choosing $\rho \in M^\xi \cap \pi$, we get $\xi \in C(\xi, \rho)$; whence $\xi \in C(m(\pi), \pi) \cap m(\pi)$.

On the other hand, $\xi \in C(m(\pi), \pi) \cap m(\pi)$ implies that $M^\xi$ is stationary in $\pi$ since $\pi \in M^{m(\pi)}$ by (i)(b).

(iii) follows from 4.12, 4.15, 4.16, 4.17, and 4.19. $\square$

To conceive of $\langle T(\mathcal{K}), < \rangle$ as a primitive recursive ordinal notation system, we need to be able to determine whether an arbitrary term, composed of the symbols $0, \mathcal{K}, +, \varphi, \Omega, \Xi, \Psi$, denotes an ordinal from $T(\mathcal{K})$, and, moreover, given two terms denoting ordinals from $T(\mathcal{K})$, the order between the denoted ordinals should be computable from the order of ordinals denoted by proper subterms. An important step towards such a decision procedure is taken in the following definition.

**Definition 5.3** By induction on the definition of $\alpha \in T(\mathcal{K})$, $K_\delta(\alpha)$ is defined as follows.

(K1) $K_\delta(\mathcal{K}) = \emptyset$.

(K2) If $\alpha = NF \alpha_1 + \ldots + \alpha_n$ or $\alpha = NF \varphi \alpha_1 \alpha_2$, then $K_\delta(\alpha) = \bigcup_{1 \leq i \leq n} K_\delta(\alpha_i)$.

(K3) If $\alpha = \Omega_\xi$ with $0 < \xi < \Omega_\xi < \mathcal{K}$, then $K_\delta(\alpha) = K_\delta(\xi)$.

(K4) If $\alpha = \Xi(\beta)$, then

$$K_\delta(\alpha) = \begin{cases} \emptyset & \text{if } \alpha < \delta \\ K_\delta(\beta) \cup \{\beta\} & \text{else} \end{cases}.$$

(K5) If $\alpha = NF \Psi_\kappa^\sigma(\beta)$, then

$$K_\delta(\alpha) = \begin{cases} \emptyset & \text{if } \alpha < \delta \\ K_\delta(\kappa) \cup K_\delta(\sigma) \cup K_\delta(\beta) \cup \{\beta\} & \text{else} \end{cases}.$$

**Lemma 5.4** If $\alpha \in T(\mathcal{K})$ and $\delta, \gamma$ are arbitrary ordinals, then

$$\alpha \in C(\gamma, \delta) \iff K_\delta(\alpha) < \gamma.$$

**Proof.** This is straightforwardly verified by induction on $\alpha \in T(\mathcal{K})$. $\square$

Given $\alpha, \xi, \pi \in T(\mathcal{K})$, Lemma 5.4 enables us to check all the conditions demanded in (T6) of Definition 5.1, solely, by inspecting the inductive generation that $\alpha, \xi, \pi$ have as elements of $T(\mathcal{K})$. Therefore, in conjunction with the recursive characterization of the $<$-relation of the previous Section, we are led to a primitive recursive description of $\langle T(\mathcal{K}), < \rangle$, when we identify the elements of $T(\mathcal{K})$ with the terms denoting them. However, there is no reason to write out such a primitive recursive definition in detail since it does not convey any more insights.
6 The Calculus $RS(\mathcal{K})$

It is well known that the axioms of Peano Arithmetic, $PA$, can be derived in a sequent calculus, $PA_\omega$, augmented by an infinitary rule, the so-called $\omega$–rule$^9$

$$\frac{\Gamma, A(\bar{n}) \text{ for all } n}{\Gamma, \forall x A(x)}.$$

An ordinal analysis for $PA$ is then attained as follows:

- Each $PA$–proof can be "unfolded" into a $PA_\omega$–proof of the same sequent.
- Each such $PA_\omega$–proof can be transformed into a cut–free $PA_\omega$–proof of the same sequent of length $< \varepsilon_0$.

In order to obtain a similar result for set theories like $KP$, we have to work a bit harder. Guided by the ordinal analysis of $PA$, we would like to invent an infinitary rule which, when added to $KP$, enables us to eliminate cuts. As opposed to the natural numbers, it is not clear how to bestow a canonical name to each element of the set–theoretic universe. However, within the confines of the constructible universe, which is made from the ordinals, it is pretty obvious how to "name" sets once we have names for ordinals at our disposal. Recall that $L_\alpha$, the $\alpha^{th}$ level of Gödel's constructible hierarchy $L$, is defined by $L_0 = \emptyset$, $L_\lambda = \bigcup\{L_\beta : \beta < \lambda\}$ for limits $\lambda$, and $L_{\beta+1} = \{X : X \subseteq L_\beta; X \text{ definable over } \langle L_\beta, \in \rangle\}$. So any element of $L$ of level $\alpha$ is definable from elements of $L$ with levels $< \alpha$ and $L_\alpha$.

6.1 The Language of $RS(\mathcal{K})$

Henceforth, we shall restrict ourselves to ordinals from $T(\mathcal{K})$.

**Definition 6.1** We extend the language of set theory, $L$, by new unary predicate symbols $Ad^\alpha$ for every $\alpha \in T(\mathcal{K})$. The augmented language will be denoted by $L_{Ad}$.

The atomic formulae of $L_{Ad}$ are those of either form $(a \in b)$, $\neg(a \in b)$, $Ad^\alpha(a)$, or $\neg Ad^\alpha(a)$. The $L_{Ad}$–formulae are obtained from atomic ones by closing off under $\land, \lor, (\exists x \in a), (\forall x \in a), \exists x$, and $\forall x$.

**Definition 6.2** The $L_{RS(\mathcal{K})}$–terms and their levels are generated as follows.

1. For each $\alpha$, $L_\alpha$ is an $L_{RS(\mathcal{K})}$–term of level $\alpha$.
2. The formal expression $[x \in L_\alpha : F[x, s_1, \cdots s_n]^{l_\alpha}]$ is an $L_{RS(\mathcal{K})}$–term of level $\alpha$ if $F[a, b_1, \cdots, b_n]$ is an $L_{Ad}$–formula and $s_1, \cdots, s_n$ are $L_{RS(\mathcal{K})}$–terms with levels $< \alpha$.

$^9\bar{n}$ stands for the $n^{th}$ numeral
We shall denote the level of an $\mathcal{L}_{RS(K)}$-term $t$ by $|t|$; $t \in \text{Term}(\alpha)$ stands for $|t| < \alpha$ and $t \in \text{Term}$ for $t \in \text{Term}(K)$.

The $\mathcal{L}_{RS(K)}$-formulae are the expressions of the form $F[s_1, \ldots, s_n]^\alpha$, where $F[a_1, \ldots, a_n]$ is an $\mathcal{L}_{Ad}$-formula and $s_1, \ldots, s_n \in \text{Term}$.

For technical convenience, we let $\neg A$ be the formula which arises from $A$ by (i) putting $\neg$ in front of each atomic formula, (ii) replacing $\land, \lor, (\forall x \in a), (\exists x \in a)$ by $\lor, \land, (\exists x \in a), (\forall x \in a)$, respectively, and (iii) dropping double negations.

**Convention:** In the sequel, $\mathcal{L}_{RS(K)}$-formulae will be referred to as formulae. The same usage applies to $\mathcal{L}_{RS(K)}$-terms.

**Definition 6.3** If $\mathfrak{r}$ is a term or a formula, then

$$k(\mathfrak{r}) := \{\alpha : \mathbb{L}_\alpha \text{ occurs in } \mathfrak{r}\}.$$

Here any occurrence of $\mathbb{L}_\alpha$, i.e. also those inside of terms, has to be considered. For technical convenience, we put $k(0) := k(1) := 0$.

We set $|\mathfrak{r}| := \max(k(\mathfrak{r}) \cup \{0\})$ and $|0| := |1| := 0$.

If $\mathcal{X}$ is a finite set consisting of objects of the above kind, put

$$k(\mathcal{X}) := \bigcup\{k(\mathfrak{r}) : \mathfrak{r} \in \mathcal{X}\}$$

and

$$|k(\mathcal{X})| := \sup\{|k(\mathfrak{r})| : \mathfrak{r} \in \mathcal{X}\}.$$

**Definition 6.4** We use the relation $\equiv$ to mean syntactical identity. For terms $s, t$ with $|s| < |t|$ we set

$$s^* t \equiv \begin{cases} B(s) & \text{if } t \equiv [x \in \mathbb{L}_\beta : B(x)] \\ s \notin \mathbb{L}_0 & \text{if } t \equiv \mathbb{L}_\beta. \end{cases}$$

Observe that $s^* t$ and $s^* t$ have the same truth value under the standard interpretation in the constructible hierarchy.

**6.2 The Rules of $RS(K)$**

Next we introduce a calculus, $RS(K)$, with infinitary rules. $A, B, C, \ldots, F(t), G(t), \ldots$ range over $\mathcal{L}_{RS(K)}$-formulae. We denote by upper case Greek letters $\Gamma, \Delta, \Lambda, \ldots$ finite sets of $\mathcal{L}_{RS(K)}$-formulae. The intended meaning of $\Gamma = \{A_1, \ldots, A_n\}$ is the disjunction $A_1 \lor \cdots \lor A_n$. $\Gamma, A$ stands for $\Gamma \cup \{A\}$ etc.. We also use the shorthands $r \neq s := \neg(r = s)$ and $r \notin t := \neg(r \in t)$. 
An $\mathcal{L}_{RS}$–formula is said to be $\Delta_0(\alpha)$ if it contains only terms with levels $< \alpha$. An $\mathcal{L}_{RS}$–formula $A$ is $\Pi_k(\alpha)$ if it has the form

$$(\forall x_1 \in \mathbb{L}_\alpha) \cdots (Q_k x_k \in \mathbb{L}_\alpha) F(x_1, \ldots, x_k),$$

where the $k$ quantifiers in front are alternating and $F(\mathbb{L}_0, \ldots, \mathbb{L}_0)$ is $\Delta_0(\alpha)$. Analogously, one defines $\Sigma_k(\alpha)$–formulae.

Given an $\mathcal{L}_{RS}$–formulae $A$ and terms $s, t$, we denote by $A(s, t)$ the formula which arises from $A$ by replacing all the quantifiers $(\exists x \in t)$ and $(\forall x \in t)$ by $(\exists x \in s)$ and $(\forall x \in s)$, respectively. To economize on subscripts, we also write $A(s, \alpha)$ for $A(s, \mathbb{L}_\alpha)$ and $A(\beta, \alpha)$ instead of $A(\mathbb{L}_\beta, \mathbb{L}_\alpha)$.

**Definition 6.5** The rules of $RS(K)$ are:

1. $$(\wedge) \quad \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \land A'}$$
2. $$(\lor) \quad \frac{\Gamma, A_i}{\Gamma, A_0 \lor A_1} \quad \text{if } i = 0 \text{ or } i = 1$$
3. $$(\forall) \quad \frac{\ldots \Gamma, s \in t \to F(s) \cdots (s \in \text{Term}(|t|))}{\Gamma, (\forall x \in t) F(x)}$$
4. $$(\exists) \quad \frac{\Gamma, s \in t \land F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } s \in \text{Term}(|t|)$$
5. $$(\forall) \quad \frac{\ldots \Gamma, s \in t \to r \neq s \cdots \cdots (s \in \text{Term}(|t|))}{\Gamma, r \not\in t}$$
6. $$(\in) \quad \frac{\Gamma, s \in t \land r = s}{\Gamma, r \in t} \quad \text{if } s \in \text{Term}(|t|)$$
7. $$(-Ad^\alpha) \quad \frac{\ldots \Gamma, \mathbb{L}_\rho \neq t \cdots (\rho \in M^a; \rho \leq |t|)}{\Gamma, \neg Ad^\alpha(t)}$$
8. $$(-Ad^\alpha) \quad \frac{\Gamma, \mathbb{L}_\rho = t}{\Gamma, Ad^\alpha(t)} \quad \text{if } \rho \in M^a \text{ and } \rho \leq |t|$$
9. $$(\text{Cut}) \quad \frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}$$
10. $$(\text{Ref}_K) \quad \frac{\Gamma, A}{\Gamma, (\exists z \in \mathbb{L}_K) [(\forall z \in \mathbb{L}_K) F(z) \land z \neq \emptyset \land A(z, z, k)]} \quad \text{if } A \in \Pi_3(K)$$
11. $$(\text{Ref}_{\pi}) \quad \frac{\Gamma, F(s)}{\Gamma, (\exists z \in \mathbb{L}_\pi) [Ad^\pi(z) \land (\exists u \in z) F(u)^{z, \pi}]} \quad \text{if } F(s) \in \Pi_2(\pi),$$
where \((Ref^\xi_\pi)\) comes with the proviso that \(M^\xi\) be stationary in \(\pi\).

**Remark 6.6** At first glance, the rule \((Ref^\xi_\pi)\) might loom complicated. As a matter of fact, instead, we could have adopted the rule:

\[
(Ref^\xi_\pi)^* \quad \frac{\Gamma, A}{\Gamma, (\exists z \in L_\pi) [\text{Ad}^\xi(z) \land A^{\{z,\pi\}}]} \quad \text{if } A \in \Pi_2(\pi).
\]

But latter on (cf. Lemma 8.12), we will need to derive \(\Sigma_3(\pi)\)-reflection and this can be accomplished more easily with \((Ref^\xi_\pi)\) at our disposal.

### 6.3 \(\mathcal{H}\)-controlled derivations

If we dropped the rules \((Ref^\xi_\pi)\) and \((Ref^\xi_\pi)\) from \(RS(\mathcal{K})\), the remaining calculus would enjoy full cut elimination owing to the symmetry of the pairs of rules \(\langle (\wedge), (\vee) \rangle\), \(\langle (\forall), (\exists) \rangle\), \(\langle (\varnothing), (\in) \rangle\), \(\langle (\text{Ad}^\alpha), (\neg\text{Ad}^\alpha) \rangle\). However, partial cut elimination for \(RS(\mathcal{K})\) can be attained by delimiting a collection of derivations of a very uniform kind.

To define uniform derivations, we shall find it useful to apply the notion of operator controlled derivations of Buchholz [1993].

**Definition 6.7** Let \(P(On) = \{X : X \text{ is a set of ordinals}\}\).

A class function \(\mathcal{H} : P(On) \rightarrow P(On)\) will be called *operator* if the following conditions are met for all \(X, X' \in P(On)\):

\begin{align*}
\text{(H0)} & \quad 0 \in \mathcal{H}(X). \\
\text{(H1)} & \quad \text{For } \alpha = NF \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}, \\
& \quad \alpha \in \mathcal{H}(X) \iff \alpha_1, \ldots, \alpha_n \in \mathcal{H}(X). \\
& \quad \text{(In particular, (H1) implies that } \mathcal{H}(X) \text{ will be closed under } + \text{ and } \sigma \mapsto \omega^\sigma, \text{i.e., if } \alpha, \beta \in \mathcal{H}(X), \text{ then } \alpha + \beta, \omega^\alpha \in \mathcal{H}(X).) \\
\text{(H2)} & \quad X \subseteq \mathcal{H}(X) \\
\text{(H3)} & \quad X' \subseteq \mathcal{H}(X) \implies \mathcal{H}(X') \subseteq \mathcal{H}(X). \\
\end{align*}

**Definition 6.8** (i) When \(f\) is a mapping \(f : On^k \rightarrow On\), then \(\mathcal{H}\) is said to be *closed* under \(f\), if, for all \(X \in P(On)\) and \(\alpha_1, \ldots, \alpha_k \in \mathcal{H}(X)\),

\[
f(\alpha_1, \ldots, \alpha_k) \in \mathcal{H}(X).
\]

(ii) \(\alpha \in \mathcal{H} := \alpha \in \mathcal{H}(\emptyset); \quad s \in \mathcal{H} := k(s) \subset \mathcal{H}.
\]
(iii) \( X \subseteq \mathcal{H} := X \subseteq \mathcal{H}(\emptyset) \).

(iv) For \( s \in \text{Term} \) let \( \mathcal{H}[s] \) denote the operator

\[
\left( X \mapsto \mathcal{H}(k(s) \cup X) \right)_{X \in \mathcal{P}(\text{On})}.
\]

(v) If \( \mathcal{X} \) is set consisting of terms, formulae, and possibly elements from \( \{0, 1\} \), then

\[
\mathcal{H}[\mathcal{X}](X) := \mathcal{H}(k(\mathcal{X}) \cup X).
\]

We shall also write \( \mathcal{H}[\mathcal{X}, s_1, \ldots, s_n] \) for \( \mathcal{H}[\mathcal{X} \cup \{s_1, \ldots, s_n\}] \), and occasionally \( \mathcal{H}[\mathcal{X}, \pi] \) instead of \( \mathcal{H}[\mathcal{X}, L_\pi] \).

The next Lemma garners some simple properties of operators.

**Lemma 6.9** If \( \mathcal{H} \) is an operator, then:

(i) \( \mathcal{H}[\mathcal{X}] \) is an operator.

(ii) \( k(\mathcal{X}) \subset \mathcal{H} = \mathcal{H}[\mathcal{X}] \).

(iii) \( \forall X, X' \in \mathcal{P}(\text{On})[X' \subseteq X \implies \mathcal{H}(X') \subseteq \mathcal{H}(X)] \).

**Definition 6.10** To each \( L_{\text{RS}(K)} \)–formula \( A \) we assign either a (possibly infinite) disjunction \( \bigvee (A_\iota)_{\iota \in J} \) or conjunction \( \bigwedge (A_\iota)_{\iota \in J} \) of \( L_{\text{RS}(K)} \)–formulae. This assignment will be indicated by \( A \approx \bigvee (A_\iota)_{\iota \in J} \) and \( A \approx \bigwedge (A_\iota)_{\iota \in J} \), respectively.

- \( r \in t \approx (s \circ t \land r = s)_{s \in \text{Term}(|t|)} \)
- \( Ad^\alpha(t) \approx \bigvee (\mathbb{L}_\eta = t)_{\eta \in J} \), where \( J := \{ \mathbb{L}_\eta : \eta \in M^\alpha ; \eta \leq |t| \} \)
- \( (\exists x \in t) F(x) \approx \bigvee (s \circ t \land F(s))_{s \in \text{Term}(|t|)} \)
- \( A_0 \lor A_1 \approx \bigvee (A_\iota)_{\iota \in \{0, 1\}} \)
- \( \neg A \approx \bigwedge (\neg A_\iota)_{\iota \in J} \), if \( A \approx \bigvee (A_\iota)_{\iota \in J} \).

Using this representation of formulae, we can define the subformulae of a formula as follows.\(^{10}\) When \( A \approx \bigwedge (A_\iota)_{\iota \in J} \) or \( A \approx \bigvee (A_\iota)_{\iota \in J} \), then \( B \) is a subformula of \( A \) if \( B \equiv A \) or, for some \( \iota \in J \), \( B \) is a subformula of \( A_\iota \).

Since we also want to keep track of the complexity of cuts appearing in derivations, we endow each formula with an ordinal rank.

\(^{10}\)That this constitutes a legitime inductive definition will follow from Lemma 6.12
Definition 6.11 The *rank* of formulae and terms is determined as follows.

1. $rk(\mathbb{L}_\alpha) := \omega \cdot \alpha$.
2. $rk([x \in \mathbb{L}_\alpha : F(x)]) := \max\{\omega \cdot \alpha + 1, rk(F(\mathbb{L}_0)) + 2\}$.
3. $rk(s \in t) := \max\{rk(s) + 6, rk(t) + 1\}$.
4. $rk(Ad^\alpha(s)) := rk(\neg Ad^\alpha(s)) := rk(s) + 5$.
5. $rk((\exists x \in t)F(x)) := \max\{rk(t), rk(F(\mathbb{L}_0)) + 2\}$.
6. $rk(A \land B) := \max\{rk(A), rk(B)\} + 1$.

There is plenty of leeway in designing the actual rank of a formula. However, it is crucial that it satisfies the following property.

Lemma 6.12 If $A \cong \bigvee_{i \in J} (A_i)_{i \in J}$ or $A \cong \bigwedge_{i \in J} (A_i)_{i \in J}$, then

$$(\forall i \in J) [rk(A_i) < rk(A)].$$

A proof for Lemma 6.12 is given in Buchholz [1993], Lemma 1.9. □

Using the formula representation of Definition 6.10, notwithstanding the many rules of $RS(K)$, the notion of $H$–controlled derivability can be defined concisely. We shall use $J \upharpoonright \alpha$ to denote the set $\{i \in J : |i| < \alpha\}$.

Definition 6.13 Let $H$ be an operator and let $\Gamma$ be a finite set of $RS(K)$–formulae. $H \upharpoonright_{\alpha} \Gamma$ is defined by recursion on $\alpha$ via

$$\{\alpha\} \cup k(\Gamma) \subset H$$

and the following inductive clauses:
(V)  \[ \frac{H \frac{\alpha_0}{\rho} \Lambda, A_{\alpha_0}}{H \frac{\alpha}{\rho} \Lambda, \bigvee (A_i)_{i \in J}} \]  
\[ \alpha_0 < \alpha \]  
\[ \iota_0 \in J | \alpha \]

(Λ)  \[ \frac{H \frac{\alpha_0}{\rho} \Lambda_A \text{ for all } \iota \in J}{H \frac{\alpha}{\rho} \Lambda, (A_i)_{i \in J}} \]  
\[ |\iota| \leq \alpha_\iota < \alpha \]

(Cut)  \[ \frac{H \frac{\alpha_0}{\rho} \Lambda, B, H \frac{\alpha_0}{\rho} \Lambda, \neg B}{H \frac{\alpha}{\rho} \Lambda} \]  
\[ \alpha_0 < \alpha \]  
\[ rk(B) < \rho \]

(Refₖ)  \[ \frac{H \frac{\alpha_0}{\rho} \Lambda, A}{H \frac{\alpha}{\rho} \Lambda, (\exists z \in L_\xi)[Tran(z) \land z \neq \emptyset \land A^{(z, \pi)}]} \]  
\[ \alpha_0, \xi < \alpha \]  
\[ A \in \Pi_3(\xi) \]

(Ref₂)  \[ \frac{H \frac{\alpha_0}{\rho} \Lambda, F(s)}{H \frac{\alpha_0}{\rho} \Lambda, (\exists z \in L_\pi)[Ad^\xi(z) \land (\exists u \in z) F(u)^{(z, \pi)}]} \]  
\[ \alpha_0 + 1, \pi < \alpha \]  
\[ \xi \in H \]  
\[ F(s) \in \Pi_2(\pi) \]  
\[ stat(\xi, \pi) \]

where \( stat(\xi, \pi) \) means that \( M^\xi \) is stationary in \( \pi \); according to 5.2(ii) this is equivalent to \( \xi \in C(m(\pi), \pi) \cap m(\pi) \), and thus is a decidable property by 5.4.

**Remark 6.14** In \( (Ref_\xi) \) we can assume that \( s \in H \), for if \( s \) occurs in \( F(s) \) then this is a consequence of \( k(\Lambda, F(s)) \subseteq H \), and if \( s \) does not occur in \( F(s) \), then \( F(s) \equiv F(L_0) \) so that we could assume \( s \equiv L_0 \) which would also entail \( s \in H \).

Henceforth, we shall tacitly make this assumption when dealing with \( (Ref_\xi) \).

The following observations are easily established by induction on \( \alpha \).

**Lemma 6.15**  
(i)  \[ H \frac{\alpha}{\rho} \Gamma \land \alpha \leq \alpha' \in H \land \rho \leq \rho' \land k(\Lambda) \subseteq H \implies H \frac{\alpha'}{\rho'} \Gamma, \Lambda. \]

(ii)  \[ H \frac{\alpha}{\rho} \Gamma, A \lor B \implies H \frac{\alpha}{\rho} \Gamma, A, B. \]

(iii)  \[ H \frac{\alpha}{\rho} \Gamma, (\forall x \in L_\beta) F(x) \land \gamma \in H \land \gamma \leq \beta \implies H \frac{\alpha}{\rho} \Gamma, (\forall x \in L_\gamma) F(x). \]

**7 Predicative Cut Elimination and Bounding**

Cuts in \( RS(\Lambda) \)-derivations whose cut formulae have not been introduced previously by a \( \Pi_3 \) or \( \Pi_2 \)-reflection inference will be called **uncritical**. Applying the usual cut elimination procedure for infinitary logic, uncritical cuts can be replaced by cuts with lesser rank.
In this Section we will deal with elimination of uncritical cuts in $L_{RS}$ in its quantitative as太平. Since these results have literally the same proofs as their counterparts in Buchholz [1993], we refrain from repeating them here.

Besides cut elimination results, we show that existential quantifiers in $L_{RS}$-derivations can always be “bounded” by the length of the derivation.

**Lemma 7.1** (Inversion)

\[
H \models^\rho_\alpha \Gamma, \bigwedge (A_i)_{i \in J} \implies (\forall i \in J) H[i] \models^\rho_\alpha \Gamma, A_i
\]

*Proof.* Use induction on $\alpha$. \hfill $\Box$

The next Lemma relates the rank of a formula $A$, to its level, $|A|$ (see 6.3).

**Lemma 7.2** Let $A, B$ be formulae and $s, t$ be terms.

1. $rk(A) = \omega \cdot |A| + n$ for some $n < \omega$.
2. $rk(s) = \omega \cdot |s| + m$ for some $m < \omega$.
3. $|A| < |B| \implies rk(A) < rk(B)$.
4. $|s| < |t| \implies rk(s) < rk(t)$.

*Proof.* See Buchholz [1993], Lemma 1.9. \hfill $\Box$

**Lemma 7.3** (Reduction Lemma) Let $A \equiv \bigvee (A_i)_{i \in J}$. Assume $\rho \notin \text{Reg} \cup \{K\}$, where $\rho := rk(A)$. Then:

\[
H \models^\alpha \rho \Lambda, \neg A \land H \models^\beta \rho \Gamma, A \implies H \models^\rho \alpha + \beta \Lambda, \Gamma
\]

*Proof.* Use induction on $\beta$. For details see Buchholz [1993], Lemma 3.14. \hfill $\Box$

**Theorem 7.4** (Predicative cut elimination) Let $H$ be closed under $\phi$. If $H \models^\beta \rho + \omega_\alpha \Gamma$, $[\rho, \rho + \omega_\alpha \cap (\text{Reg} \cup \{K\}) = \emptyset$, and $\alpha \in H$, then

\[
H \models^\rho \alpha + \beta \Gamma.
\]

*Proof.* By main induction on $\alpha$ and subsidiary induction on $\beta$ (cf. Buchholz [1993], Theorem 3.16). \hfill $\Box$

**Corollary 7.5** $H \models^\beta \rho + 1 \Gamma \land \rho \notin \text{Reg} \cup \{K\} \implies H \models^\rho \omega_\beta \Gamma$. 

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Lemma 7.6 (Bounding Lemma) Let $\mu \in \text{Reg} \cup \{K\}$ and $\beta \in \mathcal{H}$. If $\alpha \leq \beta < \mu$ and $B \in \Sigma_1(\mu)$, then

$$\mathcal{H} \mid^\beta_\rho \Gamma, B \implies \mathcal{H} \mid^{\beta,\mu}_\rho \Gamma, B(\beta,\mu).$$

Proof by induction on $\alpha$. Since $\alpha < \mu$, $B$ cannot be the principal formula of an inference $(\text{Ref}^{\mu}_{\rho})$ or $(\text{Ref}^{\mu}_{\rho})$.

If $B$ is not the principal formula of the last inference, the assertion follows by using the inductive assumption on its premisses and reapplying the same inference. Let $B$ be the principal formula of the last inference, which then must be $(\exists)$. $B$ has the form $(\exists x \in \mathbb{L}_\mu)F(x)$ with $\Delta_0(\mu)$–formula $F(\mathbb{L}_\mu)$. Also,

$$\mathcal{H} \mid^{\alpha_0}_{\rho} \Gamma, B, s \in \mathbb{L}_\mu \land F(s)$$

for some $\alpha_0 < \alpha$ and $s \in \text{Term}(\mu)$ with $|s| < \alpha$. By the induction hypothesis,

$$\mathcal{H} \mid^{\alpha_0}_{\rho} \Gamma, B(\beta,\mu), s \in \mathbb{L}_\mu \land F(s).$$

Since $|s| < \beta, \mu$, we have $s \in \mathbb{L}_\beta \equiv s \in \mathbb{L}_\mu$. Thus, applying $(\exists)$, the assertion follows. $\Box$

8 Embeddings

The first part of this Section deals with an embedding of $KP + \Pi^3 – \text{Ref}$ into $RS(\mathcal{K})$. Regarding proofs, we will be drawing on Buchholz [1993] when the proof is literally the same.

Furthermore, we shall show, by virtue of reflection for $\Pi^2(\pi)$–formulae, that reflection provably propagates to $\Sigma^3(\pi)$–formulae. This is not very surprising, however, we will also need to control the quantitative repercussions which $\Sigma^3(\pi)$–reflection causes on the ordinal bounds of a given derivation. All these results will be needed in Section 10.

Definition 8.1 For $\Gamma = \{A_1, \ldots, A_n\}$ let

$$\text{no}(\Gamma) := \omega^{rk(A_1)} \# \ldots \# \omega^{rk(A_n)}.$$ 

We define

$$\models \Gamma :\iff \text{for all operators } \mathcal{H}, \mathcal{H}[\Gamma] \mid^{\text{no}(\Gamma)}_{\rho} \Gamma$$

and

$$\models^{\xi}_{\rho} \Gamma :\iff \text{for all operators } \mathcal{H}, \mathcal{H}[\Gamma] \mid^{\text{no}(\Gamma)\#^{\xi}}_{\rho} \Gamma.$$ 

Lemma 8.2 Let $s \subseteq t$ stand for the formula $(\forall x \in s)(x \in t)$.
\[(i) \vdash A, \neg A.\]

\[(ii) \vdash s \notin s.\]

\[(iii) \vdash s \subseteq s.\]

\[(iv) \vdash s \notin t, s^\circ \in t \text{ for } s \in \text{Term}(|t|).\]

\[(v) \vdash s \neq t, t = s.\]

**Proof.** Buchholz [1993], Lemma 2.4, Lemma 2.5.

**Lemma 8.3**

\[\vdash [s_1 \neq t_1], \ldots, [s_n \neq t_n], \neg A(s_1, \ldots, s_n), A(t_1, \ldots, t_n).\]

**Proof.** Buchholz [1993], Lemma 2.7.

**Corollary 8.4** (Equality and Extensionality)

\[\vdash s_1 \neq t_1, \ldots, s_n \neq t_n, \neg A(s_1, \ldots, s_n), A(t_1, \ldots, t_n).\]

**Proof.** Buchholz [1993], Theorem 2.9.

**Lemma 8.5** (Foundation)

\[\vdash (\forall x \in \mathbb{L}_\alpha)[(\forall y \in x)F(y) \rightarrow F(x)] \rightarrow (\forall x \in \mathbb{L}_\alpha)F(x).\]

**Proof.** Fix an operator \( H \). Let \( A \equiv (\forall x \in \mathbb{L}_\alpha)[(\forall y \in x)F(y) \rightarrow F(x)] \). First, we show, by induction on \(|s|\), that if \( s \in \text{Term}(\alpha) \), then

\[(+) \quad H[A, s] \overset{\omega^{|s|+1} \omega}{\vdash} \neg A, F(s).\]

So assume that

\[H[A, t] \overset{\omega^{|t|+1} \omega}{\vdash} \neg A, F(t)\]

for all \( t \in \text{Term}(|s|) \). Using (\( \lor \)), this yields

\[H[A, s, t] \overset{\omega^{|s|+1} \omega}{\vdash} \neg A, t \in s \rightarrow F(t)\]

for all \( t \in \text{Term}(|s|) \), and hence

\[(1) \quad H[A, s] \overset{\omega^{|s|+2} \omega}{\vdash} \neg A, (\forall y \in s)F(y)\]
via (∀). Set \( \eta := \omega^{rk(A)} \# |s| + 2 \). By Lemma 8.2(i), \( \mathcal{H}[A, s] \models_0 \neg F(s), F(s) \); therefore, using (1) and (∧),

\[
\mathcal{H}[A, s] \models_0 \eta^+ \neg A, (\forall y \in s)F(y) \land \neg F(s), F(s).
\]

From the latter we obtain

\[
\mathcal{H}[A, s] \models_0 \eta^+ \neg A, s \in \mathbb{L}_\alpha \land [(\forall y \in s)F(y) \land \neg F(s)], F(s),
\]

and hence \( \mathcal{H}[A, s] \models_0 \eta^+ \neg A, (\exists x \in \mathbb{L}_\alpha) [(\forall y \in x)F(y) \land \neg F(x)], F(s) \) via (∃). This shows (+).

Finally, (+) enables us to deduce \( \mathcal{H}[A, s] \models_0 \eta^+ A, s \in \mathbb{L}_\alpha \rightarrow F(s) \) from which the assertion follows by applying (∀) and (∨).

\[\square\]

**Lemma 8.6** (Infinity Axiom) If \( \lambda \) be a limit ordinal > \( \omega \), then

\[ \models (\text{Infinity Axiom})^{\lambda \omega}, \]

i.e.,

\[ \models (\exists x \in \mathbb{L}_\lambda)[z \neq \emptyset \land (\forall y \in x)(\exists z \in x)(y \in z)]. \]

**Proof.** Buchholz [1993], Theorem 2.9. \[\square\]

**Lemma 8.7** (\( \Delta_0 \)-Separation) Let \( A[a, b_1, \ldots, b_n] \) be a \( \Delta_0 \)-formula of \( \mathcal{L}_{Ad} \). If \( \lambda \in \text{Lim} \) and \( s, t_1, \ldots, t_n \in \text{Term}(\lambda) \), then

\[ \models (\exists y \in \mathbb{L}_\lambda)[(\forall x \in y)(x \in s \land A[x, t_1, \ldots, t_n]) \land (\forall x \in s)(A[x, t_1, \ldots, t_n] \rightarrow x \in y)]. \]

More concisely, we can express this by “\( \models (\Delta_0 \text{-separation})^{\lambda \omega} \)”.

**Proof.** Buchholz [1993], Theorem 2.9. \[\square\]

**Lemma 8.8** (Pair and Union) Assume \( \lambda \in \text{Lim} \) and \( s, t \in \text{Term}(\lambda) \).

(i) \( \models (\exists z \in \mathbb{L}_\lambda)(s \in z \land t \in z). \)

(ii) \( \models (\exists z \in \mathbb{L}_\lambda)(\forall y \in s)(\forall x \in y)(x \in z). \)

**Proof.** Buchholz [1993], Theorem 2.9. \[\square\]

**Definition 8.9** The sequent calculus \( GML \) (“GML” stands for “Grundmengenlehre”) is defined as follows. The language of \( GML \) is \( \mathcal{L}_{Ad} \). With the exception of \( \Delta_0 \)-collection, \( GML \) has the same axiom schemes as \( KP \). (However, it is understood that the axiom schemes are defined with regard to \( \mathcal{L}_{Ad} \). To be precise, \( GML \) comprises the axiom scheme of \( \Delta_0(\mathcal{L}_{Ad}) \)-separation, whereas \( \Delta_0(\mathcal{L}_{Ad}) \)-collection is not an axiom scheme of \( GML \).)
Lemma 8.10 Assume $\rho = \omega^\varphi \leq K$. Let $\Gamma[\vec{a}] = \{A_1[\vec{a}], \ldots, A_k[\vec{a}]\}$ be a set of $L_{Adm}$-formulae, where $\vec{a} = a_1, \ldots, a_n$. If $GML \vdash \Gamma[\vec{a}]$, then there exists $m < \omega$ such that, for all $s = s_1, \ldots, s_n \in \text{Term}(\rho)$,

$$\mathcal{H}[\Gamma[s]^{1+\varphi}, \rho] \vdash_{\rho+m} \Gamma[s]^{1+\varphi}.$$  

Here $\Gamma[s]^{1+\varphi}$ stands for $\{A_1[s]^{1+\varphi}, \ldots, A_k[s]^{1+\varphi}\}$.

Proof by induction on $GLM$ derivations. As to the axioms of $GLM$, the claim follows easily from previous results of this Section. The inferences of $GLM$ are dealt with in the same manner as in Buchholz [1993], Theorem 3.12.  

Theorem 8.11 Let $\Gamma[\vec{a}] = \{A_1[\vec{a}], \ldots, A_k[\vec{a}]\}$ be a set of $L$-formulae with $\vec{a} = a_1, \ldots, a_n$. When $KP + \Pi_3 - \text{Ref} \vdash \Gamma[\vec{a}]$, then there exists $m < \omega$ such that, for all $s = s_1, \ldots, s_n \in \text{Term}$,

$$\mathcal{H}[\Gamma[s]^{1+\varphi}, \mathcal{K}] \vdash_{\mathcal{K}+m} \Gamma[s]^{1+\varphi}.$$  

Proof. Compared to Lemma 8.10, there is only one new inference, namely $(\Pi_3 - \text{Ref})$. But $(\Pi_3 - \text{Ref})$ is taken care of by $(\text{Ref}_\mathcal{K})$.  

Convention: We shall also write $\exists x^\varphi$ and $\forall x^\varphi$ instead of $((\exists x \in \mathbb{L}_\varphi))$ and $(\forall x \in \mathbb{L}_\varphi)$, respectively.

Lemma 8.12 Assume $\xi \in C(m(\pi), \pi) \cap m(\pi)$, $\xi \in \mathcal{H}$, and $F(\mathbb{L}_0, \mathbb{L}_0, \mathbb{L}_0) \in \Delta_0(\pi)$. If

$$\mathcal{H} \vdash_{\rho} \Gamma, \exists u \forall v \exists y \pi F(u, x, y)$$

then

$$\mathcal{H} \vdash_{\rho} \Gamma, \exists z \pi [Ad^\xi(z) \land (\exists u \in z)(\forall v \in z)(\exists y \in z)F(u, x, y)].$$

Note that $\pi^{1+\alpha} = (\omega^\varphi)^{1+\alpha} = \omega^{\pi(1+\alpha)}$.

Proof. We proceed by induction on $\alpha$. Put $C \equiv \exists u \forall v \exists y \pi F(u, x, y)$. If $C$ is not the principal formula of the last inference, then use the induction hypothesis on the premisses and subsequently apply the same inference.

Assume that $C$ is the principal formula. Then the last inference must be $\exists$, and we have

$$\mathcal{H} \vdash_{\rho} \Gamma, C, \forall v \exists y \pi F(s, x, y)$$

for some $\alpha_0 < \alpha$ and $s \in \text{Term}(\pi)$. Inductively we get

$$\mathcal{H} \vdash_{\rho} \Gamma, \exists z \pi [Ad^\xi(z) \land (\exists u \in z)(\forall v \in z)(\exists y \in z)F(u, x, y)], \forall v \exists y \pi F(s, x, y).$$

Note that $\pi^{1+\alpha} + 1, \pi < \pi^{1+\alpha}$. So, using $(\text{Ref}_\mathcal{K})$, we obtain

$$\mathcal{H} \vdash_{\rho} \Gamma, \exists z \pi [Ad^\xi(z) \land (\exists u \in z)(\forall v \in z)(\exists y \in z)F(u, x, y)].$$  

□
Lemma 8.13 Let $\xi \in C(m(\pi), \pi) \cap m(\pi)$ and $\xi > 0$. Assume that $A_1, \ldots, A_k$ are subformulae of $\Sigma_3(\pi)$-formelae, $\xi \in H$, $\pi + \omega \leq \rho$, and $\pi < \alpha = \omega^\alpha$. Then,

$$\mathcal{H} \models_\rho \Gamma, A_1 \land \ldots \land A_k \implies \mathcal{H} \models_\rho^{\pi+2} \Gamma, \exists \bar{z}[\text{Ad}^\xi(z) \land A_1^{(z, \pi)} \land \ldots \land A_k^{(z, \pi)}].$$

Proof. $A_i$ has the form $B_i[\bar{s}]^{L^\xi}$ with $B_i[\bar{a}]$ being a $L_{Ad}$-formula. Putting $B[\bar{a}] = B_i[\bar{a}] \land \ldots \land B_n[\bar{a}]$, we have $A_1 \land \ldots \land A_n = B[\bar{a}]^{L^\xi}$. By going to prenex normal form, coding adjacent quantifiers of the same sort into one quantifier, and, if necessary, inserting dummy quantifiers, we can transform $B[\bar{a}]$ into a $\Sigma_3$-formula, say $C[\bar{a}]$. The equivalence of $C[\bar{a}]$ and $B[\bar{a}]$ is provable in $GLM$ since coding tuples of sets just requires Pairing and Extensionality. Therefore, the equivalence of $C[\bar{a}]$ and $B[\bar{a}]$ still holds when we relativize all the quantifiers to a nonempty transitive set which is a model of Pairing; and this can be proved in $GLM$. So, letting $Pairing := \forall x \forall y \exists u(u = \{x, y\})$, we get

$$GML \vdash \neg B[\bar{a}], C[\bar{a}]$$

and

$$GML \vdash \neg [\text{Tran}(b) \land b \not= \emptyset \land (\text{Pairing})^b], \neg C[\bar{a}]^b, B[\bar{a}]^b.$$ (2)

From (1), using Lemma 8.10, we obtain

$$\mathcal{H} \models_\rho \Gamma, \exists \bar{z}[\text{Ad}^\xi(z) \land C[\bar{z}]^{L^\xi}].$$ (3)

for some $0 < m < \omega$. Employing Lemma 8.12, (3) yields

$$\mathcal{H} \models_\rho^{\pi+\omega^m} \Gamma, \exists \bar{z}[\text{Ad}^\xi(z) \land C[\bar{z}]^{L^\xi}].$$ (4)

Using (Cut) on (4) and $\mathcal{H} \models_\rho \Gamma, B[\bar{s}]^{L^\xi}$, and noting that $\pi + \omega \leq \rho$, one obtains

$$\mathcal{H} \models_\rho^{\pi+1} \Gamma, \exists \bar{z}[\text{Ad}^\xi(z) \land C[\bar{z}]^{L^\xi}].$$ (5)

According to Lemma 8.10, (2) implies

$$\mathcal{H}[\rho] \models_\pi \neg [\text{Tran}(\mathbb{L}_\rho) \land \mathbb{L}_\rho \not= \emptyset \land (\text{Pairing})^{L^\rho}], \neg C[\bar{s}]^{L^\rho}, B[\bar{s}]^{L^\rho}.$$ (6)

for all $\rho \in M^{\xi} \cap \pi$ since $\omega^\rho = \rho$ due to $\xi > 0$. But, by Lemma 8.10, we also have, for all $\rho \in M^{\xi} \cap \pi$,

$$\mathcal{H}[\rho] \models_\pi \text{Tran}(\mathbb{L}_\rho) \land \mathbb{L}_\rho \not= \emptyset \land (\text{Pairing})^{L^\rho},$$

whence (6) implies

$$\mathcal{H}[\rho] \models_\pi^{\pi+1} \neg C[\bar{s}]^{L^\rho}, B[\bar{s}]^{L^\rho}.$$ (7)

This is the only reason why we introduced $GML$. 

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for all $\rho \in M^\xi \cap \pi$. From (7) one deduces
\[
\mathcal{H}[\rho] \frac{\mathcal{H}[\rho] |_\pi^{\pi+2} - C[\vec{s}] \sqcup \rho, Ad^\xi(\mathbb{L}_\rho) \land B[\vec{s}] \sqcup \rho}
\]
whence
\[
\mathcal{H}[\rho] \frac{\mathcal{H}[\rho] |_\pi^{\pi+3} - C[\vec{s}] \sqcup \rho, \exists z^\pi(Ad^\xi(z) \land B[\vec{s}] \sqcup)}{\text{(8)}}
\]
for all $\rho \in M^\xi \cap \pi$. Since, by Corollary 8.4,
\[
\mathcal{H}[\rho, t] \frac{\mathcal{H}[\rho, t] |_\ell^{\ell_0} \mathbb{L}_\rho \neq t, - C[\vec{s}] \sqcup t, C[\vec{s}] \sqcup \rho}{(\text{Cut})}
\]
yields
\[
\mathcal{H}[\rho, t] \frac{\mathcal{H}[\rho, t] |_\pi^{\pi+4} \mathbb{L}_\rho \neq t, - C[\vec{s}] \sqcup t, \exists z^\pi(Ad^\xi(z) \land B[\vec{s}] \sqcup t)}{\text{for all } \rho \in M^\xi \cap \pi \text{ und } t \in \text{Term}(\pi)}.\]
Whence, via ($-Ad^\xi$),
\[
\mathcal{H}[t] \frac{\mathcal{H}[t] |_\pi^{\pi+5} - Ad^\xi(t), - C[\vec{s}] \sqcup t, \exists z^\pi(Ad^\xi(z) \land B[\vec{s}] \sqcup t)}{\text{for all } t \in \text{Term}(\pi)}.\]
Therefore, employing ($\lor$) und ($\forall$),
\[
\mathcal{H}[\rho] \frac{\mathcal{H}[\rho] |_\pi^{\pi+8} \forall z^\pi[- Ad^\xi(z) \lor - C[\vec{s}] \sqcup t], \exists z^\pi(Ad^\xi(z) \land B[\vec{s}] \sqcup t)}{\text{(9)}}
\]
Finally, by linking (5) and (9) via (Cut),
\[
\mathcal{H} \frac{\mathcal{H} |_\pi^{\pi+2} \Delta, \exists z^\pi(Ad^\xi(z) \land B[\vec{s}] \sqcup t)}{\text{for all } t \in \text{Term}(\pi)}.\]

\section{The Operators $\mathcal{H}_{\gamma}$}

In order to be able to remove critical cuts, i.e. cuts which were introduced by ($Ref_K$) or ($Ref_{\pi}$) inferences, we have to forgo arbitrary operators. We shall need operators $\mathcal{H}$ such that an $\mathcal{H}$-controlled derivation that satisfies certain extra conditions can be “collapsed” into a derivation with much smaller ordinal labels.

\textbf{Definition 9.1} The operator $\mathcal{H}_\delta$ is defined by
\[
\mathcal{H}_\delta(X) = \bigcap \{C(\alpha, \beta) : X \subseteq C(\alpha, \beta) \land \delta < \alpha\}
\]

\textbf{Lemma 9.2} (i) $\mathcal{H}_\delta$ is an operator.

(ii) $\delta < \delta' \implies \mathcal{H}_\delta(X) \subseteq \mathcal{H}_{\delta'}(X)$.
Definition 9.3 (i) $H_\delta$ is closed under $\varphi$ and $(\sigma \mapsto \Omega_\sigma)$$_{\sigma < \kappa}$.

(ii) $\xi, \pi, \alpha \in H_\delta(X) \land \xi \leq \alpha \leq \delta \implies \Psi_\pi^{\xi}(\alpha) \in H_\delta(X)$.

(iii) $\beta \leq \delta \land \beta \in H_\delta(X) \implies \Xi(\beta) \in H_\delta(X)$.

(iv) $\Omega_\sigma \leq \eta \leq \Omega_{\sigma+1} < \kappa \land \eta \in H_\delta(X) \implies \sigma, \Omega_\sigma, \Omega_{\sigma+1} \in H_\delta(X)$.

Proof. (i) follows from Lemma 4.18. (ii) holds by Lemma 4.11(i). (iii) follows from closure of any $C(\alpha, \beta)$ under these functions.

(iv): From $\xi, \pi, \alpha \in H_\delta(X)$, $X \subseteq C(\alpha', \beta)$ and $\xi \leq \alpha \leq \delta < \alpha'$, it follows $\Psi_\pi^{\xi}(\alpha) \in C(\alpha', \beta)$; thus $\Psi_\pi^{\xi}(\alpha) \in H_\delta(X)$.

The proof of (v) is similar to (iv).

(vi): Suppose $X \subseteq C(\alpha, \beta)$ with $\delta < \alpha$. Then we have to show $\sigma \in C(\alpha, \beta)$. Note that $\eta \in C(\alpha, \beta)$. By induction on $n$, one verifies

$$\Omega_\sigma \leq \eta \leq \Omega_{\sigma+1} \land \eta \in C_n(\alpha, \beta) \implies \sigma \in C(\alpha, \beta),$$

yielding $\sigma \in C(\alpha, \beta)$. If $\eta = \Omega_\sigma$, then $\sigma \in C(\alpha, \beta)$ by 4.18(iii). Otherwise, there is only one case when (*) is not immediate by the induction hypothesis, namely when $\eta = \Psi_\pi^{\xi}(\gamma) \in C_n(\alpha, \beta) \setminus C_{n-1}(\alpha, \beta)$ with $\xi, \pi, \gamma \in C_{n-1}(\alpha, \beta)$. According to 4.19(i), we then must have $\xi = 0$ and $\pi = \Omega_{\sigma+1}$; consequently, by Lemma 4.18, $\sigma \in C(\alpha, \beta)$. □

Roughly speaking, the process of collapsing a proof tree, which we will be using in the next Section, involves pruning, grafting, and relabelling the tree with smaller ordinals. The relabelling will be done by applying a variant of $\Xi$ or variants of the functions $\Psi_\pi^{\xi}$ to the ordinal labels of the original tree. We are compelled to pass to variants of these functions because $\Xi$ or $\Psi_\pi^{\xi}$ may not preserve the order of the ordinals of the given tree, and further $\Psi_\pi^{\xi}(\alpha) < \pi$ may fail to be the case for some ordinal $\alpha$ of the tree. But that the relabelling be done in an order preserving way, is necessary if this procedure is meant to transform proof trees into proof trees.

To handle the aforementioned difficulties, we will be needing several technical results, the meaning of which will emerge only gradually in the proofs of Theorem 10.1 and Theorem 10.3. I have preferred to ban these “side calculations” from the proofs of the main theorems since the danger is to be feared that they may obscure the central ideas underlying the cut elimination and collapsing procedure.

Definition 9.3 (i) $NF(\alpha, \beta)$ means that $\alpha_n \geq \beta_1$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ are the respective Cantor normal forms.

(ii) $\mathfrak{B}(X; \gamma) : \iff \gamma \in H_\gamma[X] \land k(X) \subseteq C(\gamma + 1, \Xi(\gamma + 1))$.

Lemma 9.4 Assume $\mathfrak{B}(X; \gamma)$, $\pi \in M^\hat{\alpha}$, $\alpha \in H_\gamma[X]$, and $NF(\gamma, \omega^\kappa \alpha)$, where $\hat{\alpha} := \gamma + \omega^\kappa$.

For arbitrary $\alpha_0$, let $\alpha_0 := \gamma + \omega^\kappa \alpha_0$.

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(i) \(\mathcal{H}_\gamma[X](\emptyset) \cap \mathcal{K} \subseteq \Xi(\gamma + 1)\).

(ii) \(\Xi(\hat{\alpha} + \pi) \in \mathcal{H}_{\hat{\alpha} + \pi}[X, \pi]\).

(iii) \(\alpha_0 \in \mathcal{H}_\gamma[X] \land \alpha_0 < \alpha \implies \Xi(\hat{\alpha}_0 + \pi) < \Xi(\hat{\alpha} + \pi)\).

(iv) Suppose \(t \in \text{Term}, \ |t| \leq \alpha_t < \alpha_t \in \mathcal{H}_\gamma[X, t]\). If \(\gamma_t := \gamma + \omega^{\mathcal{K} - \alpha_t + 1}\) and \(\beta_t := \gamma_t + \omega^{\mathcal{K} - \alpha_t}\), then

\[\mathfrak{B}(\mathfrak{X} \cup \{t\}; \gamma_t) \text{ and } \beta_t \in \mathcal{H}_{\gamma_t}[X, t].\]

If in addition \(t \in \text{Term}(\pi)\), then also

\[\Xi(\beta_t + \pi) < \Xi(\hat{\alpha} + \pi) \land \pi \in M^\beta.\]

Proof. (i) follows from \(k(X) \subseteq C(\gamma + 1, \Xi(\gamma + 1))\) in view of the definition of \(\mathcal{H}_\gamma[X]\).

(ii): Since \(\gamma, \alpha, \pi \in \mathcal{H}_{\hat{\alpha} + \pi}[X, \pi]\), (ii) follows from 9.2(v).

(iii): \(\hat{\alpha} + \pi \in C(\hat{\alpha} + \pi, \Xi(\hat{\alpha} + \pi))\) and \(NF(\gamma, \omega^{\mathcal{K} - \alpha})\) imply \(\gamma \in C(\hat{\alpha} + \pi, \Xi(\hat{\alpha} + \pi))\) by 4.18. Therefore,

\(\alpha_0 \in \mathcal{H}_\gamma[X] \subseteq C(\gamma + 1, \Xi(\gamma + 1)) \subseteq C(\hat{\alpha} + \pi, \Xi(\hat{\alpha} + \pi))\).

Thence, \(\hat{\alpha}_0 + \pi \in C(\hat{\alpha} + \pi, \Xi(\hat{\alpha} + \pi)) \cap \hat{\alpha} + \pi\); thus \(\Xi(\hat{\alpha}_0 + \pi) < \Xi(\hat{\alpha} + \pi)\).

(iv): \(\gamma \in \mathcal{H}_\gamma[X]\) ensures \(\gamma_t, \beta_t \in \mathcal{H}_{\gamma_t}[X; t]\). \(NF(\gamma, \omega^{\mathcal{K} - \alpha})\) and \(\alpha_t < \alpha\) yield \(NF(\gamma, \omega^{\mathcal{K} - \alpha_t + 1})\).

Hence, from \(\gamma_t \in C(\gamma_t, \Xi(\gamma_t))\), we can deduce \(\gamma, |t| \subseteq C(\gamma_t, \Xi(\gamma_t))\) and therefore, \(C(\gamma + 1, \Xi(\gamma + 1)) \subseteq C(\hat{\alpha} + \pi, \Xi(\hat{\alpha} + \pi))\). This shows \(\mathfrak{B}(\mathfrak{X} \cup \{t\}; \gamma_t)\).

Now suppose \(t \in \text{Term}(\pi)\). From \(NF(\gamma, \omega^{\mathcal{K} - \alpha})\) it follows \(\gamma \in C(\hat{\alpha}, \Xi(\hat{\alpha}))\) and hence \(k(X \cup \{t\}) \subseteq C(\hat{\alpha}, \pi)\) as \(\Xi(\hat{\alpha}) \leq \pi\) holds because of \(\pi \in M^\hat{\alpha}\). Whence, \(\beta_t \in C(\hat{\alpha}, \pi) \cap \hat{\alpha}\). This implies

\[\beta_t + \pi \in C(\hat{\alpha} + \pi, \Xi(\hat{\alpha} + \pi)) \land \hat{\alpha} + \pi;\]

thus

\[\Xi(\beta_t + \pi) < \Xi(\hat{\alpha} + \pi).\]

Finally, from \(\beta_t \in C(\hat{\alpha}, \pi) \cap \hat{\alpha}\) and \(\pi \in M^\hat{\alpha}\) we obtain, by 4.11(vi), \(\pi \in M^\beta\).

\[\square\]

Definition 9.5

(i) \(\text{Card} := \{\mathcal{K}\} \cup \{\Omega_\sigma : 0 < \sigma < \mathcal{K}\}\).

(ii) For \(\mu \in \text{Card}\), put

\[\mu_1 = \begin{cases} \mu + 1 & \text{if } \mu \in \text{Reg} \cup \{\mathcal{K}\} \\
\mu & \text{otherwise.} \end{cases}\]

(iii) Let \(\mathfrak{B}(X; \gamma, \pi, \xi, \mu)\) stand for

\[\mathfrak{B}(X; \gamma) \land \gamma, \pi, \xi, \mu \in \mathcal{H}_\gamma[X] \land \xi \in C(m(\pi), \pi) \cap m(\pi)\]

\[\land k(X) \subseteq C(\gamma + 1, \Psi^0_\pi(\gamma + 1)) \land \pi \in \cap \{C(\delta, \Psi^0_\pi(\delta)) : \delta > \gamma; \tau > \pi\}\]

\[\land \xi \leq \gamma \land \mu \in \text{Card} \land \pi \leq \mu.\]
Lemma 9.6 Assume $\mathfrak{A}(\Sigma; \gamma, \pi, \xi, \mu)$, $NF(\gamma, \omega^\mu \alpha)$, and $\alpha \in \mathcal{H}_\gamma[\Sigma]$. For arbitrary $\beta$, let $\hat{\beta} := \gamma + \omega^\mu \beta$. Then the following properties hold.

(i) $\Psi_\pi^\xi(\hat{\alpha}) \in \mathcal{H}_\alpha[\Sigma] \land \Psi_\pi^\xi(\hat{\alpha}) \in M^\xi \cap \pi$.

(ii) $\mathcal{H}_\gamma[\Sigma](\emptyset) \subseteq C(\gamma + 1, \Psi_\pi^0(\gamma + 1))$.

(iii) $\alpha_0 \in \mathcal{H}_\gamma[\Sigma] \land \alpha_0 < \alpha \implies \Psi_\pi^\xi(\hat{\alpha}_0) < \Psi_\pi^\xi(\hat{\alpha})$.

(iv) Suppose $\sigma \in \mathcal{H}_\gamma[\Sigma]$, $\sigma \leq \gamma, \sigma \in C(m(\pi), \pi) \cap m(\pi)$ and $t \in \text{Term}(\pi)$. If $\gamma_t = \gamma + \omega^\mu \alpha + |t|$, then

$$\mathfrak{A}(\Sigma \cup \{t\}; \gamma_t, \pi, \sigma, \mu).$$

(v) If $\alpha_0 < \alpha$, $\alpha_0, \tau \in \mathcal{H}_\gamma[\Sigma]$ and $\pi \leq \tau \leq \mu$, then

$$\mathfrak{A}(\Sigma; \gamma, \tau, 0, \mu) \land \mathfrak{A}(\Sigma; \alpha_0, \tau, 0, \mu).$$

Proof. (i): $\hat{\alpha} \in \mathcal{H}_\alpha[\Sigma]$ is obvious. Therefore, $\Psi_\pi^\xi(\hat{\alpha}) \in \mathcal{H}_\alpha[\Sigma]$ by 9.2(iv). Since $\mathcal{H}_\gamma[\Sigma](\emptyset) \subseteq C(\gamma + 1, \Psi_\pi^0(\gamma + 1)) \subseteq C(\hat{\alpha}, \pi)$, we get $\xi, \pi, \hat{\alpha}, \sigma \in C(\hat{\alpha}, \pi)$. Since also $\xi \in C(m(\pi), \pi) \cap m(\pi)$, we obtain $\Psi_\pi^\xi(\hat{\alpha}) \in M^\xi \cap \pi$ using 4.16.

(ii): Immediate as $k(\Sigma) \subseteq C(\gamma + 1, \Psi_\pi^0(\gamma + 1))$.

(iii): Since $\hat{\alpha}, \pi \in C(\hat{\alpha}, \Psi_\pi^\xi(\hat{\alpha}))$ by (i), and $NF(\gamma, \omega^\mu \alpha)$ involves $\gamma \in C(\hat{\alpha}, \Psi_\pi^\xi(\hat{\alpha}))$, it follows $\Psi_\pi^0(\gamma + 1) \subseteq C(\hat{\alpha}, \Psi_\pi^\xi(\hat{\alpha}))$. From (ii) we get $\Psi_\pi^0(\gamma + 1) < \pi$. Therefore, $\Psi_\pi^0(\gamma + 1) < \Psi_\pi^\xi(\hat{\alpha})$. In view of (ii), this yields $\mathcal{H}_\gamma[\Sigma](\emptyset) \subseteq C(\hat{\alpha}, \Psi_\pi^\xi(\hat{\alpha}))$ and hence $\alpha_0 \in C(\hat{\alpha}, \Psi_\pi^\xi(\hat{\alpha}))$. $\Psi_\pi^\xi(\hat{\alpha}_0) < \pi$ follows by replacing $\alpha$ with $\alpha_0$ in the proof of (i). Consequently, in view of the above, $\Psi_\pi^\xi(\hat{\alpha}_0) < \Psi_\pi^\xi(\hat{\alpha})$.

(iv): $\alpha, \mu, \gamma \in \mathcal{H}_\gamma[\Sigma]$ guarantees $\mu, \gamma, |t|, \alpha_t \in \mathcal{H}_\gamma[\Sigma, t]$. Therefore,

$$\gamma_t \in \mathcal{H}_\gamma[\Sigma, t].$$

We claim that

$$(*) \quad k(\Sigma \cup \{t\}) \subseteq C(\gamma_t + 1, \Psi_\pi^0(\gamma_t + 1))$$

By (ii), $\alpha, \gamma \in C(\gamma_t + 1, \pi)$ and hence $\gamma_t \in C(\gamma_t + 1, \pi)$, which implies $\gamma_t \in C(\gamma_t + 1, \Psi_\pi^0(\gamma_t + 1))$. As $NF(\gamma, \omega^\mu \alpha)$, this shows $\gamma \in C(\gamma_t + 1, \Psi_\pi^0(\gamma_t + 1))$, yielding (note that $\pi \in C(\gamma + 1, \pi)$ by (ii)) $\Psi_\pi^0(\gamma + 1) < \Psi_\pi^0(\gamma + 1)$. So we obtain $k(\Sigma) \subseteq C(\gamma_t, \Psi_\pi^0(\gamma_t + 1))$ and hence (*).

Finally, from (*) and $\mathcal{H}_\gamma[\Sigma] \subseteq \mathcal{H}_\gamma[\Sigma, t]$ and $\gamma_t \in \mathcal{H}_\gamma[\Sigma, t]$, we get $\mathfrak{A}(\Sigma \cup \{t\}; \gamma_t, \pi, \sigma, \mu)$.

(v): As $\tau \in \mathcal{H}_\gamma[\Sigma]$, we get $\tau \in C(\gamma + 1, \pi)$ due to (ii). If now $\kappa > \tau$ and $\delta > \gamma$, then $\pi \in C(\delta, \Psi_\pi^0(\delta))$; whence $\tau \in C(\delta, \Psi_\pi^0(\delta))$. In this case, $\gamma_t \in C(\gamma, \tau, 0, \mu)$ is now immediate.

To see $\mathfrak{A}(\Sigma; \alpha_0, \tau, 0, \mu)$, it suffices to verify $C(\gamma + 1, \Psi_\pi^0(\gamma + 1)) \subseteq C(\alpha_0 + 1, \Psi_\pi^0(\alpha_0 + 1))$. This is trivial if $\tau > \pi$. In case $\tau = \pi$, we get $\gamma \in C(\alpha_0 + 1, \Psi_\pi^0(\alpha_0 + 1))$ from $NF(\gamma, \omega^\mu \alpha)$ and $\alpha_0 \in C(\alpha_0 + 1, \Psi_\pi^0(\alpha_0 + 1))$. Thus $\Psi_\pi^0(\gamma + 1) \subseteq C(\alpha_0 + 1, \Psi_\pi^0(\alpha_0 + 1))$. As $\Psi_\pi^0(\gamma + 1) < \pi$, the latter yields the claim. \qed
10 Impredicative cut elimination and collapsing

In general, the usual cut elimination procedure does not apply when the cut formula has been introduced by a reflection inference. This is, for instance, the case when

\[ H \alpha^{K+1}_{\Gamma} \]

results from

\[ H \frac{\alpha^{K}_{\Gamma}, A}{\Gamma, \exists z^{K}[\text{Tran}(z) \land z \neq \emptyset \land A^z]} (\text{Ref}_K) \]

and

\[ \cdots H[s] \frac{\alpha^{K}_{\Gamma}, \neg[\text{Tran}(s) \land s \neq \emptyset \land A^s] \cdots (s \in \text{Term})}{\Gamma, \forall z^{K}[\text{Tran}(z) \land z \neq \emptyset \land A^z]} (\forall) \]

using (Cut), where \( A \) is a \( \Pi_3(K) \)–formula. In this situation, the usual procedure of replacing an instance of (Cut) with cuts of lesser rank does not work. In order to overcome this problem, the proof tree has to undergo more radical transformations.

**Theorem 10.1** Suppose \( \mathcal{B}(\mathcal{X}; \gamma) \) and \( NF(\gamma, K^{\alpha}) \). Let \( \Gamma \) be a set of \( RS(K) \)–formulae each of which is a subformula of a \( \Pi_2(K) \)–formula or \( \Pi_2(K) \)–formula. Furthermore, suppose

\[ H \gamma^{K}_{\Gamma} \]

then, for all \( \pi \in M^k \),

\[ H_{\hat{\alpha} + \pi}^{\xi(\gamma, K^\alpha, K^{\alpha})} \]

where \( \hat{\alpha} = \gamma + K^{\alpha} = \gamma + \omega^{K^{\alpha}}. \)

**Proof** by induction on \( \alpha \).

**Case 1:** The last inference is (\( \forall \)) with principal formula \( \forall x F(x) \in \Gamma \). Then, for all \( t \in \text{Term} \), there exists \( \alpha_t \) satisfying \( |t| \leq \alpha_t < \alpha \) and

\[ H_{\gamma}^{K+1}_{\Gamma, F(t)} \]

Define \( \gamma_t := \gamma + \omega^{K^{\alpha_t}} \) and \( \beta_t := \gamma_t + K^{\alpha_t} = \gamma_t + \omega^{K^{\alpha_t}} \). Then \( NF(\gamma_t, K^{\alpha_t}) \). Also \( \mathcal{B}(\mathcal{X} \cup \{t\}, \gamma_t) \) by 9.4(iv). Therefore, using the induction hypothesis on (10),

\[ H_{\beta_t + \pi}^{\xi(\beta_t + \pi)} \]

An appropriate name for this collapsing technique would be *stationary collapsing* since in order for this procedure to work, a single derivation has to be collapsed into a “stationary” family of derivations.
holds for all \( t \in \text{Term} \) and \( \pi \in M^\beta \). If \( \pi \in M^\beta \) and \( t \in \text{Term}(\pi) \), then, by Lemma 9.4(iv), \( \pi \in M^\beta \) and \( \Xi(\beta_t + \pi) < \Xi(\hat{\alpha} + \pi) \). Therefore, from (11), we can conclude

\[
\mathcal{H}_{\hat{\alpha} + \pi}[\mathcal{X}, \pi] \models^{\Xi}(\hat{\alpha} + \pi) \Gamma^{(\pi, \mathcal{K})}, \forall x^\pi F(x)^{\pi, \mathcal{K}}
\]

by means of (\forall). Since \( \Gamma^{(\pi, \mathcal{K})}, \forall x^\pi F(x)^{\pi, \mathcal{K}} = \Gamma^{(\pi, \mathcal{K})} \), this provides the desired result.

**Case 2:** The last inference is (\land) but does not fall under the previous Case. This implies that the principal formula has a rank \( < \mathcal{K} \) or is of the form \( A_0 \land A_1 \). The assertion then follows by simplifying the considerations of the previous Case.

**Case 3:** The last inference is (\lor) with principal formula \( C \equiv \lor(C_i)_{i \in J} \in \Gamma \). Thus \( \mathcal{H}_{\gamma}[\mathcal{X}] \models^{\alpha_0}_{K+1} \Gamma, C_i \) for some \( i_0 \in J \mid \alpha \) satisfying \( |i_0| < \alpha \) and \( k(i_0) \subset \mathcal{H}_{\gamma}[\mathcal{X}] \). Hence, by the induction hypothesis, for all \( \pi \in M^{\alpha_0} \),

\[
\mathcal{H}_{\hat{\alpha} + \pi}[\mathcal{X}, \pi] \models^{\Xi}(\hat{\alpha} + \pi) \Gamma^{(\pi, \mathcal{K})}, C_i^{(\pi, \mathcal{K})}.
\]

The conditions on \( i_0 \) ensure that \( |i_0| < \Xi(\hat{\alpha} + \pi) \). As \( M^{\hat{\alpha}} \subseteq M^{\alpha_0} \) is guaranteed by 4.11(vi), and \( \Xi(\hat{\alpha}_0 + \pi) < \Xi(\hat{\alpha} + \pi) \) holds by 9.4(iii), applying (\lor) yields

\[
\mathcal{H}_{\hat{\alpha} + \pi}[\mathcal{X}, \pi] \models^{\Xi}(\hat{\alpha} + \pi) \Gamma^{(\pi, \mathcal{K})}, C^{(\pi, \mathcal{K})} \quad (= \Gamma^{(\pi, \mathcal{K})})
\]

for \( \pi \in M^{\hat{\alpha}} \).

**Case 4:** The last inference is (Cut). Then

\[
\mathcal{H}_{\gamma}[\mathcal{X}] \models^{\alpha_0}_{K+1} \Gamma, A
\]

and

\[
\mathcal{H}_{\gamma}[\mathcal{X}] \models^{\alpha_0}_{K+1} \Gamma, \neg A
\]

for some \( \alpha_0 < \alpha \) and RS(\mathcal{K})--formul\ae \( A, \neg A \) with \( rk(A) \leq \mathcal{K} \). Since then \( A \) as well as \( \neg A \) are subformulae of \( \Pi_3(\mathcal{K}) \cup \Pi_2(\mathcal{K}) \) formul\ae, we can apply the induction hypothesis to both derivations. Whence, for all \( \pi \in M^{\alpha_0} \),

\[
\mathcal{H}_{\hat{\alpha}_0 + \pi}[\mathcal{X}, \pi] \models^{\Xi}(\hat{\alpha}_0 + \pi) \Gamma^{(\pi, \mathcal{K})}, A^{(\pi, \mathcal{K})}
\]

and

\[
\mathcal{H}_{\hat{\alpha}_0 + \pi}[\mathcal{X}, \pi] \models^{\Xi}(\hat{\alpha}_0 + \pi) \Gamma^{(\pi, \mathcal{K})}, \neg A^{(\pi, \mathcal{K})}.
\]

We also have \( M^{\hat{\alpha}} \subseteq M^{\hat{\alpha}_0} \) and

\[
 rk(A^{(\pi, \mathcal{K})}), \Xi(\hat{\alpha}_0 + \pi) < \Xi(\hat{\alpha} + \pi).
\]

So the desired derivation is obtained by (Cut).
Case 5: The last inference is \((Ref_k)\). Then
\[
H_\gamma[X] \models_{K+1} \Gamma, \forall x^K \exists y^K \forall z^K F(x, y, z)
\]
for some \(\alpha_0 < \alpha\) and a formula \(C \in \Gamma\) of the form
\[
C \equiv \exists u^K[Tran(u) \land u \neq \emptyset \land (\forall x \in u)(\exists y \in u)(\forall z \in u)F(x, y, z)].
\]
Set \(B \equiv \forall x^K \exists y^K \forall z^K F(x, y, z)\). From the induction hypothesis we then obtain, for all \(\tau \in M^{\hat{\alpha}_0}\),
\[
H^{\hat{\alpha}_0+\tau}[X, \tau] \models_{\hat{\alpha}_0+\tau} \bigg[\exists u^{\hat{\alpha}_0+\tau} \big[\Gamma(\tau, K), B(\tau, K)\big]\bigg] \quad (12)
\]
In the sequel, fix \(\pi \in M^{\hat{\alpha}}\). If \(\tau \in M^{\hat{\alpha}_0}\), then
\[
\models \Box Tran(L_\tau) \land L_\tau \neq \emptyset;
\]
therefore, using (12),
\[
H^{\hat{\alpha}_0+\pi}[X, \pi, \tau] \models_{\hat{\alpha}_0+\pi} \bigg[\exists u^{\hat{\alpha}_0+\pi} \big[\Gamma(\tau, K), B(u, K)\big]\bigg] \quad (13)
\]
for all \(\tau \in M^{\hat{\alpha}_0} \cap \pi\).
Now let \(s \in Term(\pi)\). In view of Corollary 8.4, we get
\[
\models \Box s \neq \bigwedge \neg \Gamma(\tau, K), \bigvee \Gamma(s, K).
\]
Using (13) and \((\lor)\),
\[
H^{\hat{\alpha}_0+\pi}[X, \pi, s, \tau] \models_{\hat{\alpha}_0+\pi} \bigg[\exists u^{\hat{\alpha}_0+\pi} \big[\neg Ad^{\hat{\alpha}_0}(s), \bigwedge \Gamma(s, K), C(\pi, K)\big]\bigg] \quad (14)
\]
holds for all \(\tau \in M^{\hat{\alpha}_0}\) satisfying \(\tau \leq |s|\). Thence, applying \((\neg Ad^{\hat{\alpha}_0})\), we get
\[
H^{\hat{\alpha}_0+\pi}[X, \pi, s] \models_{\hat{\alpha}_0+\pi} \bigg[\exists u^{\hat{\alpha}_0+\pi} \big[Ad^{\hat{\alpha}_0}(s), \bigwedge \Gamma(s, K), C(\pi, K)\big]\bigg] \quad (15)
\]
for \(s \in Term(\pi)\). Putting to use \((\lor)\) and subsequently \((\forall)\), we arrive at
\[
H^{\hat{\alpha}_0+\pi}[X, \pi] \models_{\hat{\alpha}_0+\pi} \big[\forall u^{\pi} [Ad^{\hat{\alpha}_0}(u) \rightarrow \bigwedge \Gamma(u, K)], C(\pi, K)\big] \quad (16)
\]
Furthermore,
\[
\models \Gamma(\pi, K), \bigwedge \neg \Gamma(\tau, K)
\]
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by 8.2(i). \( \land \lnot \Gamma^{(\pi,K)} \) is a conjunction of subformulae of \( \Sigma_3(\pi) \)-formulae. As a consequence, we can apply 8.13, yielding\(^\text{13}\)

\[
\mathcal{H}_{\hat{\alpha} + \pi}[X, \pi] \left|_{\Xi(\hat{\alpha}_0 + \pi) \Gamma^{(\pi,K)}} \Phi^{[\hat{\alpha}_0 + \pi]}[\pi] \wedge \Gamma^{(\pi,K)} \right].
\]

(17)

Since \( \Xi(\hat{\alpha}_0 + \pi) < \Xi(\hat{\alpha} + \pi) \), \((\text{Cut})\) can be applied on (16) and (17). Hence,

\[
\mathcal{H}_{\hat{\alpha} + \pi}[X, \pi] \left|_{\Xi(\hat{\alpha}_0 + \pi) \Gamma^{(\pi,K)}} \right. \Gamma^{(\pi,K)} = \Gamma^{(\pi,K)}.
\]

(18)

Case 6: The last inference is \((\text{Ref}_{\sigma \tau})\). Thus

\[
\mathcal{H}_{\gamma}[X] \left|_{\xi,\mu_{\hat{\alpha}}} \Gamma, A(s) \right.,
\]

where \( \alpha_0 + 1, \tau < \alpha, A(s) \in \Pi_2(\tau), \sigma \in \mathcal{H}_{\gamma}, \exists z^{\tau}[A^\sigma(z) \wedge (\exists u : z)A(u)^{z,\tau}] \in \Gamma, \) and \( \sigma \in C(m(\tau), \tau) \cap m(\tau) \).

Here the induction hypothesis provides us with

\[
\mathcal{H}_{\hat{\alpha}_0 + \pi}[X, \pi] \left|_{\Xi(\hat{\alpha}_0 + \pi) \Gamma^{(\pi,K)}} \right. \Gamma^{(\pi,K)}, A(s)
\]

for all \( \pi \in M_{\hat{\alpha}} \subseteq M_{\hat{\alpha}_0} \). Since also \( \Xi(\hat{\alpha}_0 + \pi) + \tau < \Xi(\hat{\alpha} + \pi) \), because of \( \tau < \Xi(\gamma) < \Xi(\hat{\alpha}_0 + \pi) \), applying \((\text{Ref}_\sigma^\tau)\) gives the assertion. \( \square \)

Corollary 10.2 The passage from \( \mathcal{H}_{\gamma}[X] \left|_{\xi,\mu_{\hat{\alpha}}} \Gamma \right. \) to \( \mathcal{H}_{\hat{\alpha} + \pi}[X, \pi] \left|_{\Xi(\hat{\alpha}_0 + \pi) \Gamma^{(\pi,K)}} \right. \Gamma^{(\pi,K)} \) (for \( \pi \in M_{\hat{\alpha}} \)) only introduces inferences \((\text{Ref}_\sigma^\tau)\) such that \( \sigma < \hat{\alpha} \).

Proof. New instances of \((\text{Ref}_\sigma^\tau)\) were only introduced when we removed an instance of \((\text{Ref}_K)\) and those satisfied \( \sigma < \hat{\alpha} \). \( \square \)

Theorem 10.3 Suppose \( \mathfrak{A}(X; \gamma, \pi, \xi, \mu), NF(\gamma, \omega^{\mu^{\alpha}}) \), and \( \Gamma \subset \Sigma_1(\pi) \cup \Delta_\omega(\pi) \). Furthermore, assume that

\[
\mathcal{H}_{\gamma}[X] \left|_{\xi,\mu_{\hat{\alpha}}} \Gamma \right.
\]

and that all the inferences of the form \((\text{Ref}_\sigma^\tau)\) that appear in this derivation satisfy \( \sigma \leq \gamma \). Then, for \( \hat{\alpha} = \gamma + \omega^{\mu^{\alpha}} \),

\[
\mathcal{H}_{\hat{\alpha}}[X] \left|_{\xi(\hat{\alpha}),\mu_{\hat{\alpha}}} \Gamma \right.
\]

\(^{13}\text{This is exactly the place, where the removal of an instance of } (\text{Ref}_K) \text{ forces us to introduce an instance of } (\text{Ad}^{\hat{\alpha}_0}).\)
Proof. We proceed by main induction on $\mu$ and subsidiary induction on $\alpha$.

**Case 1:** The last inference is $\left(\text{Ref}_\pi^\gamma\right)$. Then

$$\mathcal{H}_\gamma[X] \models_{\eta/\mu} \Gamma, A(s),$$

where $\alpha_0 + 1, \pi < \alpha$, $A(s) \equiv \forall x^\pi \exists y^\pi G(x, y, s) \in \Pi_2(\pi)$, $\sigma, s \in \mathcal{H}_\gamma$, $\sigma \leq \gamma$, and

$$\exists x^\pi [A^\pi(z) \land (\exists u \in z) A(u)^{\langle x, y \rangle}] \in \Gamma,$$

and $\sigma \in C(m(\pi), \pi) \cap m(\pi)$. Applying Inversion, i.e. 7.1, we have, for all $t \in Term(\pi)$,

$$\mathcal{H}_\gamma[X, t] \models_{\eta/\mu} \exists y^\pi G(t, y, s) \tag{19}$$

For $t \in Term(\pi)$ and $t \gamma_\mu := \gamma + \omega^{\mu - \alpha_0} + [t]$, by 9.6(iv), it holds $\mathfrak{A}(X \cup \{t\}; \gamma_\mu, \pi, \sigma, \mu)$ and also $\gamma_\mu \in \mathcal{H}_\gamma[X, t]$. Therefore we can apply the subsidiary induction hypothesis to (19), so that with $\gamma_\mu := \gamma_\mu + \omega^{\mu - \alpha_0}$, for all $t \in Term(\pi)$,

$$\mathcal{H}_{\gamma_\mu}[X, t] \models_{\eta/\mu} \exists y^\pi G(t, y, s) \tag{20}$$

Set $\delta_t := \Psi_\pi^\sigma(\gamma_\mu + \omega^{\mu - \alpha_0})$, $\gamma^* := \gamma + \omega^{\mu - \alpha_0} + \pi$ and let $\eta := \Psi_\pi^\sigma(\gamma + \omega^{\mu - \alpha_0} + \pi)$. With the aid of the Bounding Lemma, 7.6, we then obtain from (20),

$$\mathcal{H}_{\gamma^*}[X, t] \models_{\eta/\mu} \exists y^\pi G(t, y, s) \tag{21}$$

for $t \in Term(\pi)$ satisfying $\delta_t \leq \eta$. Due to $\mathfrak{A}(X; \gamma, \pi, \sigma, \mu)$ and $NF(\gamma, \omega^{\mu - \alpha_0} + \pi)$, it follows $\sigma, \pi, \gamma + \omega^{\mu - \alpha_0} + \pi \in C(\gamma + \omega^{\mu - \alpha_0} + \pi, \pi)$. Also $\sigma \in C(m(\pi), \pi) \cap m(\pi)$. Thus $M^\pi$ is stationary in $\pi$. From this we gather that $\eta = \Psi_\pi^\sigma(\gamma + \omega^{\mu - \alpha_0} + \pi) \in M^\sigma \cap \pi$. Whence,

$$\models Ad^\pi(L_\eta). \tag{22}$$

Furthermore, one computes that if $t \in Term(\eta)$, then $\delta_t < \eta$. Therefore

$$\mathcal{H}_\alpha[X] \models_{\eta/\mu} \forall x^\pi \exists y^\pi G(x, y, s) \tag{23}$$

follows from (21). (23) in conjunction with $|s| < \Psi_\pi^0(\gamma) < \eta$ yields

$$\mathcal{H}_\alpha[X] \models_{\eta/\mu} \forall x^\pi \exists y^\pi G(x, y, s). \tag{24}$$

Since $\eta < \pi$,

$$\mathcal{H}_\alpha[X] \models_{\eta/\mu} \exists x^\pi [Ad^\pi(z) \land (\exists u \in z) A(u)^{\langle x, y \rangle}] = \Gamma \tag{25}$$

by (24) and (22). Finally, it remains to verify $\eta < \Psi_\pi^\gamma(\hat{\alpha})$. We have $\gamma + \omega^{\mu - \alpha_0} + \pi < \gamma + \omega^{\mu - \alpha} = \hat{\alpha}$ as $\alpha_0 + 1, \pi < \alpha$ and $\pi \leq \mu$. From $NF(\gamma, \omega^{\mu - \alpha})$ it follows $\gamma, \mu, \pi, \sigma \in C(\hat{\alpha}, \Psi_\pi^\gamma(\hat{\alpha}))$, so $\gamma + \omega^{\mu - \alpha_0} + \pi \in C(\hat{\alpha}, \Psi_\pi^\gamma(\hat{\alpha})) \cap \hat{\alpha}$, hence $\eta < \Psi_\pi^\gamma(\hat{\alpha})$. Therefore,

$$\mathcal{H}_\alpha[X] \models_{\Psi_\pi^\gamma(\hat{\alpha})} \Gamma \tag{26}$$
by (25).

**Case 2:** The last inference is \((Ref^\gamma_\kappa)\) for some \(\kappa < \pi\). Then

\[ \mathcal{H}_{\gamma}[x] \models^\alpha \Gamma, A(s), \]

where \(\alpha_0 + 1, \kappa < \alpha\), \(A(s) \equiv \forall x \exists y G(x, y, s) \in \Pi_2(\kappa), \sigma \in \mathcal{H}_\gamma, \exists z^x [Ad^\gamma(z) \land (\exists u \in z) A(u)^{1..\kappa}] \in \Gamma\), and \(\sigma \in C(m(\kappa), \kappa) \cap m(\kappa)\). Therefore \(A(s) \in \Delta_0(\pi)\) and unlike in the previous Case we can apply the subsidiary induction hypothesis directly, yielding

\[ \mathcal{H}_{a_0}[x] \models^{\xi(a_0)}_{\xi(a_0)} \Gamma, A(s). \]

Due to \(\Psi_{\xi}(a_0) + \kappa < \Psi_{\xi}(\hat{a})\), the same inference \((Ref^\gamma_\kappa)\) leads to

\[ \mathcal{H}_{\hat{a}}[x] \models^{\xi(\hat{a})}_{\xi(\hat{a})} \Gamma. \]

**Case 3:** The last inference is \((\lor)\) with principal formula \(C \equiv \lor(C_i)_{i \in J} \in \Gamma\). Then

\[ \mathcal{H}_{\gamma}[x] \models^\alpha \Gamma, C_{i_0}, \]

for some \(\alpha_0 < \alpha\) and \(i_0 \in J \upharpoonright \alpha\). By subsidiary induction hypothesis, we obtain

\[ \mathcal{H}_{a_0}[x] \models^{\xi(a_0)}_{\xi(a_0)} \Gamma, C_{i_0}, \]

whence,

\[ \mathcal{H}_{\hat{a}}[x] \models^{\xi(\hat{a})}_{\xi(\hat{a})} \Gamma, C \ (\ = \Gamma) \]

via \((\lor)\).

**Case 4:** The last inference is \((\land)\) with principal formula \(C \equiv \land(C_i)_{i \in J} \in \Gamma\). This means

\[ \mathcal{H}_{\gamma}[x, i] \models^\alpha \Gamma, C_i, \]

and \( \mid i \mid \leq \alpha_i < \alpha \) for \(i \in J\). The conditions on \(\Gamma\) force \(C \in \Delta_0(\pi)\). Due to \(k(C) \subset \mathcal{H}_{\gamma}[x](0) \cap \pi \subseteq C(\gamma + 1, \Psi_{\pi}^\gamma(\gamma + 1)) \cap \pi\), we must have \(\mid i \mid < \Psi_{\pi}^\gamma(\gamma + 1)\) for all \(i \in J\). Let \(\gamma_i := \gamma + \omega^{\mu \alpha_i} \mid i \mid\). From \(NF(\gamma_i, \omega^{\mu \alpha_i})\) it follows \(\mathfrak{A}(\mathfrak{X} \cup \{\gamma_i\}; \gamma_i, \pi, \xi, \mu)\) for all \(i \in J\). The subsidiary induction hypothesis then yields

\[ \mathcal{H}_{\delta_i}[x, i] \models^{\xi(\delta_i)}_{\xi(\delta_i)} \Gamma, C_i, \]

for all \(i \in J\), where \(\delta_i := \gamma_i + \omega^{\mu \alpha_i} \in C(\hat{a}, \Psi_{\pi}^\xi(\hat{a})). \ \mid i \mid \leq \alpha_i < \alpha \) implies \(\delta_i < \hat{a}\); thus \(\Psi_{\pi}^\xi(\delta_i) < \Psi_{\pi}^\xi(\hat{a})\). So, using \((\land)\), we conclude

\[ \mathcal{H}_{\hat{a}}[x] \models^{\xi(\hat{a})}_{\xi(\hat{a})} \Gamma. \]
Case 5: The last inference is \((Cut)\). Then there exist \(\alpha_0 < \alpha\) and an \(RS(\mathcal{K})\)-formula \(A\) with \(rk(A) < \mu\), so that

\[
\mathcal{H}_\gamma[X] \models_{\frac{\alpha_0}{\mu}} \Gamma, A
\]  
(26)

and

\[
\mathcal{H}_\gamma[X] \models_{\frac{\alpha_0}{\mu}} \Gamma, \neg A.
\]  
(27)

Subcase 5.1: Suppose \(\mu = \kappa + 1\). For \(\kappa := \Xi(\hat{\alpha}_0)\) one obtains, by applying 10.1 to (26) and (27),

\[
\mathcal{H}_{\hat{\alpha}_0 + \kappa}[X] \models_{\Xi(\hat{\alpha}_0 + \kappa)}^{\Xi(\hat{\alpha}_0 + \kappa)} \Gamma, A^{(\kappa, \mathcal{K})}
\]  
and

\[
\mathcal{H}_{\hat{\alpha}_0 + \kappa}[X] \models_{\Xi(\hat{\alpha}_0 + \kappa)}^{\Xi(\hat{\alpha}_0 + \kappa)} \Gamma, \neg A^{(\kappa, \mathcal{K})},
\]

recalling \(\Gamma^{(\kappa, \mathcal{K})} = \Gamma\) (since \(\pi < \kappa\)) and \(\kappa = \Xi(\hat{\alpha}_0) \in \mathcal{H}_{\hat{\alpha}_0 + \kappa}[X]\). Whence,

\[
\mathcal{H}_{\gamma'}[X] \models_{\Xi(\hat{\alpha}_0 + \kappa)}^{\Xi(\hat{\alpha}_0 + \kappa) + 1} \Gamma
\]  
(28)

by means of \((Cut)\), where \(\gamma' := \gamma + \omega^{K \cdot \alpha_0} \cdot 2\).

Since we have lowered the cut rank from \(\mu = \kappa + 1\) to \(\Xi(\hat{\alpha}_0 + \kappa) < \kappa\), the main induction hypothesis can be applied to (28); hence

\[
\mathcal{H}_\eta[X] \models_{\Psi^\xi(\eta)} \Gamma,
\]

where \(\eta := \gamma' + \omega^{\Xi(\hat{\alpha}_0 + \kappa)^2 + \Xi(\hat{\alpha}_0 + \kappa)} = \gamma + \omega^{K \cdot \alpha_0} + \omega^{K \cdot \alpha_0} + \omega^{\Xi(\hat{\alpha}_0 + \kappa)^2 + \Xi(\hat{\alpha}_0 + \kappa)}\). Since \(\eta < \hat{\alpha}\) und \(\Psi^\xi(\eta) < \Psi^\xi(\hat{\alpha})\), we deduce

\[
\mathcal{H}_{\hat{\alpha}}[X] \models_{\Psi^\xi(\hat{\alpha})} \Gamma.
\]

In the sequel, we shall assume \(\mu < \kappa\).

Subcase 5.2: \(rk(A) < \pi\).

Then \(rk(A) < \Psi^\xi(\hat{\alpha}_0)\) and \(A \in \Delta_0(\pi)\), hence \(\neg A \in \Delta_0(\pi)\). Therefore, applying the subsidiary induction hypothesis to (26) and (27),

\[
\mathcal{H}_{\hat{\alpha}_0}[X] \models_{\Psi^\xi(\hat{\alpha}_0)} \Gamma, A \quad \text{and} \quad \mathcal{H}_{\hat{\alpha}_0}[X] \models_{\Psi^\xi(\hat{\alpha}_0)} \Gamma, \neg A;
\]

whence

\[
\mathcal{H}_{\hat{\alpha}}[X] \models_{\Psi^\xi(\hat{\alpha})} \Gamma
\]

by means of \((Cut)\) since \(\Psi^\xi(\hat{\alpha}_0) < \Psi^\xi(\hat{\alpha})\).
Subcase 5.3: \(rk(A) > \pi\) and \(rk(A) \notin \text{Reg}\).

We can select \(\sigma \in H_{\gamma}[X]\) so that

\[\Omega_{\sigma} \leq rk(A) < \Omega_{\sigma+1}.\]

Set \(\tau := \Omega_{\sigma+1}\). Then \(\tau \leq \mu\), \(A(X;\gamma,\tau,0,\mu)\), and \(\Gamma \cup \{A,\neg A\} \subset \Delta_0(\tau)\). Using the subsidiary induction hypothesis we get

\[H_{\hat{\alpha}_0}[X] \frac{\Psi^0(\hat{\alpha}_0)}{\Psi^0(\hat{\alpha}_0)} \Gamma, A\]

and

\[H_{\hat{\alpha}_0}[X] \frac{\Psi^0(\hat{\alpha}_0)}{\Psi^0(\hat{\alpha}_0)} \Gamma, \neg A,\]

whence,

\[H_{\hat{\alpha}_0}[X] \frac{\Psi^0(\hat{\alpha}_0)+1}{\Psi^0(\hat{\alpha}_0)} \Gamma,\]

as \(rk(A) < \Psi^0(\hat{\alpha}_0)\). Employing predicative cut elimination, 7.4, we obtain

\[H_{\hat{\alpha}_0}[X] \frac{\varphi_{\eta}(\eta+1)}{\varphi_{\eta}(\eta)} \Gamma\]

with \(\eta := \Psi^0(\hat{\alpha}_0)\) and \(\nu := \Omega_{\sigma}\). Note that \(\pi \leq \nu\). Furthermore, \(A(X;\hat{\alpha}_0,\pi,\xi,\nu)\) and \(\text{NF}(\hat{\alpha}_0,\omega^{\nu-\varphi_{\eta}(\eta+1)})\). Also \(\nu < \mu\). Therefore, letting \(\zeta := \hat{\alpha}_0 + \omega^{\nu-\varphi_{\eta}(\eta+1)}\), we can use the main induction hypothesis on (30) to conclude

\[H_{\hat{\alpha}}[X] \frac{\Psi^0(\hat{\alpha})}{\Psi^0(\hat{\alpha})} \Gamma.\]

Noting that \(\zeta < \hat{\alpha}\) and \(\Psi^0(\zeta) < \Psi^0(\hat{\alpha})\), this implies

\[H_{\hat{\alpha}}[X] \frac{\Psi^0(\hat{\alpha})}{\Psi^0(\hat{\alpha})} \Gamma.\]

Subcase 5.4: \(rk(A) \geq \pi\) and \(rk(A) \in \text{Reg}\).

Let \(\tau := rk(A)\). Then either \(A\) or \(\neg A\) is of the form \(\exists x^* F(x)\) with \(F(\mathbb{L}_0) \in \Delta_0(\tau)\).

If \(\alpha_0 < \tau\), then \(\neg A\) never gets used as a principal formula of an inference in \(H_{\gamma}[X]\) \(\frac{l_0}{l_1} \Gamma, \neg A,\)

and therefore, \(H_{\gamma}[X] \frac{l_0}{l_1} \Gamma\). Thus, by subsidiary induction hypothesis , \(H_{\hat{\alpha}_0}[X] \frac{\Psi^0(\hat{\alpha}_0)}{\Psi^0(\hat{\alpha}_0)} \Gamma,\)

whence \(H_{\hat{\alpha}}[X] \frac{\Psi^0(\hat{\alpha})}{\Psi^0(\hat{\alpha})} \Gamma\) since \(\Psi^0(\hat{\alpha}_0) < \Psi^0(\hat{\alpha})\).

Now assume \(\tau \leq \alpha_0\). Observe that \(A(X;\gamma,\tau,0,\mu)\) and \(\Gamma, A \subset \Delta_0(\tau) \cup \Sigma_1(\tau)\). Applying the subsidiary induction hypothesis to (26) and using the Bounding Lemma 7.6, we obtain

\[H_{\hat{\alpha}_0}[X] \frac{\Psi^0(\hat{\alpha}_0)}{\Psi^0(\hat{\alpha}_0)} \Gamma, A(\Psi^0(\hat{\alpha}_0)).\]
From (27), by employing 6.15(iii) and \( \Psi^0_\tau(\hat{\alpha}_0) \in \mathcal{H}_{\hat{\alpha}_0}[x] \), we get
\[
\mathcal{H}_{\hat{\alpha}_0}[x] \vdash_{\text{G}}^{\alpha_0} \Gamma, \neg A(\Psi^0_\tau(\hat{\alpha}_0), \tau).
\] (32)
Since \( \mathfrak{A}(x; \hat{\alpha}_0, \tau, 0, \mu) \) and \( NF(\hat{\alpha}_0, \omega^{\mu \cdot \alpha_0}) \), the subsidiary induction hypothesis can be used on (32), furnishing
\[
\mathcal{H}_{\hat{\alpha}_0}[x] \vdash_{\text{G}}^{\alpha_0} \Gamma, \neg A(\Psi^0_\tau(\hat{\alpha}_0), \tau),
\] (33)
where \( \delta := \hat{\alpha}_0 + \omega^{\mu \cdot \alpha_0} \). Using (Cut) on (31) and (32), we obtain
\[
\mathcal{H}_{\delta}[x] \vdash_{\text{G}}^{\delta + 1} \Gamma,
\] (34)
If \( \tau = \pi \), then (34) implies
\[
\mathcal{H}_{\hat{\alpha}}[x] \vdash_{\text{G}}^{\hat{\alpha}} \Gamma,
\]
noting that \( \Psi^0_\pi(\delta) < \Psi^\xi_\pi(\hat{\alpha}) \).

From now on, let \( \pi < \tau \). Again, we can select \( \sigma \in \mathcal{H}_\gamma[x] \) so that \( \Omega_\sigma \leq \Psi^0_\tau(\delta) < \Omega_{\sigma + 1} \leq \tau \). Through the use of predicative cut elimination, (34) yields
\[
\mathcal{H}_{\delta}[x] \vdash_{\text{G}}^{\delta + 1} \Gamma,
\] (35)
where we put \( \eta := \varphi \Psi^0_\tau(\delta)(\Psi^0_\tau(\delta) + 1) \) and \( \nu := \Omega_\sigma \). Set \( \gamma' := \delta + \omega^{\mu \cdot \alpha_0} \). Then \( \delta < \gamma' \) and \( NF(\gamma', \omega^{\nu \cdot \eta}) \) since \( \nu < \mu \) as well as \( \eta < \nu \leq \alpha_0 \). Since \( \pi < \tau \) and \( \pi \in C(\gamma + 1, \Psi^0_\tau(\gamma + 1)) \), we get \( \pi < \Psi^0_\tau(\delta) \); thence \( \pi \leq \nu \). Note that \( \mathfrak{A}(x; \gamma', \pi, \xi, \nu) \). Since \( \nu < \mu \), we can use the main induction hypothesis on (35), so that with \( \rho := \gamma' + \omega^{\nu \cdot \eta} \),
\[
\mathcal{H}_{\rho}[x] \vdash_{\text{G}}^{\rho} \Gamma.
\] (36)
One readily verifies \( \rho < \hat{\alpha} \) and \( \rho \in C(\hat{\alpha}, \Psi^\xi_\pi(\hat{\alpha})) \). Therefore, by (36),
\[
\mathcal{H}_{\hat{\alpha}}[x] \vdash_{\text{G}}^{\hat{\alpha}} \Gamma.
\]
\[\Box\]

**Theorem 10.4** Let \( \rho_0 := 1 \) and \( \rho_{n + 1} := K^{\rho_n} \).

(i) \(^{14} \) If \( A \) is a \( \Pi_3 \)-sentence of \( \mathcal{L} \) and \( KP + \Pi_3-\text{Ref} \vdash A \), then there is an \( n < \omega \) such that, for all \( \pi \in M^{\rho_n} \),
\[
\mathcal{H}_{\rho_n + 1} \vdash_{\text{G}}^{\rho_n + \pi} A^{1-\pi}.
\]

\(^{14} \) The meaning of (i) can be greatly enhanced by developing the collapsing functions on the basis of a \( \Pi_3 \)-reflecting ordinal, say \( \kappa_0 \). It will then be possible, given a proof of a \( \Pi_3 \)-sentence in \( KP + \Pi_3-\text{Ref} \), to determine a \( \kappa_0 \)-recursively stationary set of reflection points; thereby providing an Herbrand analysis for provable \( \Pi_3 \)-sentences of \( KP + \Pi_3-\text{Ref} \).
(ii) The property of being an admissible set above $\omega$ can be expressed by a $\Delta_0$-formula. (For definiteness, let this be the formula displayed in Aczel and Richter [1974].) If $B$ is a $\Sigma_1$-sentence and

$$KP + \Pi_3\text{-}Ref \vdash \forall x[\text{Ad}(x) \rightarrow B^x],$$

then there is a $k < \omega$ such that

$$H_{\rho_k} \frac{\Psi^0_\Omega_1(\rho_k)}{\Phi^0_\Omega_1(\rho_k)} B^x.$$ 

Proof. (i) According to Theorem 8.11, there is an $m < \omega$ satisfying

$$H_0 \frac{\xi_\omega^m}{\xi_\omega} A^\xi.$$ 

Applying Corollary 7.5 several times, we get

$$H_0 \frac{\xi_{\omega+1}^{m+2}}{\xi_\omega} A^\xi.$$ 

Letting $\gamma := \rho_{m+4}$, we have $NF(\gamma, K^{\rho_{m+2}})$ and $\mathfrak{B}(\emptyset; \gamma)$. So we can apply Theorem 10.1 to get

$$H_{\rho_n + \pi} \frac{\Xi(\rho_n + \pi_0)}{\Xi(\rho_n + \pi)} A^\xi$$

for all $\pi \in M^{\rho_n}$, provided that $n > m + 4$.

(ii): By the same procedure as in (i), we obtain an $n < \omega$ satisfying

$$H_{\rho_n + \pi_0} \frac{\Xi(\rho_n + \pi_0)}{\Xi(\rho_n + \pi_0)} \neg \text{Ad}(L_{\Omega_1}), B^{L_{\Omega_1}},$$

where $\pi_0 := \Xi(\rho_n)$. Since

$$H_0 \frac{\Omega_\omega}{\Omega_\omega} \text{Ad}(L_{\Omega_1}),$$

it follows

$$H_{\rho_n + \pi_0} \frac{\Xi(\rho_n + \pi_0) + 1}{\Xi(\rho_n + \pi_0)} B^{L_{\Omega_1}}.$$ (37)

Letting $\gamma := \rho_{n+2}$, $\alpha := \Xi(\rho_n + \pi_0) + 1$ and $\mu := \Xi(\rho_n + \pi_0)$, we have $\gamma, \alpha \in H_\gamma, NF(\gamma, \omega^\mu), \text{ and } \mathfrak{B}(\emptyset; \gamma, \Omega_1, 0, \mu)$. Also, by Corollary 10.2, $\sigma < \gamma$ holds for all inferences $(Ref^*_\gamma)$ appearing in (37). Therefore, by Theorem 10.3, we obtain

$$H_{\delta} \frac{\delta}{\delta} B^{L_{\Omega_1}}$$

for $\alpha := \gamma + \omega^\alpha$ and $\delta := \Xi_\Omega_1(\alpha)$. Using predicative cut elimination, Theorem 7.4, this leads to

$$H_{\delta} \frac{\delta}{\delta} B^{L_{\Omega_1}}.$$
For \( k := n + 3 \), one easily verifies \( \dot{\alpha} < \rho_k \) and \( \varphi \delta \delta < \Psi_{\Omega_1}^0 (\rho_k) \). Hence, \[ H_{\rho_k} \models_{\Omega_1} B_{\Omega_1}^\mathcal{I}. \]

\[ \square \]

**Corollary 10.5**

\[ |KP + \Pi_3 - \text{Ref}| \leq \Psi_{\Omega_1}^0 (\varepsilon_{\kappa + 1}). \]

(\( |KP + \Pi_3 - \text{Ref}| \) denotes the proof-theoretic ordinal of \( KP + \Pi_3 - \text{Ref} \).)

\[ \square \]

**Remark 10.6** The bound given in 10.5 is indeed sharp. But we will not give a proof for that in this paper.

### 11 Conclusions

A notation system which is suitable for an ordinal analysis of \( KP + \Pi_{n+2} - \text{reflection} \) \((n > 1)\) can be derived from collapsing functions based on \( \Pi_m^1 \) indescribable cardinals, where \( 0 < m \leq n \). Here one employs the thinning-operation

\[ M_{k+1}(X) = \{ \pi \in X : \pi \text{ is } \Pi_k^1 \text{ indescribable on } X \}, \]

where \( \pi \) is \( \Pi_k^1 \text{ indescribable on } X \) if for all \( U_1, \ldots , U_i \subseteq V_\pi \) and every \( \Pi_k^1 \) sentence \( F \), whenever \( \langle V_\pi, \varepsilon, U_1, \ldots , U_i \rangle \models F \), then there exists a \( \rho \in X \cap \pi \) such that

\[ \langle V_\rho, \varepsilon, U_1 \cap V_\rho, \ldots , U_i \cap V_\rho \rangle \models F. \]

As a matter of fact, if \( \kappa \) is \( \Pi_{k+1}^1 \) indescribable and \( X \subseteq \kappa \) is stationary in \( \kappa \) then \( M_k(X) \) is also stationary in \( \kappa \). So, analogously to Definition 4.8, given a \( \Pi_{n+1}^1 \) indescribable cardinal \( \mathcal{R} \), one defines a hierarchy of subsets \( M_n^{\mathcal{R}, \alpha} \) of \( \mathcal{R} \) (using \( M_n \) in place of \( M \)) which induces a collapsing function \( \Xi_{n+1}^{\mathcal{R}, \alpha} \) by letting

\[ \Xi_{n+1}^{\mathcal{R}, \alpha} (\alpha) = \text{least } \nu \in M_n^{\mathcal{R}, \alpha}. \]

We have already pointed out that the use of large cardinals in the development of collapsing functions is merely an exaggeration that simplifies proofs, but could be avoided by employing their recursively large analogues (see Rathjen [1993c]). However, regarding a consistency proof for \( KP + \Pi_3 - \text{Ref} \) (or, more generally, \( KP + \Pi_{n+2} - \text{reflection} \)) we would like to have some kind of constructive justification for the well-foundedness of \( \langle T(K), \prec \rangle \). First, let us delimit in which metatheory such a consistency proof can be accomplished. A rough estimate would be first order arithmetic augmented by the scheme of transfinite
induction along the ordering of $\mathcal{T}(\mathcal{K})$. To see this, note that $\langle \mathcal{T}(\mathcal{K}), < \rangle$ is primitive recursive (after some coding) and that recursive $RS(\mathcal{K})$ derivations suffice for the results of Sections 6 through 10. Now, recursive $RS(\mathcal{K})$ derivations can be formalized in first order arithmetic (see Schwichtenberg [1977]). But we can do even better. For a particular arithmetic theorem of $KP + \Pi_3$–Ref, say $A$, an $n$ can be determined (depending on the proof of $A$) such that there is a cut free controlled recursive derivation of $A$ that utilizes solely ordinals from $\mathcal{T}_n(\mathcal{K}) = C(\rho_n, 0)$, where $\rho_0 = 1$ and $\rho_{k+1} = K^{\rho_k}$. So the upshot is that any arithmetic theorem of $KP + \Pi_3$–Ref is provable in first order arithmetic augmented by the schemes of transfinite induction for all the orderings $<_n$ arising by restricting $<$ to $\mathcal{T}_n(\mathcal{K})$.

Finally, by results of Friedman and Sheard [1993], Theorem 4.5, the consistency (even the 1–consistency) of the latter theory is provable in primitive recursive arithmetic plus a scheme expressing that there is no infinite primitive recursive\footnote{This holds even for elementary functions.} descending sequence in the notation system determined by $C(\varepsilon_{\kappa+1}, 0) \subseteq \mathcal{T}(\mathcal{K})$.

By now we have managed to reduce the consistency of $KP + \Pi_3$–Ref to the principle (say $FT(<)$) that every concrete strictly decreasing sequence of members of $C(\varepsilon_{\kappa+1}, 0)$ terminates in a finite number of steps. How can we assure ourselves of the validity of $FT(<)$? Takeuti (see [1985],[1987]) refers to such proofs as accessibility proofs. In his work he has given accessibility proofs for the ordinal diagrams that he used for his consistency proof of $\Pi_1^1$ comprehension. As to the methods allowed for such proofs, Takeuti delimits a kind of concrete constructivity. In the words of Takeuti [1987, p.96]: “We believe that our standpoint is a natural extension of Hilbert’s finitist standpoint, similar to that introduced by Gentzen, and we call it the Hilbert–Gentzen finitist standpoint.”

However, Takeuti does not formally lay bare what he counts as acceptable from his stance, this especially applies to what he calls (using Hilbert’s jargon) “performing a Gedankenexperiment”. Of course, ultimately, justification can only come about by halting at some intuitively convincing grounds, and no explanation can substitute for each individuals understanding. Incidentally, the author convinced himself of the accessibility of $\mathcal{T}(\mathcal{K})$ along the lines delineated by Takeuti.

Nonetheless, it might be desirable to obtain different accessibility proofs based on different styles of constructivity. There are prospects that extensions of Martin–Löf’s intuitionistic type theory with higher universes can provide a uniform setting for consistency proofs. Palmgren (in [1990]) has outlined an intuitionistic theory of types with transfinite universes that provides a means of understanding constructive Mahlo numbers.

References


APT K.R. and MAREK W. [1974], Second order arithmetic and related topics, Annals
of Mathematical Logic 6, 117–229.
BUCHHOLZ W. and SCHÜTTE K., [1988], Proof theory of impredicative subsystems of analysis, Bibliopolis, Naples.
JÄGER G. [1986], Theories for admissible sets: A unifying approach to proof theory, Bibliopolis, Naples.
JÄGER G. and POHLERS W. [1982], Eine beweistheoretische Untersuchung von $(\Delta^1_2 - CA) + BI$ und verwandter Systeme, Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathematisch–Naturwissenschaftliche Klasse.
PALMGREN E. [1990], An intuitionistic theory of transfinite types, preliminary draft.
RATHJEN M. [1990], Ordinal notations based on a weakly Mahlo cardinal, Arch. Math. Logic 29, 249–263.
RATHJEN M. [1991b], An interpretation of $KPM$ in second order arithmetic and a characterization of $1$–section (superjump), Preprint, Universität Münster.
RATHJEN M. [1993a], How to develope proof–theoretic ordinal functions on the basis of admissible ordinals, to appear in: Mathematical Logic Quarterly (formerly Zeitschrift für Mathematische Logik und Grundlagen der Mathematik.)
RATHJEN M. [1993b], Admissible proof theory and beyond, to appear in: 9th Interna-


SCHÜTTE K. [1993] Zur Beweistheorie von KP + \( \Pi_3 \)-Rel, type–written manuscript.


TAKEUTI G. and YASUGI M. [1973], The ordinals of the systems of second order arithmetic with the provably \( \Delta_1^1 \)-comprehension and the \( \Delta_2^1 \)-comprehension axiom respectively, Japan J. Math. 41, 1–67.