PROOF THEORY OF REFLECTION

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Abstract

The paper contains proof–theoretic investigations on extensions of Kripke–Platek set theory, *KP*, which accommodate first order reflection. Ordinal analyses for such theories are obtained by devising cut elimination procedures for infinitary calculi of ramified set theory with Π_n reflection rules. This leads to consistency proofs for the theories $KP + \Pi_n$*–reflection* using a small amount of arithmetic (PRA) and the well–foundedness of a certain ordinal notation system with respect to primitive recursive descending sequences.

Regarding future work, we intend to avail ourselves of these new cut elimination techniques to attain an ordinal analysis of Π^1_2 comprehension by approaching Π^1_2 comprehension through transfinite levels of reflection.

1 Introduction

Since 1967, when Takeuti obtained a consistency proof for the subsystem of analysis based on impredicative Π^1_1 comprehension, great progress[1] has been made in the proof theory of impredicative systems, culminating in the *"Admissible Proof Theory"* originating with Jäger and Pohlers in the early 80's. In essence, admissible proof theory is a gathering of cut elimination techniques for infinitary calculi of ramified set theory with Σ and/or Π_2 reflection rules[2] that lends itself to ordinal analyses of theories of the form $KP +$ *"there are x many admissibles"* or $KP +$ *"there are many admissibles"*. By way of illustration, the subsystem of analysis with Δ^1_2 comprehension and Bar induction can be couched in such terms, for it is naturally interpretable in the set theory $KPi := KP + \forall y \exists z (y \in z \wedge z \textit{ is admissible})$ (cf. Jäger and Pohlers [1982]). Nonetheless, the advanced techniques of admissible proof theory are way too weak for dealing with significantly stronger theories like Π^1_2 analysis, let alone full analysis. An ordinal analysis of Π^1_2 comprehension would inherently involve one for all the theories $KP + \Pi_n$*reflection*, and, therefore, a first step to be taken towards this end consists in devising ordinal notation systems that give rise to cut elimination procedures for infinitary calculi with Π_n reflection rules.

In this paper we focus on the ordinal analysis of Π_3 reflection. This means no genuine loss of generality, as the removal of Π_3 reflection rules in derivations already exhibits the pattern of cut elimination that applies for arbitrary Π_n reflection rules as well.

As regards the advance achieved in this paper, it should be pointed out that we cherish much higher expectations than just moving a tiny step towards Π^1_2 comprehension. The idea is that Π^1_2 comprehension can be fathomed by going through transfinite levels of

[1]See Takeuti and Yasugi [1973], Schütte [1977], Buchholz et al. [1981], Jäger and Pohlers [1982], Pohlers [1982], Jäger [1986], Takeuti [1987], Pohlers [1987], Pohlers [1991].

[2]Recall that the salient feature of admissible sets is that they are models of Δ_0 collection and that Δ_0 collection is equivalent to Σ reflection on the basis of the other axioms of *KP* (see Barwise [1975]). Furthermore, admissible sets of the form L_α also satisfy Π_2 reflection.

reflection; and thus an ordinal analysis for it should be attainable via an, admittedly, considerable extension of the machinery laid out in this paper.

The paper is organized as follows: Section 2 introduces set–theoretic reflection and situates it with regard to non-monotone inductive definitions, subsystems of analysis with β–model reflection and Π^1_2 comprehension. Section 3 provides a formalization of KP as sequent calculus. In Section 4, so–called collapsing functions are developed which give rise to a strong ordinal notation system $\mathcal{T}(\mathcal{K})$. $\mathcal{T}(\mathcal{K})$ is introduced in Section 5. In Section 6, we define an infinitary calculus $RS(\mathcal{K})$ with Π_3 and Π_2 relection. Here we draw on Buchholz's [1993] approach to local predicativity, in particular, the notion of operator controlled derivations. Section 7 deals with the elimination of uncritical cuts in $RS(\mathcal{K})$ derivations, i.e. cuts whose cut formulae have not been introduced by reflection rules. Section 8 is devoted to interpreting $KP + \Pi_3$–*Ref* in $RS(\mathcal{K})$. Section 9 and 10 are concerned with the removal of critical cuts in $RS(\mathcal{K})$ derivation. Finally, in Section 11, we indicate how ordinal analyses for arbitrary Π_n reflections can be obtained. This Section also contains some remarks on consistency proofs.

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2 Set–theoretic reflection and related principles

This Section provides some background information and contains (almost) no proofs. Its theorems will not be used in later Sections.

We shall consider set-theoretic reflection on the basis of *Kripke–Platek set theory*, KP, which arises from ZF by omitting[4] the power set axiom and restricting the axiom schemes of comprehension and collection to absolute predicates, i.e. Δ_0 predicates.

Definition 2.1 A set–theoretic formula is said to be Π_n (respectively Σ_n) if it consists of a string of n alternating quantifiers beginning with a universal one (respectively existential one), followed by a Δ_0 formula. By Π_n *reflection* we mean the scheme

$$F \rightarrow \exists z[Tran(z) \wedge z \neq \emptyset \wedge F^z],$$

[3]Meanwhile, Kurt Schütte has given another presentation of the ordinal analysis of $KP + \Pi_3$–*Ref* using the calculus of positive and negative forms (cf. Schütte [1993]).

[4]This contrasts with Barwise [1975], where the infinity axiom is not included in KP.

where F is Π_n and $Tran(z)$ expresses that z is a transitive set; F^z denotes the formula that arises from F by restricting the unbounded quantifiers to z, i.e. $\forall x$ gets replaced with $(\forall x \in z)$ and $\exists x$ with $(\exists x \in z)$.

An ordinal $\alpha > 0$ is said to be Π_n*–reflecting* if $L_\alpha \models \Pi_n$*reflection.*

Σ_n reflection and Σ_n–reflecting are defined analogously.

Note that if κ is Π_n–reflecting and $n \geq 2$, then κ must be a limit ordinal $> \omega$. Therefore L_κ is a model of all the axioms of KP other than Δ_0 collection. But Δ_0 collection issues from Π_n reflection, and hence $L_\kappa \models KP + \Pi_n$*reflection*.

Π_n–reflecting ordinals have interesting points of contact with non–monotone inductive definitions.

Definition 2.2 A function Γ from the power set of $\mathbb{N}$ into itself is called an operator on $\mathbb{N}$. Γ determines a transfinite sequence $\langle \Gamma^\xi : \xi \in ON \rangle$ of subsets of $\mathbb{N}$,

$$\Gamma^\lambda = \Gamma^{<\lambda} \cup \Gamma(\Gamma^{<\lambda}),$$

where $\Gamma^{<\lambda} = \bigcup_{\xi<\lambda} \Gamma^\xi$.

The *closure ordinal* $|\Gamma|$ of Γ is the least ordinal ρ such that $\Gamma^{\rho+1} = \Gamma^\rho$. Γ is said to be Π^0_k when there is an arithmetic Π^0_k formula $F(U, u)$ with second order variable U such that, for all $X \subseteq \mathbb{N}$,

$$\Gamma(X) = \{n \in \mathbb{N} : F(X, n)\}.$$

Let $\mid \Pi^0_k \mid := sup\{\mid \Gamma \mid : \Gamma \text{ is } \Pi^0_k\}$.

Owing to Aczel and Richter [1974], we have the following characterization.

Theorem 2.3 *For $k > 0$,*

$$|\Pi^0_k| = \text{ first } \Pi_{k+1}\text{–reflecting ordinal.}$$

Several notions of recursively large ordinals are modelled upon notions of large cardinals. This is especially true of notions like "recursively inaccessible ordinal" and "recursively Mahlo ordinal". It turns out that the least Π_3–reflecting ordinal is greater than the least recursively Mahlo ordinal, indeed much greater than any transfinite iteration of recursive "Mahloness" from below. For instance, every Π_3–reflecting ordinal κ is recursively κ–Mahlo.

Definition 2.4 An ordinal κ is *recursively Mahlo* if for every κ–recursive function $f : \kappa \longrightarrow \kappa$ there exists an admissible ordinal $\rho < \kappa$ that is closed under f.

A recursively Mahlo ordinal κ is *recursively α–Mahlo* if for every κ–recursive function $f : \kappa \longrightarrow \kappa$ there exists an admissible ordinal $\rho < \kappa$ closed under f such that ρ is recursively β–Mahlo for all $\beta < \alpha$.

Regarding a notion of large cardinal to which Π_3–reflecting ordinals provide the recursive counterparts, Aczel and Richter [1974] have convincingly argued that this should be the weakly compact (or Π^1_1 indescribable) cardinals. By the same token, for $n > 1$, Π_{n+2}–reflecting ordinals should be regarded as the recursive analogues of Π^1_n indescribable cardinals.

Since subsystems of analysis appear to be the most common measure for the calibration of proof–theoretic strength of theories, we shall also give a characterization of $KP + \Pi_n$*reflection* (for $n > 2$) in terms of subsystems of analysis. However, Π_n*reflection* does not simply translate into familiar levels of comprehension of the projective hierarchy. In proof–theoretic strength, the theories $KP + \Pi_n$*reflection* ($n > 2$) are strictly between Δ^1_2 comprehension plus Bar–induction and Π^1_2 comprehension. It turns out that set–theoretic reflection by transitive sets is related to β–model reflection.

Via coding, any set of natural numbers X gives rise to a countable collection of subsets of $\mathbb{N}$, $\{(X)_k : k\in\mathbb{N}\}$, where $(X)_k = \{m : 2^k 3^m \in X\}$. The structure

$$\mathcal{B}_X = \langle \mathbb{N}, \{(X)_k : k\in\mathbb{N}\}, 0, 1, +, \cdot, =, \in\rangle$$

(where the first order part is standard) is a *β–model* if, for any Π^1_1 sentence A with parameters from $\mathcal{B}_X$, A holds in $\mathcal{B}_X$ iff A is true (or, equivalently, the notion of well–foundedness is absolute with regard to $\mathcal{B}_X$). We shall refer to $\mathcal{B}_X$ as the the model coded by X. The notion of *countably coded β–model* can be formalized in analysis. Hereditarily countable sets can be identified with certain well–founded trees on $\mathbb{N}$ and thus can be modelled in second order arithmetic (see Apt and Marek [1974]). Let ACA denote the subsystem of second order arithmetic with comprehension restricted to arithmetic predicates. We use $Z\in X$ as an abbreviation for $\exists k[Z = (X)_k]$. The following characterization can be obtained (Rathjen [1991b]).

Theorem 2.5 *For $n > 2$,* $\mathrm{KP} + \Pi_n\mathrm{reflection}$ *proves the same Π^1_4 sentences of second order arithmetic as ACA plus Bar–induction augmented by the scheme*

$$\forall Z_1, \ldots, Z_k\, [A(Z_1, \ldots, Z_k) \;\rightarrow\; \exists X\, [Z_1\in X \wedge \ldots \mathrm{Z}_k\in X \;\wedge\; \mathcal{B}_X \models A(Z_1, \ldots, Z_k)\,]],$$

where A ranges over the Π^1_{n+1} formulae of second order arithmetic and the free second order variables of A are among the ones shown. It is readily shown that Δ^1_2 comprehension is derivable in the latter theory

Next, we are going to explain why an ordinal analysis of Π^1_2 comprehension, unlike Δ^1_2 comprehension, has to exceed the methods of admissible proof theory. On the set–theoretic side, Π^1_2 comprehension corresponds to Σ_1 *separation*, i.e. the scheme

$$\exists z(z = \{x\in a : F(x)\})$$

for all Σ_1 formulae $F(x)$ in which z does not occur free. The precise relationship reads as follows.

Theorem 2.6 *$KP + \Sigma_1$ separation and $(\Pi^1_2 - CA) + BI$ prove the same theorems of second order arithmetic.*[5]

The ordinals κ such that $L_\kappa \models KP + \Sigma_1$*separation* are familiar from ordinal recursion theory (see Barwise [1975], Hinman [1978]). An admissible ordinal κ is said to be *nonprojectible* if there is no (total) κ–recursive function mapping κ one–one into some $\beta < \kappa$.

The key to the "largeness" properties of nonprojectible ordinals is the following.

Theorem 2.7 *For any nonprojectible ordinal κ, L_κ is a limit of Σ_1–elementary substructures*[6]*, i.e. for every $\beta < \kappa$ there exists a $\beta < \rho < \kappa$ such that L_ρ is a Σ_1–elementary substructure of L_κ (written $L_\rho \prec_1 L_\kappa$).*

Ordinals ρ satisfying $L_\rho \prec_1 L_\kappa$ for some $\kappa > \rho$ have strong reflecting properties. For instance, if $L_\rho \models F$ for some set–theoretic sentence F (possibly containing parameters from L_ρ), then there exists a $\gamma < \rho$ such that $L_\gamma \models F$ because from $L_\rho \models F$ we can infer $L_\kappa \models \exists\gamma F^{L_\gamma}$ which yields $L_\rho \models \exists\gamma F^{L_\gamma}$ using $L_\rho \prec_1 L_\kappa$.

The last remark makes it clear that an ordinal analysis of Π^1_2 comprehension would necessarily involve a proof–theoretic treatment of reflections.

3 A sequent calculus for *KP*

Since later on we are going to interpret *KP* in an infinitary sequent calculus $RS(\mathcal{K})$, we will furnish *KP* in sequent calculus style. For technical reasons we shall treat equality as a defined symbol and assume that formulae are in negation normal form. Also bounded quantifiers will be treated syntactically as quantifiers in their own right.

The *language* of *KP*, $\mathcal{L}$, consists of: *free variables* $a_1, a_2, a_3, \ldots$; *bound variables* $x_1, x_2, x_3, \ldots$; the *predicate symbol* $\in$; the *logical symbols* $\neg, \wedge, \vee, \forall, \exists$; and parenthesis.

The *atomic formulae* are those of the form $(a \in b)$ with free variables a, b. *Formulae* are built from atomic and *negated* atomic formulae by means of the connectives $\wedge, \vee$ and the following construction step: If b is a free variable and $F(a)$ is a formula in which the bound variable x does not occur, then $(\forall x \in b)F(x), (\exists x \in b)F(x), \forall x F(x), \exists x F(x)$ are formulae.

A formula which contains only bounded quantifiers, i.e. quantifiers of the form $(\forall x \in b), (\exists x \in b)$, is said to be a Δ_0*–formula*. The *negation*, $\neg A$, of a non–atomic formula A is defined to be the formula obtained from A by (i) putting $\neg$ in front any atomic subformula, (ii) replacing $\wedge, \vee, (\forall x \in b), (\exists x \in b), \forall x, \exists x$ by $\vee, \wedge, (\exists x \in b), (\forall x \in b), \exists x, \forall x$, respectively, and (iii) dropping double negations.

[5]For this result to hold it is crucial that Infinity is among the axioms of *KP*.

[6]L_ρ is a Σ_1–elementary substructure of L_κ if every Σ_1 sentence with parameters from L_ρ that holds in L_κ holds in L_ρ as well.

Equality is defined by $a = b :\Leftrightarrow (\forall x \in a)(x \in b) \ \wedge \ (\forall x \in b)(x \in a)$. As a result of this, we will have to state the Axiom of Extensionality in a different way than usually.

We use $A, B, C, ..., F(a), G(a), ..$ as meta–variables for formulae. Upper case Greek letters $\Delta, \Gamma, \Lambda, ...$ range over finite sets of formulae. The meaning of $\{A_1, \dots, A_n\}$ is the disjunction $A_1 \vee \cdots \vee A_n$. Γ, A stands for $\Gamma \cup \{A\}$. As usual, $A \to B$ abbreviates $\neg A \vee B$. We shall write $b = \{y \in a : F(y)\}$ for $(\forall y \in b)[y \in a \wedge F(y)] \wedge (\forall y \in a)[F(y) \to y \in x]$.

For any Γ and formula A,

$$\Gamma, A, \neg A$$

is a *logical axiom* of KP.

The *set–theoretic axioms* of KP are:

Extensionality:	$\Gamma, a = b \to [F(a) \leftrightarrow F(b)]$ for all formulae $F(a)$.
Foundation:	$\Gamma, \exists x G(x) \to \exists x[G(x) \wedge (\forall y \in x)\neg G(y)]$ for all formulae $G(b)$.
Pairing:	$\Gamma, \exists x \ (x = \{a, b\})$.
Union:	$\Gamma, \exists x \ (x = \bigcup a)$.
Infinity:	$\Gamma, \exists x \ \big[x \neq \emptyset \ \wedge \ (\forall y \in x)(\exists z \in x)(y \in z)\big]$.
Δ_0*–Separation:*	$\Gamma, \exists x \ \big(x = \{y \in a : F(y)\}\big)$ for all Δ_0–formulae $F(b)$
Δ_0*–Collection:*	$\Gamma, (\forall x \in a)\exists y G(x, y) \to \exists z (\forall x \in a)(\exists y \in z) G(x, y)$ for all Δ_0–formulae $G(b)$.

The *logical rules of inference* are:

$$(\wedge) \ \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'} \qquad (\vee) \ \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \ \text{ if } i \in \{0, 1\}$$

$$(b\forall) \ \frac{\Gamma, a \in b \to F(a)}{\Gamma, (\forall x \in b) F(x)} \qquad (\forall) \ \frac{\Gamma, F(a)}{\Gamma, \forall x F(x)}$$

$$(b\exists) \ \frac{\Gamma, a \in b \wedge F(a)}{\Gamma, (\exists x \in b) F(x)} \qquad (\exists) \ \frac{\Gamma, F(a)}{\Gamma, \exists x F(x)}$$

$$(Cut) \ \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

where in $(\forall)$ and $(b\forall)$ the free variable a is not to occur in the conclusion of the inference.

We formalize Π_n–reflection as an inference rule.

Definition 3.1 The sequent calculus $KP + \Pi_n$–*Ref* arises from KP by adjoining the Π_n–reflection rule of inference

$$(\Pi_n - Ref) \ \frac{\Gamma, A}{\Gamma, \exists z[Tran(z) \wedge z \neq \emptyset \wedge A^z]}$$

for all Π_n–formulae A.

4 Collapsing functions

We are going to develop so–called *collapsing functions* which give rise to a strong ordinal notation system $\mathcal{T}(\mathcal{K})$. Rather than developing such functions on the basis of Π_3 reflecting ordinals, we build them by employing a weakly compact cardinal. This is not a far–fetched assumption since Π_3 reflecting ordinals are the recursive analogues of weakly compact cardinals (see Aczel and Richter [1974]). Proceeding this way, allows us to develop the right intuitions about these functions and to side–step fiddly and delicate ordinal recursion theory (cf. Rathjen [1993a] and [1993c]). Of course, another option would be to abstain completely from set theory by directly defining the primitive recursive notation system. However, nude ordinal notation systems without any set-theoretic interpretation tend to be hard to grasp.

Firstly, we remind the reader of some set–theoretical notions and take this as an opportunity to fix some notations.

Definition 4.1 Let On denote the class of ordinals and let Lim be the class of limit ordinals. The *cumulative hierarchy*, $V = \bigcup\{ V_\alpha : \alpha \in On\}$, is defined by: $V_0 = \emptyset$, $V_{\alpha+1} = \{X : X \subseteq V_\alpha\}$, $V_\lambda = \bigcup\{V_\xi : \xi < \lambda\}$ for $\lambda \in Lim$.

Let $\mathfrak{A} = \langle A, U_1, \ldots, f_1, \ldots, c_1, \ldots\rangle$ be a structure for a language. The extension of $\mathcal{L}$ to second order, denoted $\mathcal{L}_2$, is given as follows. Besides symbols of $\mathcal{L}$, a formula of $\mathcal{L}_2$ may contain second order quantifiers $\forall X$, $\exists X$, and atomic formulae $X(t)$, where X is a second order variable and t is a term of $\mathcal{L}$.

Satisfaction of sentences of $\mathcal{L}_2$ in $\mathfrak{A}$ is defined as follows. Variables of first order range over elements of A. Variables of second order range over the full power set of A. A formula $X(t)$ is interpreted as $t \in X$.

A formula of $\mathcal{L}_2$ is Π^1_n if it is of the form

$$\forall X_1 \exists X_2 \cdots Q X_n F(X_1, \cdots, X_n),$$

where $F(X_1, \cdots, X_n)$ does not contain second order quantifiers and the n second order quantifiers in $\forall X_1 \exists X_2 \cdots Q X_n$ are alternating.

Definition 4.2 A cardinal κ is Π^1_n–*indescribable*, if whenever $U_1, \ldots, U_m \subseteq V_\kappa$ and F is a Π^1_n sentence of the language of $\langle V_\kappa, \in, U_1, \ldots, U_m\rangle$ such that

$$\langle V_\kappa, \in, U_1, \ldots, U_m\rangle \models F$$

then, for some $0 < \alpha < \kappa$,

$$\langle V_\alpha, \in, U_1 \cap V_\alpha, \ldots, U_m \cap V_\alpha\rangle \models F.$$

Definition 4.3 A class of ordinals C is *unbounded* in $\alpha \in Lim$ if $(\forall \xi < \alpha)(\exists \delta \in C)(\xi < \delta \wedge \delta < \alpha)$.

Let κ be a regular cardinal $> \omega$. A class C of ordinals is *closed* in κ if whenever λ is a limit ordinal $< \kappa$ such that C is unbounded in λ, then $\lambda \in C$.

A class of ordinals S is *stationary* in κ if, for all C which are closed and unbounded in κ, $S \cap C \neq \emptyset$.

κ is *Mahlo on* $X \subseteq On$ if $\kappa \in X$ and X is stationary in κ. The *Mahlo thinning–operation* M is defined as follows

$$M(X) = \{\alpha \in X : X \text{ ist stationary in } \alpha\}.$$

The Π^1_1 indescribable cardinals are also called (or proved to be the same as) the *weakly compact cardinals* (see Jech [1979]). To give an inkling as to the strength of weakly compact cardinals, we introduce the notion of Mahlo cardinal. A cardinal is called *Mahlo cardinal* (respectively, *weakly Mahlo cardinal*) if, for every function $f : \kappa \longmapsto \kappa$, there exists an inaccessible cardinal (respectively, weakly inaccessible cardinal) $\rho < \kappa$ such that ρ is closed under f. Equivalently, κ is Mahlo (respectively, weakly Mahlo) iff the inaccessible cardinals (respectively, weakly inaccessible cardinals) are stationary in κ.

Remark 4.4 *If κ is weakly compact, then κ is Mahlo and the Mahlo cardinals are stationary in κ.*

The Veblen–function figures prominently in predicative proof theory (cf. Feferman [1968], Schütte [1977], Sec.13 and Pohlers [1989].) We are going to incorporate this function in our notation system.

Definition 4.5 The *Veblen–function* $\varphi\alpha\beta := \varphi_\alpha(\beta)$ is defined by transfinite recursion on α by letting φ_α be the function that enumerates the class of ordinals

$$\{\omega^\gamma : \gamma \in On \wedge (\forall \xi < \alpha)[\varphi_\xi(\omega^\gamma) = \omega^\gamma]\}.$$

Corollary 4.6 *(i)* $\varphi 0\beta = \omega^\beta$.

(ii) $\xi, \eta < \varphi\alpha\beta \Longrightarrow \xi + \eta < \varphi\alpha\beta$.

(iii) $\xi < \zeta \Longrightarrow \varphi\alpha\xi < \varphi\alpha\zeta$.

(iv) $\alpha < \beta \Longrightarrow \varphi\alpha(\varphi\beta\xi) = \varphi\beta\xi$.

Definition 4.7 To save space, we introduce some abbreviations. $fun(g)$ abbreviates that g is a function. $dom(g)$ and $ran(g)$ denote the domain and the range of g, respectively. $g''x$ stands for the set $\{g(u) : u \in x \cap dom(g)\}$. Let $pow(a) := \{x : x \subseteq a\}$. For U a second order variable, let $club(U)$ be the formula expressing that U is closed and unbounded in On, i.e. $\forall\alpha(\exists\beta \in U)(\alpha < \beta) \wedge (\forall \lambda \in Lim)[(\forall \xi < \lambda)(\exists \delta \in U)(\xi < \delta < \lambda) \rightarrow \lambda \in U]$.

For classes G, one defines $fun(G)$, $ran(G)$ and $dom(G)$ analogously.

Let

$$\Omega_\xi = \begin{cases} \aleph_\xi & \text{if } \xi > 0 \\ 0 & \text{otherwise.} \end{cases}$$

General assumption: From now on, we assume that there exists a weakly compact cardinal, denoted $\mathcal{K}$.

Reg denotes the set of uncountable regular cardinals $< \mathcal{K}$. We shall use the variables $\kappa, \pi, \tau, \kappa', \pi', \tau'$ exclusively for elements of *Reg*.

Definition 4.8 By recursion on α, we define sets $C(\alpha,\beta)$ and M^α, and ordinals Ξ_κ und $\Psi^\xi_\pi(\alpha)$ as follows[7]

$$C(\alpha,\beta) = \begin{cases} \quad \textit{closure of } \beta \cup \{0, \mathcal{K}\} \\ \quad \textit{under } +, \\ (\xi\eta \mapsto \varphi\xi\eta), \\ (\xi \mapsto \Omega_\xi)_{\xi<\mathcal{K}}, \\ (\xi \longmapsto \Xi(\xi))_{\xi<\alpha} \\ (\xi\pi\delta \longmapsto \Psi^\xi_\pi(\delta))_{\xi\le\delta<\alpha} \end{cases}$$

$M^0 = \mathcal{K} \cap Lim$, and, for $\alpha > 0$,

$$M^\alpha = \left\{ \begin{array}{r} \pi < \mathcal{K} : \; C(\alpha,\pi) \cap \mathcal{K} = \pi \;\wedge\; (\forall \xi \in C(\alpha,\pi) \cap \alpha)[M^\xi \textit{ stationary in } \pi] \\ \wedge\; \alpha \in C(\alpha,\pi) \end{array} \right\}$$

$$\Xi(\alpha) = min(M^\alpha \cup \{\mathcal{K}\}).$$

For $\xi \le \alpha$,

$$\Psi^\xi_\pi(\alpha) = min(\{\rho \in M^\xi \cap \pi : \; C(\alpha,\rho) \cap \pi = \rho \;\wedge\; \pi, \alpha \in C(\alpha,\rho)\} \cup \{\pi\}).$$

Note that in the above definition, we tacitly assume, in keeping with our convention, that π ranges over regular cardinals.

Remark 4.9 To gain a better picture of the sets M^α, it is instructive to study some initial cases. It is readily verified that any $\kappa \in M^1$ is weakly inaccessible since κ is regular and closed under Ω. Therefore, M^1 consists of the weakly inaccessible cardinals below $\mathcal{K}$. Subsequently, we come to see that, for any $\pi \in M^2$, M^1 is stationary in π and hence π is weakly Mahlo. This pattern continues for quite a while, i.e., M^3 consists of the weakly hyper–Mahlo cardinals below $\mathcal{K}$, M^4 consists of the weakly hyper–hyper–Mahlo cardinals below $\mathcal{K}$ and so forth. However, only for weakly $\alpha < \mathcal{K}$, M^α can be couched in terms of α–hyper–Mahloness. By way of contrast, $M^{\mathcal{K}}$ is obtained by diagonalizing over the sequence $(M^\alpha)_{\alpha<\mathcal{K}}$.

[7]Closure of $C(\alpha,\beta)$ under $(\xi \mapsto \Omega_\xi)_{\xi<\mathcal{K}}$ is only demanded for technical convenience. This closure property does not contribute to the strength of the intended ordinal notation system. Likewise, it would suffice to demand only closure under $\xi \mapsto \omega^\xi$ instead of φ.

Remark 4.10 The inductive generation of $C(\alpha,\beta)$ is completed after ω stages. Therefore $C(\alpha,\beta)$ can be depicted as $C(\alpha,\beta) = \bigcup_{n<\omega} C_n(\alpha,\beta)$, where $C_n(\alpha,\beta)$ consists of the elements constructed up to stage n. We emphazise this build–up of $C(\alpha,\beta)$ since we will be proving properties of the elements of this set by induction on stages $C_n(\alpha,\beta)$.

Lemma 4.11 *(i)* $\alpha \leq \alpha' \wedge \beta \leq \beta' \implies C(\alpha,\beta) \subseteq C(\alpha',\beta')$.

(ii) $\beta < \pi \implies |C(\alpha,\beta)| < \pi$.

(iii) $\lambda \in Lim \implies C(\alpha,\lambda) = \bigcup_{\eta<\lambda} C(\alpha,\eta) \ \wedge \ C(\lambda,\alpha) = \bigcup_{\eta<\lambda} C(\eta,\alpha)$.

(iv) $C(\alpha,\Xi(\alpha)) \cap \mathcal{K} = \Xi(\alpha)$.

(v) $C(\alpha,\Psi^{\zeta}_{\pi}(\alpha)) \cap \pi = \Psi^{\zeta}_{\pi}(\alpha)$.

(vi) If $\pi \in M^{\alpha}$ *and* $\zeta \in C(\alpha,\pi) \cap \alpha$*, then* $\pi \in M^{\zeta}$.

(vii) If M^{ξ} *is stationary in* π*, then* $\pi \in M^{\xi}$.

Proof. (i)–(v) are obvious.
(vi): The assumptions imply $C(\alpha,\pi) \cap \mathcal{K} = \pi$ and $(\forall \xi \in C(\alpha,\pi) \cap \alpha)[M^{\xi}$ *stationary in* $\pi]$; hence, a fortiori, $C(\zeta,\pi) \cap \mathcal{K} = \pi$ and $(\forall \xi \in C(\zeta,\pi) \cap \zeta)[M^{\xi}$ *stationary in* $\pi]$. Since M^{ζ} is also stationary in π, we get $\zeta \in C(\zeta,\pi)$. Therefore, $\pi \in M^{\zeta}$.

(vii): Let $\rho \in M^{\xi} \cap \pi$. Then $\xi \in C(\xi,\rho)$, whence $\xi \in C(\xi,\pi)$. Since M^{ξ} is unbounded in π it follows $C(\xi,\pi) \cap \mathcal{K} = (\bigcup\{C(\xi,\rho) : \rho \in M^{\xi} \cap \pi\}) \cap \mathcal{K} = \bigcup\{C(\xi,\rho) \cap \mathcal{K} : \rho \in M^{\xi} \cap \pi\} = \bigcup\{\rho : \rho \in M^{\xi} \cap \pi\} = \pi$.

Now suppose that $\eta \in C(\xi,\pi) \cap \xi$, and let $U \subseteq \pi$ be closed and unbounded in π. Since M^{ξ} is stationary in π, we may select a $\rho \in M^{\xi} \cap \pi$ so that $\eta \in C(\xi,\rho)$ and U is already closed and unbounded in ρ. M^{η} being stationary in ρ implies $U \cap M^{\eta} \cap \rho \neq \emptyset$; thus $U \cap M^{\eta} \cap \pi \neq \emptyset$. Thence, M^{η} is stationary in π. □

Let $\mathcal{K}^{\Gamma}$ denote the least ordinal $\alpha > \mathcal{K}$ satisfying $(\forall \xi,\eta < \alpha)(\varphi\xi\eta < \alpha)$.

Theorem 4.12 *For all* $\alpha < \mathcal{K}^{\Gamma}$*,* M^{α} *is stationary in* $\mathcal{K}$ *and hence* $\Xi(\alpha) < \mathcal{K}$.

Proof. Each ordinal $\mathcal{K} < \beta < \mathcal{K}^{\Gamma}$ has a unique representation of either form $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$ with $\beta > \beta_1 \geq \cdots \geq \beta_n$ and $n > 0$, or $\beta = \varphi\beta_1\beta_2$ with $\beta > \beta_1, \beta_2$, denoted $\beta =_{NF} \omega^{\beta_1} + \cdots + \omega^{\beta_n}$ and $\beta =_{NF} \varphi\beta_1\beta_2$, respectively. Due to uniqueness, we can define an injective mapping

$$f : \mathcal{K}^{\Gamma} \longrightarrow L_{\mathcal{K}}$$

by letting[8]

$$f(\beta) = \begin{cases} \beta & \text{if } \beta < \mathcal{K} \\ \{1\} & \text{if } \beta = \mathcal{K} \\ \langle 2, f(\beta_1), \ldots, f(\beta_n)\rangle & \text{if } \beta =_{NF} \omega^{\beta_1} + \cdots + \omega^{\beta_n} \text{ and } \mathcal{K} < \beta \\ \langle 3, f(\beta_1), f(\beta_2)\rangle & \text{if } \beta =_{NF} \varphi\beta_1\beta_2 \text{ and } \mathcal{K} < \beta. \end{cases}$$

Putting

$$f(\alpha) \lhd f(\beta) :\iff \alpha < \beta,$$

$\lhd$ defines a well–ordering on a subset of $L_{\mathcal{K}}$ of order type $\mathcal{K}^{\Gamma}$.

To show the Theorem, we proceed by induction on α, or, equivalently, by induction on $\lhd$.

Assume that E is closed and unbounded in $\mathcal{K}$. We have to verify that $M^{\alpha} \cap E \neq \emptyset$. Since $\alpha < \mathcal{K}^{\Gamma}$, we may utilize the above representations to see that there are finitely many ordinals $\alpha_1 \ldots, \alpha_n < \mathcal{K}$ such that α is in the closure of $\{\alpha_1 \ldots, \alpha_n, \mathcal{K}\}$ under $+$ and φ. Therefore we can pick a $\rho_0 < \mathcal{K}$ with $\alpha \in C(\alpha, \rho_0)$. Since $E \backslash \rho_0$ is also closed and unbounded in $\mathcal{K}$, we may assume that $E \cap \rho_0 = \emptyset$. Using the induction hypothesis, for all $\beta < \alpha$, M^{β} is stationary in $\mathcal{K}$. Define

$$U_1 := \{f(\alpha)\}, \;\; U_2 := \{\langle x, y\rangle : x \lhd y\}, \text{ and } U_3 := \bigcup_{\beta<\alpha} (M^{\beta} \times \{f(\beta)\}).$$

The following sentences are satisfied in the structure $\langle V_{\mathcal{K}}, \in, U_1, U_2, U_3, E\rangle$:

(1) $\forall G \forall \delta[fun(G) \wedge dom(G) = \delta \wedge ran(G) \subseteq On \;\rightarrow\; \exists\gamma(G''\delta \subseteq \gamma)]$

(2) $\forall a \exists b \exists \beta \exists g[b = pow(a) \wedge fun(g) \wedge dom(g) = b \;\wedge\; ran(g) = \beta \wedge\;$ g injective $]$

(3) $U_1 \neq \emptyset \;\wedge\; \forall\gamma\exists\delta(\gamma < \delta \wedge E(\delta))$

(4) $\forall X \forall s \forall t[U_1(t) \wedge U_2(\langle s, t\rangle) \wedge club(X) \;\rightarrow\; \{y : U_3(\langle y, s\rangle)\} \cap X \neq \emptyset]$

Employing the Π^1_1–indescribability of $\mathcal{K}$, there exists $\pi < \mathcal{K}$ such that the structure

$$\langle V_{\pi}, \in, U_1 \cap \pi, U_2 \cap \pi, U_3 \cap \pi, E \cap \pi\rangle$$

satisfies:

(a) $\forall G \forall \delta[fun(G) \wedge dom(G) = \delta \wedge ran(G) \subseteq On \;\rightarrow\; \exists\gamma(G''\delta \subseteq \gamma)]$

(b) $\forall a \exists b \exists \beta \exists g[b = pow(a) \;\wedge\; fun(g) \;\wedge\; dom(g) = b \;\wedge\; ran(g) = \beta \wedge\;$ g injective $]$

(c) $U_1 \cap \pi \neq \emptyset \;\wedge\; \forall\gamma\exists\delta(\gamma < \delta \;\wedge\; \delta \in E \cap \pi)$

[8] $\langle x, y\rangle := \{\{x\}, \{x, y\}\}$; $\langle x_1, \ldots, x_{n+1}\rangle := \langle\langle x_1, \ldots, x_n\rangle, x_{n+1}\rangle$ for $n > 2$.

(d) $\forall X \forall s \forall t[t \in U_1 \cap \pi \wedge (\langle s,t\rangle) \in U_2 \cap \pi \wedge club(X) \ \rightarrow \{y : \langle y,s\rangle \in U_3 \cap \pi\} \cap X \neq \emptyset]$

By virtue of (a), observing that $\forall G$ is second order, and (b), π must be inaccessible. Due to (c), $f(\alpha) \in V_\pi$ and E is unbounded in π; whence $\pi \in E$. (d) forces that

$$(*) \ \ (\forall \beta < \alpha)[f(\beta) \in V_\pi \rightarrow M^\beta \text{ stationary in } \pi].$$

Next, we want to verify

$$(+) \ \ (\forall \eta \in C(\alpha,\pi))[f(\eta) \in V_\pi].$$

Set $X := \{\eta \in C(\alpha,\pi) : f(\eta) \in V_\pi\}$. Clearly, $\pi \cup \{0, \mathcal{K}\} \subseteq X$. If $\eta =_{NF} \omega^{\eta_1} + \cdots + \omega^{\eta_n}$ and $\eta_1, \ldots, \eta_n \in X$, then $\eta \in X$ since π is closed under $+$ and $\zeta \mapsto \omega^\zeta$ and V_π is closed under $\langle \cdot,\cdot \rangle$. Likewise, π being closed under φ implies that X is closed under φ.

For $\sigma \in X \cap \mathcal{K}$, $f(\sigma) = \sigma \in V_\pi$; thus $\sigma < \pi$ and hence $\Omega_\sigma < \pi$ because π is inaccessible.

If $\beta \in X \cap \alpha$, then, according to $(*)$, M^β is stationary in π, yielding $\Xi(\beta) = f(\Xi(\beta)) < \pi$.

If $\kappa, \xi, \delta \in X$ und $\xi \leq \delta < \alpha$, then $f(\kappa) = \kappa < \pi$ and therefore $\Psi^\xi_\kappa(\delta) < \pi$. So it turns out that X enjoys all the closure properties defining $C(\alpha,\pi)$. This verifies (+).

From $\pi \in E$ it follows $\alpha \in C(\alpha,\pi)$. Using $(*)$ and $(+)$, we obtain

$$(\forall \beta \in C(\alpha,\pi) \cap \alpha)[M^\beta \text{ is stationary in } \pi].$$

Whence, $\pi \in M^\alpha \cap E$. □

Corollary 4.13 *When $\alpha < \mathcal{K}^\Gamma$, then $\alpha \in C(\alpha, \Xi(\alpha))$ and $\Xi(\alpha) < \mathcal{K}$.*

Agreement: For the remainder of this Section, we shall only consider ordinals $< \mathcal{K}^\Gamma$.

Lemma 4.14 $\Xi(\alpha) < \Xi(\beta)$ *iff either*

$$(1) \ \ \alpha < \beta \ \wedge \ \alpha \in C(\beta, \Xi(\beta))$$

or

$$(2) \ \ \beta < \alpha \ \wedge \ \beta \notin C(\alpha, \Xi(\alpha)).$$

Proof. First, let $\Xi(\alpha) < \Xi(\beta)$ be the case. If $\alpha < \beta$, then $\alpha \in C(\alpha, \Xi(\alpha)) \subseteq C(\beta, \Xi(\beta))$; thus (1). If, however, $\beta < \alpha$, then $\beta \in C(\alpha, \Xi(\alpha))$ is impossible since this would entail $\Xi(\beta) \in C(\alpha, \Xi(\alpha))$ and consequently, $\Xi(\beta) < \Xi(\alpha)$; thence in this case (2) is satisfied.

For the reverse implication, note that (1) yields $\Xi(\alpha) \in C(\beta, \Xi(\beta))$ and hence $\Xi(\alpha) < \Xi(\beta)$. (2) entails $\beta \notin C(\beta, \Xi(\alpha))$ and therefore, utilizing $\beta \in C(\beta, \Xi(\beta))$, $\Xi(\alpha) < \Xi(\beta)$. □

Corollary 4.15 $\alpha \neq \beta \Longrightarrow \Xi(\alpha) \neq \Xi(\beta)$.

Proposition 4.16 *Let M^ξ be stationary in π. Assume that $\xi \leq \alpha$ und $\xi, \pi, \alpha \in C(\alpha, \pi)$. Then,*

$$\Psi^\xi_\pi(\alpha) \in M^\xi \cap \pi.$$

Moreover, if $\xi > 0$, then M^ξ is not stationary in $\Psi^\xi_\pi(\alpha)$ and, for all $\beta > \xi$, $\Psi^\xi_\pi(\alpha) \notin M^\beta$.

Proof. Since $\xi, \pi, \alpha \in C(\alpha, \pi)$ and $\pi \in Lim$, we may select a $\mu_0 < \pi$ so that already $\xi, \pi, \alpha \in C(\alpha, \mu_0)$.

Letting $E := \{\rho < \pi : \mu_0 \leq \rho \wedge C(\alpha, \rho) \cap \pi = \rho\}$, we claim that E is closed and unbounded in π.

Unboundedness: Fix δ such that $\mu_0 \leq \delta < \pi$. For $\delta_0 := \delta + 1$ and $\delta_{n+1} := sup(C(\alpha, \delta_n) \cap \pi)$, one obtains, by Lemma 4.11(ii) and the regularity of π, $\delta < \delta_n \leq \delta_{n+1} < \pi$. The regularity of π also ensures $\delta^* := sup_{n<\omega}\delta_n < \pi$. Since $C(\alpha, \delta_n) \cap \pi \subseteq \delta_{n+1} \subseteq C(\alpha, \delta_{n+1}) \cap \pi$ issues from the definition of δ_{n+1}, it follows

$$C(\alpha, \delta^*) \cap \pi = \bigcup_{n<\omega} (C(\alpha, \delta_n) \cap \pi) = \delta^*.$$

Therefore, $\delta < \delta^* \in E$.

Closedness: Let $\lambda \in Lim \cap \pi$ and suppose that E is unbounded in λ. Then $C(\alpha, \lambda) = \bigcup_{\eta \in E \cap \lambda} C(\alpha, \eta)$, and consequently $\lambda \in E$ follows from

$$C(\alpha, \lambda) \cap \pi = \bigcup_{\eta \in E \cap \lambda} (C(\alpha, \eta) \cap \pi) = sup(E \cap \lambda) = \lambda.$$

By assumption, M^ξ is stationary in π, so there exists a $\nu \in E \cap M^\xi$. This involves $C(\alpha, \nu) \cap \pi = \nu$. Because of $\mu_0 \leq \nu$, we get $\xi, \pi, \alpha \in C(\alpha, \nu)$. Due to the definition of $\Psi^\xi_\pi(\alpha)$, this implies $\Psi^\xi_\pi(\alpha) \leq \nu < \pi$.

Now assume $\xi > 0$. Then $\Psi^\xi_\pi(\alpha)$ is regular. We want to show that M^ξ is not stationary in $\Psi^\xi_\pi(\alpha)$. Observe that $\xi, \pi, \alpha \in C(\alpha, \Psi^\xi_\pi(\alpha))$. So, if M^ξ were stationary in $\Psi^\xi_\pi(\alpha)$, by applying the same arguments as in the first part of the proof, we could verify the existence of a $\rho \in M^\xi \cap \Psi^\xi_\pi(\alpha)$ with $\xi, \pi, \alpha \in C(\alpha, \rho)$ and $C(\alpha, \rho) \cap \pi = \rho$, which would collide with the definition of $\Psi^\xi_\pi(\alpha)$.

Finally, if we had $\Psi^\xi_\pi(\alpha) \in M^\beta$ for some $\beta > \xi$, then, since $\xi \in C(\xi, \Psi^\xi_\pi(\alpha))$, we would get $\xi \in C(\beta, \Psi^\xi_\pi(\alpha)) \cap \beta$, leading to the contradiction that M^ξ is stationary in $\Psi^\xi_\pi(\alpha)$. $\square$

Proposition 4.17 *(i)* $\Psi^\xi_\pi(\alpha) < \pi \Longrightarrow \Psi^\xi_\pi(\alpha) \neq \Xi(\beta)$.

(ii) $\Psi^\xi_\pi(\alpha) < \pi \wedge \Psi^\sigma_\kappa(\beta) < \kappa \wedge \Psi^\xi_\pi(\alpha) = \Psi^\sigma_\kappa(\beta) \implies \alpha = \beta \wedge \pi = \kappa \wedge \xi = \sigma$.

Proof. (i): By way of a contradiction, suppose $\Psi^\xi_\pi(\alpha) = \Xi(\beta)$. $\Psi^\xi_\pi(\alpha) < \pi$ implies $\pi \in C(\alpha, \Psi^\xi_\pi(\alpha))$. From $\alpha \leq \beta$ we could deduce $\pi \in C(\beta, \Xi(\beta))$ and therefore the contradiction $\pi < \Xi(\beta)$. From $\beta < \alpha$ we would get $\beta \in C(\beta, \Xi(\beta)) \subseteq C(\alpha, \Psi^\xi_\pi(\alpha))$ and consequently $\Xi(\beta) \in C(\alpha, \Psi^\xi_\pi(\alpha))$, contradicting $\Psi^\xi_\pi(\alpha) \notin C(\alpha, \Psi^\xi_\pi(\alpha))$. Since in any case we are led to a contradiction, the assumption $\Psi^\xi_\pi(\alpha) = \Xi(\beta)$ must be false.

(ii): The hypotheses imply

$$\text{(a)} \quad \xi, \alpha, \pi \in C(\alpha, \Psi^\sigma_\kappa(\beta))$$

and

$$\text{(b)} \quad \sigma,\kappa,\beta \in C(\beta,\Psi^{\xi}_{\pi}(\alpha)).$$

From $\alpha<\beta$, using (a), we would get $\xi,\alpha,\pi \in C(\beta,\Psi^{\sigma}_{\kappa}(\beta))$ and hence $\Psi^{\xi}_{\pi}(\alpha) \in C(\beta,\Psi^{\sigma}_{\kappa}(\beta))$, contradicting $\Psi^{\sigma}_{\kappa}(\beta) \notin C(\beta,\Psi^{\sigma}_{\kappa}(\beta))$. Similarly, using (b), the assumption $\beta<\alpha$ leads to a contradiction. Therefore, $\alpha=\beta$.

From $\pi<\kappa$ we would get $\pi \in C(\beta,\Psi^{\sigma}_{\kappa}(\beta))\cap\kappa$ by (a); but this is impossible since $C(\beta,\Psi^{\sigma}_{\kappa}(\beta))\cap\kappa = \Psi^{\sigma}_{\kappa}(\beta) = \Psi^{\xi}_{\pi}(\alpha) < \pi$. Using (b), we can also exclude that $\kappa<\pi$. Consequently, $\pi=\kappa$.

Finally, we have to show $\xi=\sigma$. For a contradiction, assume $\xi<\sigma$. $\Psi^{\xi}_{\pi}(\alpha)<\pi$ yields $\Psi^{\xi}_{\pi}(\alpha)\in M^{\xi}$ und thus $\xi \in C(\xi,\Psi^{\xi}_{\pi}(\alpha))$. Therefore, $\xi\in C(\sigma,\Psi^{\sigma}_{\kappa}(\beta))$. Utilizing the definition of $\Psi^{\sigma}_{\kappa}(\beta)$, the latter implies that M^{ξ} is stationary in $\Psi^{\sigma}_{\kappa}(\beta)$. Letting

$$Y := \{\eta < \Psi^{\sigma}_{\kappa}(\beta) : C(\alpha,\eta)\cap\Psi^{\sigma}_{\kappa}(\beta) = \eta \ \wedge \ \alpha,\pi \in C(\alpha,\eta)\},$$

we obtain a set that is unbounded and closed in $\Psi^{\sigma}_{\kappa}(\beta)$. But then $M^{\xi}\cap Y \neq \emptyset$ and, as a consequence, $\Psi^{\xi}_{\pi}(\alpha) = min(M^{\xi}\cap Y) < \Psi^{\xi}_{\pi}(\alpha)$, contradicting $\Psi^{\xi}_{\pi}(\alpha)=\Psi^{\sigma}_{\kappa}(\beta)$. Interchanging the roles of σ and ξ in the preceeding argument, one also excludes $\sigma<\xi$. □

Lemma 4.18

(i) $\alpha =_{NF} \omega^{\alpha_1}+\cdots+\omega^{\alpha_n} \implies [\alpha\in C(\zeta,\rho) \iff \alpha_1,\dots,\alpha_n \in C(\zeta,\rho)]$.

(ii) $\alpha =_{NF} \varphi\alpha_1\alpha_2 \implies [\alpha\in C(\zeta,\rho) \iff \alpha_1,\alpha_2 \in C(\zeta,\rho)]$.

(iii) $\sigma<\mathcal{K} \implies [\sigma\in C(\zeta,\rho) \iff \Omega_{\sigma} \in C(\zeta,\rho)]$.

Proof. (i) Using induction on n, one easily shows that $\alpha\in C_n(\zeta,\rho)$ implies $\alpha_1,\dots,\alpha_n \in C_n(\zeta,\rho)$. Similarly one proves (ii) and (iii). □

Lemma 4.19 *(i)* $0<\alpha\wedge\pi\in M^{\alpha} \implies \Omega_{\pi}=\pi$.

(ii) $\pi\in M^1 \implies \Omega_{\Psi^0_{\pi}(\alpha)} = \Psi^0_{\pi}(\alpha)$.

(iii) $\pi=\Omega_{\zeta+1}\wedge\alpha\in C(\alpha,\pi) \implies \Omega_{\zeta}<\Psi^0_{\pi}(\alpha)<\Omega_{\zeta+1}$.

(iv) $\Psi^0_{\pi}(\alpha)<\pi \implies \Psi^0_{\pi}(\alpha)\notin Reg$.

Proof. (i): The hypotheses imply $C(\alpha,\pi)\cap\mathcal{K}=\pi$. Therefore π is closed under $\sigma\mapsto\Omega_{\sigma}$; whence $\Omega_{\pi}=\pi$.

(ii) follows from (i), noting that $C(\alpha,\Psi^0_{\pi}(\alpha))\cap\pi = \Psi^0_{\pi}(\alpha)$.

(iii): As $\zeta<\pi$ and $\alpha\in C(\alpha,\pi)$, there is an $\eta<\pi$ with $\alpha,\pi\in C(\alpha,\eta)$. Utilizing the regularity of π, we can find a $\rho<\pi$ so that simultaneously $\alpha,\pi\in C(\alpha,\rho)$ and

$C(\alpha,\rho)\cap\pi = \rho$. This shows $\Psi^0_\pi(\alpha) < \Omega_{\zeta+1}$. Therefore $\pi\in C(\alpha,\Psi^0_\pi(\alpha)$, and hence, by 4.18, $\zeta\in C(\alpha,\Psi^0_\pi(\alpha))$. Consequently, $\Omega_\zeta\in C(\alpha,\Psi^0_\pi(\alpha))\cap\pi = \Psi^0_\pi(\alpha)$.

(iv): $\Psi^0_\pi(\alpha) < \pi$ implies $\alpha,\pi\in C(\alpha,\Psi^0_\pi(\alpha))$. Let σ_0 be minimal with the property $\alpha,\pi\in C(\alpha,\sigma_0)$. In view of Lemma 4.11(iii), σ_0 is not a limit; hence $\sigma_0 < \Psi^0_\pi(\alpha)$.

Put $\sigma_{n+1} := sup(C(\alpha,\sigma_n)\cap\pi)$ and $\sigma^* := sup_{n<\omega}\sigma_n$. Then $\sigma_n \le \sigma_{n+1} \le \sigma^* < \pi$. Using induction on n, we come to see that $\sigma_n \le \Psi^0_\pi(\alpha)$. Since $C(\alpha,\sigma_n)\cap\pi \subseteq \sigma_{n+1}$ and $\bigcup\limits_{n<\omega} C(\alpha,\sigma_n) = C(\alpha,\sigma^*)$, we get $C(\alpha,\sigma^*)\cap\pi = \sigma^*$. Further, $\alpha,\pi\in C(\alpha,\sigma^*)$. Therefore, $\Psi^0_\pi(\alpha) \le \sigma^*$. This verifies $\Psi^0_\pi(\alpha) = \sigma^*$.

Regarding the sequence of σ_n's, there are two possible outcomes. In the first case, this sequence is strictly increasing and therefore $\Psi^0_\pi(\alpha)$ has cofinality ω, yielding that $\Psi^0_\pi(\alpha)$ is singular.

In the second case, there exists an n_0 such that $\sigma_{n_0} < \sigma_{n_0+1} = \sigma_{n_0+2}$. To see this, note that σ_0 is not a limit whereas $\sigma_n \in Lim$ for $n > 0$. In this case we also have $\sigma_{n_0+1} = \sigma^* = \Psi^0_\pi(\alpha)$. Further, $\mid C(\alpha,\sigma_{n_0})\cap\pi\mid = max(\omega,\mid\sigma_{n_0}\mid) < \sigma_{n_0+1}$. On the other hand, $\sigma_{n_0+1} = sup(C(\alpha,\sigma_{n_0})\cap\pi)$, so σ_{n_0+1} must be singular. Whence, $\Psi^0_\pi(\alpha)\notin Reg$. $\square$

In the rest of this Section, we provide "recursive" $<$–comparisons for ordinals which are presented in terms of Ψ and Ξ.

Proposition 4.20 *Suppose that $\Psi^\xi_\pi(\alpha) < \pi$, $\Psi^\sigma_\kappa(\beta) < \kappa$, and $\Psi^\sigma_\kappa(\beta) < \pi$. Then*

$$\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\kappa(\beta)$$

iff one of the following cases holds:

(1) $\alpha < \beta \wedge \alpha,\xi,\pi\in C(\beta,\Psi^\sigma_\kappa(\beta)) \wedge \Psi^\xi_\pi(\alpha) < \kappa$.

(2) $\beta \le \alpha \wedge \{\beta,\sigma,\kappa\} \nsubseteq C(\alpha,\Psi^\xi_\pi(\alpha))$.

(3) $\alpha = \beta \wedge \kappa = \pi \wedge \xi < \sigma \wedge \xi\in C(\sigma,\Psi^\sigma_\kappa(\beta))$.

(4) $\sigma < \xi \wedge \sigma\notin C(\xi,\Psi^\xi_\pi(\alpha))$.

Proof. From (1) it follows $\Psi^\xi_\pi(\alpha)\in C(\beta,\Psi^\sigma_\kappa(\beta))\cap\kappa$, whence $\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\kappa(\beta)$.

(2) yields $\{\beta,\sigma,\kappa\} \nsubseteq C(\beta,\Psi^\xi_\pi(\alpha))$; so, because of $\{\beta,\sigma,\kappa\} \subseteq C(\beta,\Psi^\sigma_\kappa(\beta))$, this becomes $\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\kappa(\beta)$.

(3) implies that M^ξ is stationary in $\Psi^\sigma_\kappa(\beta)$. As $\alpha,\pi,\xi\in C(\beta,\Psi^\sigma_\kappa(\beta))$, $\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\kappa(\beta)$ follows from 4.16.

(4) yields $\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\kappa(\beta)$ since $\sigma\in C(\sigma,\Psi^\sigma_\kappa(\beta))$.

Next, assume $\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\kappa(\beta)$. Then $\Psi^\xi_\pi(\alpha) < \kappa$. We have to show that one of (1)–(4) holds.

First, assume $\alpha < \beta$. From $\{\alpha,\xi,\pi\} \nsubseteq C(\beta,\Psi^\sigma_\kappa(\beta))$ we would get $\{\alpha,\xi,\pi\} \nsubseteq C(\alpha,\Psi^\xi_\pi(\alpha))$, contradicting $\Psi^\xi_\pi(\alpha) < \pi$. So (1) must be the case.

If $\beta < \alpha$, then $\{\beta, \sigma, \kappa\} \subseteq C(\alpha, \Psi^\xi_\pi(\alpha))$ cannot hold since this would imply $\Psi^\sigma_\kappa(\beta) \in C(\alpha, \Psi^\xi_\pi(\alpha)) \cap \pi$ and therefore $\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\kappa(\beta)$. This shows that $\beta < \alpha$ implies (2).

Finally, suppose $\alpha = \beta$. If $\kappa < \pi$, then $\kappa \notin C(\alpha, \Psi^\xi_\pi(\alpha))$; whence (2). $\pi < \kappa$ would force $\pi \in C(\alpha, \Psi^\sigma_\kappa(\beta)) \cap \kappa = \Psi^\sigma_\kappa(\beta)$, contradicting $\Psi^\sigma_\kappa(\beta) < \pi$.

So it remains to prove the assertion when $\alpha = \beta$ and $\pi = \kappa$. If $\sigma \notin C(\alpha, \Psi^\xi_\pi(\alpha))$, then (2) is satisfied. So assume $\sigma \in C(\alpha, \Psi^\xi_\pi(\alpha))$. From $\Psi^\xi_\pi(\alpha) < \pi$ we get $\Psi^\xi_\pi(\alpha) \in M^\xi$, in particular, $\xi \in C(\xi, \Psi^\xi_\pi(\alpha))$. Also, by assumption, we have $\Psi^\xi_\pi(\alpha) < \Psi^\sigma_\kappa(\beta)$. Consequently, if $\xi < \sigma$, then $\xi \in C(\sigma, \Psi^\sigma_\kappa(\beta))$, so (3) holds. 4.17 excludes that $\xi = \sigma$. Furthermore, $\sigma < \xi \wedge \sigma \in C(\xi, \Psi^\xi_\pi(\alpha))$ can be excluded since this would lead to the contradiction $\Psi^\sigma_\kappa(\beta) < \Psi^\xi_\pi(\alpha)$ by 4.16. Therefore $\sigma < \xi$ yields (4). □

Proposition 4.21

$$\Psi^\xi_\pi(\alpha) < \Xi(\beta) \iff [\pi \leq \Xi(\beta) \vee (\beta < \alpha \wedge \beta \notin C(\alpha, \Psi^\xi_\pi(\alpha)))]$$

Proof. "$\Leftarrow$" is immediate.

To verify "$\Rightarrow$", we assume $\Psi^\xi_\pi(\alpha) < \Xi(\beta)$ and $\Xi(\beta) < \pi$. We have to verify $\beta < \alpha \wedge \beta \notin C(\alpha, \Psi^\xi_\pi(\alpha))$.

$\alpha \leq \beta$ would imply $\alpha, \xi, \pi \in C(\alpha, \Psi^\xi_\pi(\alpha)) \subseteq C(\beta, \Xi(\beta))$, and hence the contradiction $\pi < \Xi(\beta)$. So we must have $\beta < \alpha$. If $\beta \in C(\alpha, \Psi^\xi_\pi(\alpha))$, then $\Xi(\beta) \in C(\alpha, \Psi^\xi_\pi(\alpha)) \cap \pi$, yielding the contradiction $\Xi(\beta) < \Psi^\xi_\pi(\alpha)$. □

5 The ordinal notation system $\mathcal{T}(\mathcal{K})$

We are going to define a set of ordinals $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{C}(\mathcal{K}^\Gamma, \prime)$ in conjunction with a function m which assigns to inaccessibles $\pi \in \mathcal{T}(\mathcal{K}) \cap \mathcal{K}$ the maximal α with $\pi \in M^\alpha$. However, $m(\pi)$ will be defined "constructively" from a normal form representation of π, and only later we shall verify the identity

$$(*) \qquad m(\pi) = sup\{\beta : \pi \in M^\beta\}.$$

We shall demand closure of $\mathcal{T}(\mathcal{K})$ under Ψ^ξ_π only when M^ξ is stationary in π (and $\xi, \pi \in \mathcal{T}(\mathcal{K})$). It will transpire that, for $\pi \in \mathcal{T}(\mathcal{K})$, stationarity of M^ξ in π is equivalent to $\xi \in C(m(\pi), \pi) \cap m(\pi)$.

Finally, by utilizing normal forms and the $<$–comparisons of the previous Section, we will come to see that $\langle \mathcal{T}(\mathcal{K}), < \rangle$ gives rise to a primitive recursive ordinal notation system.

Definition 5.1 The set of ordinals $\mathcal{T}(\mathcal{K})$ and a function

$$m : \mathcal{T}(\mathcal{K}) \cap \mathcal{R}]\} \longrightarrow \mathcal{T}(\mathcal{K})$$

are inductively defined by the following clauses.

(T1) $0, \mathcal{K} \in \mathcal{T}(\mathcal{K})$.

(T2) If $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{T}(\mathcal{K})$, then $\alpha \in \mathcal{T}(\mathcal{K})$.

(T3) If $\alpha =_{NF} \varphi\alpha_1\alpha_2$ with $\alpha_1, \alpha_2 \in \mathcal{T}(\mathcal{K})$, then $\alpha \in \mathcal{T}(\mathcal{K})$.

(T4) If $\xi \in \mathcal{T}(\mathcal{K}) \cap \mathcal{K}$ and $0 < \xi < \Omega_\xi$, then $\Omega_\xi \in \mathcal{T}(\mathcal{K})$. If further $\Omega_\xi \in Reg$, i.e. $\xi = \xi_0 + 1$ for some ξ_0, then $m(\Omega_\xi) = 1$.

(T5) If $\alpha \in \mathcal{T}(\mathcal{K})$ and $0 < \alpha$, then $\Xi(\alpha) \in \mathcal{T}(\mathcal{K})$ and $m(\Xi(\alpha)) = \alpha$.

(T6) If $\alpha, \xi, \pi \in \mathcal{T}(\mathcal{K})$ and $\alpha, \xi, \pi \in C(\alpha, \pi)$ and $\xi \le \alpha$ and $\xi \in C(m(\pi), \pi) \cap m(\pi)$, then $\Psi^\xi_\pi(\alpha) \in \mathcal{T}(\mathcal{K})$.
$m(\Psi^\xi_\pi(\alpha)) = \xi$, providing that $\xi > 0$.

We shall write $\delta =_{NF} \Psi^\xi_\pi(\alpha)$ if $\delta = \Psi^\xi_\pi(\alpha)$ and the requirements of (T6) are fulfilled.

The meaning of the function m and the condition $\xi \in C(m(\pi), \pi) \cap m(\pi)$ in (T6) are elucidated in the following Lemma.

Lemma 5.2 *Let $\delta \in \mathcal{T}(\mathcal{K})$. Then:*

(i)(a) $\delta \in C(\mathcal{K}^\Gamma, 0)$.

(i)(b) When δ is weakly inaccessible and $\delta < \mathcal{K}$, then $\delta \in M^{m(\delta)}$; moreover, $M^{m(\delta)}$ is **not** *stationary in δ and $m(\delta) = sup\{\beta : \delta \in M^\beta\}$.*

(ii) If $\pi, \xi \in \mathcal{T}(\mathcal{K})$, then M^ξ is stationary in π iff $\xi \in C(m(\pi), \pi) \cap m(\pi)$.

(iii) The clauses defining $\mathcal{T}(\mathcal{K})$ are deterministic, i.e., for each $\beta \in \mathcal{T}(\mathcal{K})$, there is only one way to get into $\mathcal{T}(\mathcal{K})$. Whence, each ordinal in $\mathcal{T}(\mathcal{K})$ can be denoted uniquely using only the symbols $0, \mathcal{K}, +, \varphi, \Omega, \Xi, \Psi$.

Proof. (i): We prove $(a), (b)$ simultaneously by induction on the definition of $\delta \in \mathcal{T}(\mathcal{K})$. During the proof, we frequently use the fact that $C(\mathcal{K}^\Gamma, 0) \subseteq \mathcal{K}^\Gamma$, which easily follows from the definition of $C(\mathcal{K}^\Gamma, 0)$.

Suppose $\delta = \Xi(\alpha)$ with $\alpha \in \mathcal{T}(\mathcal{K})$. The induction hypothesis yields $\alpha \in C(\mathcal{K}^\Gamma, 0) \cap \mathcal{K}^\Gamma$. Therefore, $\delta \in C(\mathcal{K}^\Gamma, 0)$ and $m(\delta) = \alpha$ and, according to 4.12, $\delta \in M^{\pi(\delta)}$. If $\delta \in M^\beta$ for some $\beta > \alpha$, then, as $\alpha \in C(\alpha, \delta)$, we would get $\alpha \in C(\beta, \delta) \cap \beta$ and thus the contradiction that M^α is stationary in $\Xi(\alpha)$. Hence, $m(\delta) =_{NF} sup\{\beta : \delta \in M^\beta\}$.

Suppose $\delta = \Psi^\xi_\pi(\alpha)$. The induction hypothesis yields $\alpha, \xi, \pi \in C(\mathcal{K}^\Gamma, 0)$, so $\delta \in C(\mathcal{K}^\Gamma, 0)$. Assume further that δ is weakly inaccessible. Then, by 4.19(iii) , π must be weakly inaccessible, too, and $\xi > 0$. The induction hypothesis yields $\pi \in M^{m(\pi)}$. Hence, from $\xi \in C(m(\pi), \pi) \cap m(\pi)$, it follows that M^ξ is stationary in π. So, using 4.16, we can infer that $\delta \in M^\xi$, M^ξ is not stationary in δ and $\xi = sup\{\beta : \delta \in M^\beta\}$. This gives the assertion since $m(\delta) = \xi$.

Finally, if δ enters $\mathcal{T}(\mathcal{K})$ by one of the clauses (T1),(T2),(T3),(T4), then (a) is immediate by the inductive assumption.

(ii): First, assume that M^ξ is stationary in π. Observe that (ii) is trivial for successor cardinals. So let π be weakly inaccessible. Then, using 4.11(vii), $\pi \in M^\xi$; thus $\xi < m(\pi)$ by (i)(b). Choosing $\rho \in M^\xi \cap \pi$, we get $\xi \in C(\xi, \rho)$; whence $\xi \in C(m(\pi), \pi) \cap m(\pi)$.

On the other hand, $\xi \in C(m(\pi), \pi) \cap m(\pi)$ implies that M^ξ is stationary in π since $\pi \in M^{m(\pi)}$ by $(i)(b)$.

(iii) follows from 4.12, 4.15, 4.16, 4.17, and 4.19. □

To conceive of $\langle \mathcal{T}(\mathcal{K}), < \rangle$ as a primitive recursive ordinal notation system, we need to be able to determine whether an arbitrary term, composed of the symbols $0, \mathcal{K}, +, \varphi, \Omega, \Xi, \Psi$, denotes an ordinal from $\mathcal{T}(\mathcal{K})$, and, moreover, given two terms denoting ordinals from $\mathcal{T}(\mathcal{K})$, the order between the denoted ordinals should be computable from the order of ordinals denoted by proper subterms. An important step towards such a decision procedure is taken in the following definition.

Definition 5.3 By induction on the definition of $\alpha \in \mathcal{T}(\mathcal{K})$, $K_\delta(\alpha)$ is defined as follows.

(K1) $K_\delta(\mathcal{K}) = \emptyset$.

(K2) If $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n$ or $\alpha =_{NF} \varphi\alpha_1\alpha_2$, then $K_\delta(\alpha) = \bigcup_{1 \le i \le n} K_\delta(\alpha_i)$.

(K3) If $\alpha = \Omega_\xi$ with $0 < \xi < \Omega_\xi < \mathcal{K}$, then $K_\delta(\alpha) = K_\delta(\xi)$.

(K4) If $\alpha = \Xi(\beta)$, then

$$K_\delta(\alpha) = \begin{cases} \emptyset & \textit{if } \alpha < \delta \\ K_\delta(\beta) \cup \{\beta\} & \textit{else.} \end{cases}$$

(K5) If $\alpha =_{NF} \Psi^\sigma_\kappa(\beta)$, then

$$K_\delta(\alpha) = \begin{cases} \emptyset & \textit{if } \alpha < \delta \\ K_\delta(\kappa) \cup K_\delta(\sigma) \cup K_\delta(\beta) \cup \{\beta\} & \textit{else.} \end{cases}$$

Lemma 5.4 *If $\alpha \in \mathcal{T}(\mathcal{K})$ and δ, γ are arbitrary ordinals, then*

$$\alpha \in C(\gamma, \delta) \iff K_\delta(\alpha) < \gamma.$$

Proof. This is straightforwardly verified by induction on $\alpha \in \mathcal{T}(\mathcal{K})$. □

Given $\alpha, \xi, \pi \in \mathcal{T}(\mathcal{K})$, Lemma 5.4 enables us to check all the conditions demanded in (T6) of Definition 5.1, solely, by inspecting the inductive generation that α, ξ, π have as elements of $\mathcal{T}(\mathcal{K})$. Therefore, in conjunction with the recursive characterization of the $<$–relation of the previous Section, we are led to a primitive recursive description of $\langle \mathcal{T}(\mathcal{K}), < \rangle$, when we identify the elements of $\mathcal{T}(\mathcal{K})$ with the terms denoting them. However, there is no reason to write out such a primitive recursive definition in detail since it does not convey any more insights.

6 The Calculus $RS(\mathcal{K})$

It is well known that the axioms of Peano Arithmetic, *PA*, can be derived in a sequent calculus, PA_ω, augmented by an infinitary rule, the so–called *ω–rule*[9]

$$\frac{\Gamma, A(\bar{n}) \text{ for all } n}{\Gamma, \forall x A(x)}.$$

An ordinal analysis for *PA* is then attained as follows:

- Each PA–proof can be "unfolded" into a PA_ω–proof of the same sequent.
- Each such PA_ω–proof can be transformed into a cut–free PA_ω–proof of the same sequent of length $< \varepsilon_0$.

In order to obtain a similar result for set theories like *KP*, we have to work a bit harder. Guided by the ordinal analysis of *PA*, we would like to invent an infinitary rule which, when added to *KP*, enables us to eliminate cuts. As opposed to the natural numbers, it is not clear how to bestow a canonical name to each element of the set–theoretic universe. However, within the confines of the constructible universe, which is made from the ordinals, it is pretty obvious how to "name" sets once we have names for ordinals at our disposal. Recall that L_α, the α^{th} level of Gödel's constructible hierarchy L, is defined by $L_0 = \emptyset$, $L_\lambda = \bigcup\{L_\beta : \beta < \lambda\}$ for limits λ, and $L_{\beta+1} = \big\{X : X \subseteq L_\beta;\ X \text{ definable over } \langle L_\beta, \in\rangle\big\}$. So any element of L of level α is definable from elements of L with levels $< \alpha$ and L_α.

6.1 The Language of $RS(\mathcal{K})$

Henceforth, we shall restrict ourselves to ordinals from $\mathcal{T}(\mathcal{K})$.

Definition 6.1 We extend the language of set theory, $\mathcal{L}$, by new unary predicate symbols Ad^α for every $\alpha \in \mathcal{T}(\mathcal{K})$. The augmented language will be denoted by $\mathcal{L}_{Ad}$.

The *atomic formulae* of $\mathcal{L}_{Ad}$ are those of either form $(a \in b)$, $\neg(a \in b)$, $Ad^\alpha(a)$, or $\neg Ad^\alpha(a)$. The $\mathcal{L}_{Ad}$*–formulae* are obtained from atomic ones by closing off under $\wedge, \vee, (\exists x \in a), (\forall x \in a), \exists x$, and $\forall x$.

Definition 6.2 The $\mathcal{L}_{RS(\mathcal{K})}$*–terms* and their *levels* are generated as follows.

1. For each α, $\mathbb{L}_\alpha$ is an $\mathcal{L}_{RS(\mathcal{K})}$–term of level α.
2. The formal expression $[x \in \mathbb{L}_\alpha : F[x, s_1, \cdots s_n]^{\mathbb{L}_\alpha}]$ is an $\mathcal{L}_{RS(\mathcal{K})}$–term of level α if $F[a, b_1, \cdots, b_n]$ is an $\mathcal{L}_{Ad}$–formula and $s_1, \cdots, s_n$ are $\mathcal{L}_{RS(\mathcal{K})}$–terms with levels $< \alpha$.

[9] $\bar{n}$ stands for the n^{th} numeral

We shall denote the level of an $\mathcal{L}_{RS(\mathcal{K})}$–term t by $|t|$; $t \in Term(\alpha)$ stands for $|t| < \alpha$ and $t \in Term$ for $t \in Term(\mathcal{K})$.

The $\mathcal{L}_{RS(\mathcal{K})}$*–formulae* are the expressions of the form $F[s_1, \ldots, s_n]^{\mathbb{L}_\mathcal{K}}$, where $F[a_1, \ldots, a_n]$ is an $\mathcal{L}_{Ad}$–formula and $s_1, \ldots, s_n \in Term$.

For technical convenience, we let $\neg A$ be the formula which arises from A by (i) putting $\neg$ in front of each atomic formula, (ii) replacing $\wedge, \vee, (\forall x \in a), (\exists x \in a)$ by $\vee, \wedge, (\exists x \in a), (\forall x \in a)$, respectively, and (iii) dropping double negations.

Convention: In the sequel, $\mathcal{L}_{RS(\mathcal{K})}$–formulae will be referred to as formulae. The same usage applies to $\mathcal{L}_{RS(\mathcal{K})}$–terms.

Definition 6.3 If $\mathfrak{x}$ is a term or a formula, then

$$k(\mathfrak{x}) := \{\alpha : \mathbb{L}_\alpha \text{ occurs in } \mathfrak{x}\}.$$

Here any occurrence of $\mathbb{L}_\alpha$, i.e. also those inside of terms, has to be considered. For technical convenience, we put $k(0) := k(1) := \emptyset$.

We set $|\mathfrak{x}| := max(k(\mathfrak{x}) \cup \{0\})$ and $|0| := |1| := 0$.

If $\mathfrak{X}$ is a finite set consisting of objects of the above kind, put

$$k(\mathfrak{X}) := \bigcup \{k(\mathfrak{x}) : \mathfrak{x} \in \mathfrak{X}\}$$

and

$$|k(\mathfrak{X})| := sup\{|k(\mathfrak{x})| : \mathfrak{x} \in \mathfrak{X}\}.$$

Definition 6.4 We use the relation $\equiv$ to mean syntactical identity. For terms s, t with $|s| < |t|$ we set

$$s \mathring{\in} t \equiv \begin{cases} B(s) & \text{if} \quad t \equiv [x \in \mathbb{L}_\beta : B(x)] \\ s \notin \mathbb{L}_0 & \text{if} \quad t \equiv \mathbb{L}_\beta. \end{cases}$$

Observe that $s \in t$ and $s \mathring{\in} t$ have the same truth value under the standard interpretation in the constructible hierarchy.

6.2 The Rules of $RS(\mathcal{K})$

Next we introduce a calculus, $RS(\mathcal{K})$, with infinitary rules. $A, B, C, \ldots, F(t), G(t), \ldots$ range over $\mathcal{L}_{RS(\mathcal{K})}$–formulae. We denote by upper case Greek letters $\Gamma, \Delta, \Lambda, \ldots$ finite sets of $\mathcal{L}_{RS(\mathcal{K})}$–formulae. The intended meaning of $\Gamma = \{A_1, \cdots, A_n\}$ is the disjunction $A_1 \vee \cdots \vee A_n$. Γ, A stands for $\Gamma \cup \{A\}$ etc.. We also use the shorthands $r \neq s := \neg(r = s)$ and $r \notin t := \neg(r \in t)$.

An $\mathcal{L}_{RS}$–formula is said to be $\Delta_0(\alpha)$ if it contains only terms with levels $< \alpha$. An $\mathcal{L}_{RS}$–formula A is $\Pi_k(\alpha)$ if it has the form

$$(\forall x_1 \in \mathbb{L}_\alpha) \cdots (Q_k x_k \in \mathbb{L}_\alpha) F(x_1, \ldots, x_k),$$

where the k quantifiers in front are alternating and $F(\mathbb{L}_0, \ldots, \mathbb{L}_0)$ is $\Delta_0(\alpha)$. Analoguously, one defines $\Sigma_k(\alpha)$–formulae.

Given an $\mathcal{L}_{RS}$–formulae A and terms s, t, we denote by $A^{(s,t)}$ the formula which arises from A by replacing all the quantifiers $(\exists x \in t)$ and $(\forall x \in t)$ by $(\exists x \in s)$ and $(\forall x \in s)$, respectively. To economize on subscripts, we also write $A^{(s,\alpha)}$ for $A^{(s,\mathbb{L}_\alpha)}$ and $A^{(\beta,\alpha)}$ instead of $A^{(\mathbb{L}_\beta,\mathbb{L}_\alpha)}$.

Definition 6.5 The *rules* of $RS(\mathcal{K})$ are:

$(\wedge)$ $$\frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

$(\vee)$ $$\frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$(\forall)$ $$\frac{\cdots \Gamma, s \mathring{\in} t \rightarrow F(s) \cdots (s \in Term(|t|))}{\Gamma, (\forall x \in t) F(x)}$$

$(\exists)$ $$\frac{\Gamma, s \mathring{\in} t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } s \in Term(|t|)$$

$(\notin)$ $$\frac{\cdots \Gamma, s \mathring{\in} t \rightarrow r \neq s \cdots \cdots (s \in Term(|t|))}{\Gamma, r \notin t}$$

$(\in)$ $$\frac{\Gamma, s \mathring{\in} t \ \wedge \ r = s}{\Gamma, r \in t} \quad \text{if } s \in Term(|t|)$$

$(\neg Ad^\alpha)$ $$\frac{\cdots \Gamma, \mathbb{L}_\rho \neq t \cdots (\rho \in M^\alpha; \ \rho \leq |t|)}{\Gamma, \neg Ad^\alpha(t)}$$

(Ad^α) $$\frac{\Gamma, \mathbb{L}_\rho = t}{\Gamma, Ad^\alpha(t)} \quad \text{if } \rho \in M^\alpha \text{ and } \rho \leq |t|$$

(Cut) $$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

$(Ref_{\mathcal{K}})$ $$\frac{\Gamma, A}{\Gamma, (\exists z \in \mathbb{L}_{\mathcal{K}})[Tran(z) \wedge z \neq \emptyset \wedge A^{(z,\mathcal{K})}]} \quad \text{if } A \in \Pi_3(\mathcal{K})$$

(Ref^ξ_π) $$\frac{\Gamma, F(s)}{\Gamma, (\exists z \in \mathbb{L}_\pi)[Ad^\xi(z) \wedge (\exists u \in z) F(u)^{(z,\pi)}]} \quad \text{if } F(s) \in \Pi_2(\pi),$$

where (Ref^{ξ}_{π}) comes with the proviso that M^{ξ} be stationary in π.

Remark 6.6 *At first glance, the rule (Ref^{ξ}_{π}) might loom complicated. As a matter of fact, instead, we could have adopted the rule:*

$$(Ref^{\xi}_{\pi})^* \qquad \frac{\Gamma, A}{\Gamma, (\exists z\in \mathbb{L}_{\pi})[Ad^{\xi}(z) \wedge A^{(z,\pi)}]} \quad \textit{if } A\in \Pi_2(\pi).$$

But latter on (cf. Lemma 8.12), we will need to derive $\Sigma_3(\pi)$–reflection and this can be accomplished more easily with (Ref^{ξ}_{π}) at our disposal.

6.3 $\mathcal{H}$–controlled derivations

If we dropped the rules $(Ref_{\mathcal{K}})$ and (Ref^{ξ}_{π}) from $RS(\mathcal{K})$, the remaining calculus would enjoy full cut elimination owing to the symmetry of the pairs of rules $\langle\,(\wedge),\ (\vee)\,\rangle$, $\langle\,(\forall),\ (\exists)\,\rangle$, $\langle\,(\notin),\ (\in)\,\rangle$, $\langle\,(Ad^{\alpha}),\ (\neg Ad^{\alpha})\,\rangle$. However, partial cut elimination for $RS(\mathcal{K})$ can be attained by delimiting a collection of derivations of a very uniform kind.

To define uniform derivations, we shall find it useful to apply the notion of operator controlled derivations of Buchholz [1993].

Definition 6.7 Let $P(On) = \{X : X \text{ is a set of ordinals}\}$.

A class function

$$\mathcal{H} : P(On) \to P(On)$$

will be called *operator* if the following conditions are met for all $X, X' \in P(On)$:

(H0) $0\in\mathcal{H}(X)$.

(H1) For $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$,

$$\alpha\in\mathcal{H}(X) \iff \alpha_1, ..., \alpha_n\in\mathcal{H}(X).$$

(In particular, (H1) implies that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma \mapsto \omega^{\sigma}$, i.e., if $\alpha, \beta\in\mathcal{H}(X)$, then $\alpha + \beta, \omega^{\alpha}\in\mathcal{H}(X)$.)

(H2) $X \subseteq \mathcal{H}(X)$

(H3) $X' \subseteq \mathcal{H}(X) \implies \mathcal{H}(X') \subseteq \mathcal{H}(X)$.

Definition 6.8 (i) When f is a mapping $f : On^k \longrightarrow On$, then $\mathcal{H}$ is said to be *closed* under f, if, for all $X\in P(On)$ and $\alpha_1, \ldots, \alpha_k\in\mathcal{H}(X)$,

$$f(\alpha_1, \ldots, \alpha_k)\in\mathcal{H}(X).$$

(ii) $\alpha\in\mathcal{H} := \alpha\in\mathcal{H}(\emptyset)$; $\quad s\in\mathcal{H} := k(s) \subset \mathcal{H}$.

(iii) $X \subseteq \mathcal{H} := X \subseteq \mathcal{H}(\emptyset)$.

(iv) For $s \in Term$ let $\mathcal{H}[s]$ denote the operator

$$\big(X \mapsto \mathcal{H}(k(s) \cup X)\big)_{X \in P(On)}.$$

(v) If $\mathfrak{X}$ is set consisting of terms, formulae, and possibly elements from $\{0, 1\}$, then

$$\mathcal{H}[\mathfrak{X}](X) := \mathcal{H}(k(\mathfrak{X}) \cup X).$$

We shall also write $\mathcal{H}[\mathfrak{X}, s_1, \ldots, s_n]$ for $\mathcal{H}[\mathfrak{X} \cup \{s_1, \ldots, s_n\}]$, and occasionally $\mathcal{H}[\mathfrak{X}, \pi]$ instead of $\mathcal{H}[\mathfrak{X}, \mathbb{L}_\pi]$.

The next Lemma garners some simple properties of operators.

Lemma 6.9 *If $\mathcal{H}$ is an operator, then:*

(i) $\mathcal{H}[\mathfrak{X}]$ *is an operator.*

(ii) $k(\mathfrak{X}) \subset \mathcal{H} \implies \mathcal{H}[\mathfrak{X}] = \mathcal{H}$.

(iii) $\forall X, X' \in P(On)[X' \subseteq X \Longrightarrow \mathcal{H}(X') \subseteq \mathcal{H}(X)]$.

Definition 6.10 To each $\mathcal{L}_{RS(\mathcal{K})}$–formula A we assign either a (possibly infinite) disjunction $\bigvee(A_\iota)_{\iota \in J}$ or conjunction $\bigwedge(A_\iota)_{\iota \in J}$ of $\mathcal{L}_{RS(\mathcal{K})}$–formulae. This assignment will be indicated by $A \cong \bigvee(A_\iota)_{\iota \in J}$ and $A \cong \bigwedge(A_\iota)_{\iota \in J}$, respectively.

- $r \in t \cong \bigvee(s \mathring{\in} t \wedge r = s)_{s \in Term(|t|)}$
- $Ad^\alpha(t) \cong \bigvee(\mathbb{L}_\rho = t)_{\mathbb{L}_\rho \in J}$, where $J := \{\mathbb{L}_\eta : \eta \in M^\alpha; \eta \leq |t|\}$
- $(\exists x \in t)F(x) \cong \bigvee(s \mathring{\in} t \wedge F(s))_{s \in Term(|t|)}$
- $A_0 \vee A_1 \cong \bigvee(A_\iota)_{\iota \in \{0,1\}}$
- $\neg A \cong \bigwedge(\neg A_\iota)_{\iota \in J}$, if $A \cong \bigvee(A_\iota)_{\iota \in J}$.

Using this representation of formulae, we can define the *subformulae* of a formula as follows.[10] When $A \cong \bigwedge(A_\iota)_{\iota \in J}$ or $A \cong \bigvee(A_\iota)_{\iota \in J}$, then B is a subformula of A if $B \equiv A$ or, for some $\iota \in \mathbb{J}$, B is a subformula of A_ι.

Since we also want to keep track of the complexity of cuts appearing in derivations, we endow each formula with an ordinal rank.

[10] That this constitutes a legitimite inductive definition will follow from Lemma 6.12

Definition 6.11 The *rank* of formulae and terms is determined as follows.

1. $rk(\mathbb{L}_\alpha) := \omega \cdot \alpha$.
2. $rk([x \in \mathbb{L}_\alpha : F(x)]) := max\{\omega \cdot \alpha + 1, rk(F(\mathbb{L}_0)) + 2\}$.
3. $rk(s \in t) := rk(s \notin t) := max\{rk(s) + 6, rk(t) + 1\}$.
4. $rk(Ad^\alpha(s)) := rk(\neg Ad^\alpha(s)) := rk(s) + 5$.
5. $rk((\exists x \in t)F(x)) := rk((\forall x \in t)F(x)) := max\{rk(t), rk(F(\mathbb{L}_0)) + 2\}$.
6. $rk(A \wedge B) := rk(A \vee B) := max\{rk(A), rk(B)\} + 1$.

There is plenty of leeway in designing the actual rank of a formula. However, it is crucial that it satisfies the following property.

Lemma 6.12 *If* $A \cong \bigvee(A_\iota)_{\iota \in J}$ *or* $A \cong \bigwedge(A_\iota)_{\iota \in J}$*, then*

$$(\forall \iota \in J)\ [rk(A_\iota) < rk(A)].$$

A proof for Lemma 6.12 is given in Buchholz [1993], Lemma 1.9. □

Using the formula representation of Definition 6.10, notwithstanding the many rules of $RS(\mathcal{K})$, the notion of $\mathcal{H}$–controlled derivability can be defined concisely. We shall use $J \restriction \alpha$ to denote the set $\{\iota \in J : |\iota| < \alpha\}$.

Definition 6.13 Let $\mathcal{H}$ be an operator and let Γ be a finite set of $RS(\mathcal{K})$–formulae. $\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma$ is defined by recursion on α via

$$\{\alpha\} \cup k(\Gamma) \subset \mathcal{H}$$

and the following inductive clauses:

$$(\bigvee) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vert\!\!-}} \Lambda, A_{\iota_0}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Lambda, \bigvee(A_\iota)_{\iota\in J}} \qquad \begin{array}{r}\alpha_0 < \alpha \\ \iota_0 \in J \restriction \alpha\end{array}$$

$$(\bigwedge) \quad \frac{\mathcal{H}[\iota] \overset{\alpha_\iota}{\underset{\rho}{\vert\!\!-}} \Lambda, A_\iota \textit{ for all } \iota \in J}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Lambda, \bigwedge(A_\iota)_{\iota\in J}} \qquad |\iota| \leq \alpha_\iota < \alpha$$

$$(Cut) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vert\!\!-}} \Lambda, \mathcal{B} \qquad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vert\!\!-}} \Lambda, \neg\mathcal{B}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Lambda} \qquad \begin{array}{r}\alpha_0 < \alpha \\ rk(B) < \rho\end{array}$$

$$(Ref_{\mathcal{K}}) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vert\!\!-}} \Lambda, A}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Lambda, (\exists z \in \mathbb{L}_{\mathcal{K}})[Tran(z) \wedge z \neq \emptyset \wedge A^{(z,\mathcal{K})}]} \qquad \begin{array}{r}\alpha_0, \mathcal{K} < \alpha \\ A \in \Pi_3(\mathcal{K})\end{array}$$

$$(Ref_\pi^\xi) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vert\!\!-}} \Lambda, F(s)}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Lambda, (\exists z \in \mathbb{L}_\pi)[Ad^\xi(z) \ \wedge \ (\exists u \in z) F(u)^{(z,\pi)}]} \qquad \begin{array}{r}\alpha_0 + 1, \pi < \alpha \\ \xi \in \mathcal{H} \\ F(s) \in \Pi_2(\pi) \\ stat(\xi, \pi)\end{array}$$

where $stat(\xi, \pi)$ means that M^ξ is stationary in π; according to 5.2(ii) this is equivalent to $\xi \in C(m(\pi), \pi) \cap m(\pi)$, and thus is a decidable property by 5.4.

Remark 6.14 In (Ref_π^ξ) we can assume that $s \in \mathcal{H}$, for if s occurs in $F(s)$ then this is a consequence of $k(\Lambda, F(s)) \subseteq \mathcal{H}$, and if s does not occur in $F(s)$, then $F(s) \equiv F(\mathbb{L}_0)$ so that we could assume $s \equiv \mathbb{L}_0$ which would also entail $s \in \mathcal{H}$.

Henceforth, we shall tacitly make this assumption when dealing with (Ref_π^ξ).

The following observations are easily eastablished by induction on α.

Lemma 6.15 *(i)* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Gamma \ \wedge \ \alpha \leq \alpha' \in \mathcal{H} \ \wedge \ \rho \leq \rho' \ \wedge \ k(\Lambda) \subseteq \mathcal{H} \implies \mathcal{H} \overset{\alpha'}{\underset{\rho'}{\vert\!\!-}} \Gamma, \Lambda\,.$

(ii) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Gamma, A \vee B \implies \mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Gamma, A, B\,.$

(iii) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Gamma, (\forall x \in \mathbb{L}_\beta) F(x) \ \wedge \ \gamma \in \mathcal{H} \ \wedge \ \gamma \leq \beta \implies \mathcal{H} \overset{\alpha}{\underset{\rho}{\vert\!\!-}} \Gamma, (\forall x \in \mathbb{L}_\gamma) F(x)\,.$

7 Predicative Cut Elimination and Bounding

Cuts in $RS(\mathcal{K})$–derivations whose cut formulae have not been introduced previously by a Π_3 or Π_2–reflection inference will be called *uncritical*. Applying the usual cut elimination procedure for infinitary logic, uncritical cuts can be replaced by cuts with lesser rank.

In this Section we will deal with elimination of uncritical cuts in $\mathcal{L}_{RS}$ in its quantitative aspects. Since these results have literally the same proofs as their counterparts in Buchholz [1993], we refrain from repeating them here.

Besides cut elimination results, we show that existential quantifiers in $\mathcal{L}_{RS}$–derivations can always be "bounded" by the length of the derivation.

Lemma 7.1 (Inversion)

$$\mathcal{H} \mathrel{|}\!\!\frac{\alpha}{\rho}\, \Gamma, \bigwedge(A_\iota)_{\iota\in J} \implies (\forall\iota\in J)\mathcal{H}[\iota] \mathrel{|}\!\!\frac{\alpha}{\rho}\, \Gamma, A_\iota$$

Proof. Use induction on α. □

The next Lemma relates the rank of a formula A, to its level, $|A|$ (see 6.3).

Lemma 7.2 *Let A, B be formulae and s, t be terms.*

(i) $rk(A) = \omega\cdot|A| + n$ *for some* $n < \omega$.

(ii) $rk(s) = \omega\cdot|s| + m$ *for some* $m < \omega$.

(iii) $|A| < |B| \implies rk(A) < rk(B)$.

(iv) $|s| < |t| \implies rk(s) < rk(t)$.

Proof. See Buchholz [1993], Lemma 1.9. □

Lemma 7.3 (Reduction Lemma) *Let $A \cong \bigvee(A_\iota)_{\iota\in J}$. Assume $\rho \notin Reg \cup \{\mathcal{K}\}$, where $\rho := rk(A)$. Then:*

$$\mathcal{H} \mathrel{|}\!\!\frac{\alpha}{\rho}\, \Lambda, \neg A \;\wedge\; \mathcal{H} \mathrel{|}\!\!\frac{\beta}{\rho}\, \Gamma, A \implies \mathcal{H} \mathrel{|}\!\!\frac{\alpha+\beta}{\rho}\, \Lambda, \Gamma$$

Proof. Use induction on β. For details see Buchholz [1993], Lemma 3.14. □

Theorem 7.4 (Predicative cut elimination) *Let $\mathcal{H}$ be closed under φ. If $\mathcal{H} \mathrel{|}\!\!\frac{\beta}{\rho+\omega^\alpha}\, \Gamma$, $[\rho, \rho+\omega^\alpha[\cap(Reg \cup \{\mathcal{K}\}) = \emptyset$, and $\alpha\in\mathcal{H}$, then*

$$\mathcal{H} \mathrel{|}\!\!\frac{\varphi\alpha\beta}{\rho}\, \Gamma\,.$$

Proof. By main induction on α and subsidiary induction on β (cf. Buchholz [1993], Theorem 3.16). □

Corollary 7.5 $\quad \mathcal{H} \mathrel{|}\!\!\frac{\beta}{\rho+1}\, \Gamma \;\wedge\; \rho\notin Reg \cup \{\mathcal{K}\} \implies \mathcal{H} \mathrel{|}\!\!\frac{\omega^\beta}{\rho}\, \Gamma\,.$

Lemma 7.6 (Bounding Lemma) *Let $\mu \in Reg \cup \{\mathcal{K}\}$ and $\beta \in \mathcal{H}$. If $\alpha \leq \beta < \mu$ and $B \in \Sigma_1(\mu)$, then*

$$\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma, B \implies \mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma, B^{(\beta,\mu)}\,.$$

Proof by induction on α. Since $\alpha < \mu$, B cannot be the principal formula of an inference (Ref_μ) or (Ref^ξ_μ).

If B is not the principal formula of the last inference, the assertion follows by using the inductive assumption on its premisses and reapplying the same inference. Let B be the principal formula of the last inference, which then must be $(\exists)$. B has the form $(\exists x \in \mathbb{L}_\mu) F(x)$ with $\Delta_0(\mu)$–formula $F(\mathbb{L}_0)$. Also,

$$\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma, B, s \mathring{\in} \mathbb{L}_\mu \wedge F(s)$$

for some $\alpha_0 < \alpha$ and $s \in Term(\mu)$ with $|\, s \,| < \alpha$. By the induction hypothesis,

$$\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma, B^{(\beta,\mu)}, s \mathring{\in} \mathbb{L}_\mu \wedge F(s)\,.$$

Since $|\, s \,| < \beta, \mu$, we have $s \mathring{\in} \mathbb{L}_\beta \equiv s \mathring{\in} \mathbb{L}_\mu$. Thus, applying $(\exists)$, the assertion follows. $\square$

8 Embeddings

The first part of this Section deals with an embedding of $KP + \Pi_3$–*Ref* into $RS(\mathcal{K})$. Regarding proofs, we will be drawing on Buchholz [1993] when the proof is literally the same.

Furthermore, we shall show, by virtue of reflection for $\Pi_2(\pi)$–formulae, that reflection provably propagates to $\Sigma_3(\pi)$–formulae. This is not very surprising, however, we will also need to control the quantitative repercussions which $\Sigma_3(\pi)$–reflection causes on the ordinal bounds of a given derivation. All these results will be needed in Section 10.

Definition 8.1 For $\Gamma = \{A_1, \ldots, A_n\}$ let

$$no(\Gamma) := \omega^{rk(A_1)} \# \cdots \# \omega^{rk(A_n)}.$$

We define

$$\Vdash \Gamma \quad :\Longleftrightarrow \quad \text{for all operators } \mathcal{H},\ \mathcal{H}[\Gamma] \mid\!\frac{no(\Gamma)}{0}\, \Gamma$$

and

$$\Vdash^\xi_\rho \Gamma \quad :\Longleftrightarrow \quad \text{for all operators } \mathcal{H},\ \mathcal{H}[\Gamma] \mid\!\frac{no(\Gamma)\#\xi}{\rho}\, \Gamma\,.$$

Lemma 8.2 *Let $s \subseteq t$ stand for the formula $(\forall x \in s)(x \in t)$.*

(i) $\Vdash A, \neg A$.

(ii) $\Vdash s \notin s$.

(iii) $\Vdash s \subseteq s$.

(iv) $\Vdash s \mathring{\notin} t, s \mathring{\in} t$ *for* $s \in Term(|t|)$.

(v) $\Vdash s \neq t, t = s$.

Proof. Buchholz [1993], Lemma 2.4, Lemma 2.5. □

Lemma 8.3

$$\Vdash [s_1 \neq t_1], \ldots, [s_n \neq t_n], \neg A(s_1, \ldots, s_n), A(t_1, \ldots, t_n).$$

Proof. Buchholz [1993], Lemma 2.7. □

Corollary 8.4 (Equality and Extensionality)

$$\Vdash s_1 \neq t_1, \ldots, s_n \neq t_n, \neg A(s_1, \ldots, s_n), A(t_1, \ldots, t_n).$$

Proof. Buchholz [1993], Theorem 2.9. □

Lemma 8.5 (Foundation)

$$\Vdash (\forall x \in \mathbb{L}_\alpha)[(\forall y \in x)F(y) \to F(x)] \;\longrightarrow\; (\forall x \in \mathbb{L}_\alpha)F(x).$$

Proof. Fix an operator $\mathcal{H}$. Let $A \equiv (\forall x \in \mathbb{L}_\alpha)[(\forall y \in x)F(y) \to F(x)]$. First, we show, by induction on $|s|$, that if $s \in Term(\alpha)$, then

$$(+) \qquad \mathcal{H}[A, s] \left|\frac{\omega^{rk(A)} \# \omega^{|s|+1}}{0}\right. \neg A, F(s)\,.$$

So assume that

$$\mathcal{H}[A, t] \left|\frac{\omega^{rk(A)} \# \omega^{|t|+1}}{0}\right. \neg A, F(t)$$

for all $t \in Term(|s|)$. Using $(\vee)$, this yields

$$\mathcal{H}[A, s, t] \left|\frac{\omega^{rk(A)} \# \omega^{|t|+1} + 1}{0}\right. \neg A, t \mathring{\in} s \to F(t)$$

for all $t \in Term(|s|)$, and hence

$$(1) \qquad \mathcal{H}[A, s] \left|\frac{\omega^{rk(A)} \# \omega^{|s|} + 2}{0}\right. \neg A, (\forall y \in s)F(y)$$

via $(\forall)$. Set $\eta := \omega^{rk(A)}\#\omega^{|s|}+2$. By Lemma 8.2(i), $\mathcal{H}[A,s] \overset{\eta}{\underset{0}{|}} \neg F(s), F(s)$; therefore, using (1) and $(\wedge)$,

$$\mathcal{H}[A,s] \overset{\eta+1}{\underset{0}{|}} \neg A, (\forall y\in s)F(y) \wedge \neg F(s), F(s)\,.$$

From the latter we obtain

$$\mathcal{H}[A,s] \overset{\eta+2}{\underset{0}{|}} \neg A, s\mathring{\in}\mathbb{L}_\alpha \wedge [(\forall y\in s)F(y) \wedge \neg F(s)], F(s)\,,$$

and hence $\mathcal{H}[A,s] \overset{\eta+3}{\underset{0}{|}} \neg A, (\exists x\mathring{\in}\mathbb{L}_\alpha)[(\forall y\in x)F(y) \wedge \neg F(x)], F(s)$ via $(\exists)$. This shows (+).

Finally, (+) enables us to deduce $\mathcal{H}[A,s] \overset{\omega^{rk(A)}\#\omega^{|s|+1}+1}{\underset{0}{|}} \neg A, s\mathring{\in}\mathbb{L}_\alpha \to F(s)$ from which the assertion follows by applying $(\forall)$ and $(\vee)$. □

Lemma 8.6 (Infinity Axiom) *If λ be a limit ordinal $> \omega$, then*

$$\Vdash \textit{(Infinity Axiom)}^{\mathbb{L}_\lambda},$$

i.e.,

$$\Vdash (\exists x\in\mathbb{L}_\lambda)[z \neq \emptyset \wedge (\forall y\in x)(\exists z\in x)(y\in z)].$$

Proof. Buchholz [1993], Theorem 2.9. □

Lemma 8.7 (Δ_0–Separation) *Let $A[a,b_1,\dots,b_n]$ be a Δ_0–formula of $\mathcal{L}_{Ad}$. If $\lambda \in Lim$ and $s,t_1,\dots,t_n\in Term(\lambda)$, then*

$$\Vdash (\exists y\in\mathbb{L}_\lambda)[(\forall x\in y)(x\in s \wedge A[x,t_1,\dots,t_n]) \wedge (\forall x\in s)(A[x,t_1,\dots,t_n] \to x\in y)].$$

More concisely, we can express this by “ $\Vdash (\Delta_0$–separation$)^{\mathbb{L}_\lambda}$ ”.

Proof. Buchholz [1993], Theorem 2.9. □

Lemma 8.8 (Pair and Union) *Assume $\lambda\in Lim$ and $s,t\in Term(\lambda)$.*

(i) $\Vdash (\exists z\in\mathbb{L}_\lambda)(s\in z \wedge t\in z)$.

(ii) $\Vdash (\exists z\in\mathbb{L}_\lambda)(\forall y\in s)(\forall x\in y)(x\in z)$.

Proof. Buchholz [1993], Theorem 2.9. □

Definition 8.9 The sequent calculus GML (“GML” stands for *“Grundmengenlehre”*) is defined as follows. The language of GML is $\mathcal{L}_{Ad}$. With the exception of Δ_0–collection, GML has the same axiom schemes as KP. (However, it is understood that the axiom schemes are defined with regard to $\mathcal{L}_{Ad}$. To be precise, GML comprises the axiom scheme of $\Delta_0(\mathcal{L}_{Ad})$–separation, whereas $\Delta_0(\mathcal{L}_{Ad})$–collection is not an axiom scheme of GML.)

Lemma 8.10 *Assume $\rho = \omega^\rho \le \mathcal{K}$. Let $\Gamma[\vec{a}\,] = \{A_1[\vec{a}\,], \ldots, A_k[\vec{a}\,]\}$ be a set of $\mathcal{L}_{Ad}$–formulae, where $\vec{a} = a_1, \ldots, a_n$. If $GML \vdash \Gamma[\vec{a}\,]$, then there exists $m < \omega$ such that, for all $\vec{s} = s_1, \ldots, s_n \in Term(\rho)$,*

$$\mathcal{H}[\Gamma[\vec{s}\,]^{\mathbb{L}_\rho}, \rho] \mathrel{\Big|\!\frac{\rho\cdot\omega^m}{\rho+m}} \Gamma[\vec{s}\,]^{\mathbb{L}_\rho}\,.$$

Here $\Gamma[\vec{s}\,]^{\mathbb{L}_\rho}$ stands for $\{A_1[\vec{s}\,]^{\mathbb{L}_\rho}, \ldots, A_k[\vec{s}\,]^{\mathbb{L}_\rho}\}$.

Proof by induction on GLM derivations. As to the axioms of GLM, the claim follows easily from previous results of this Section. The inferences of GLM are dealt with in the same manner as in Buchholz [1993], Theorem 3.12. $\square$

Theorem 8.11 *Let $\Gamma[\vec{a}\,] = \{A_1[\vec{a}\,], \ldots, A_k[\vec{a}\,]\}$ be a set of $\mathcal{L}$–formulae with $\vec{a} = a_1, \ldots, a_n$. When $KP + \Pi_3$–Ref $\vdash \Gamma[\vec{a}\,]$, then there exists $m < \omega$ such that, for all $\vec{s} = s_1, \ldots, s_n \in Term$,*

$$\mathcal{H}[\Gamma[\vec{s}\,]^{\mathbb{L}_\mathcal{K}}, \mathcal{K}] \mathrel{\Big|\!\frac{\mathcal{K}\cdot\omega^m}{\mathcal{K}+m}} \Gamma[\vec{s}\,]^{\mathbb{L}_\mathcal{K}}\,.$$

Proof. Compared to Lemma 8.10, there is only one new inference, namely $(\Pi_3 - Ref)$. But $(\Pi_3 - Ref)$ is taken care of by $(Ref_\mathcal{K})$. $\square$

Convention: We shall also write $\exists x^\zeta$ and $\forall x^\zeta$ instead of $(\exists x \in \mathbb{L}_\zeta)$ and $(\forall x \in \mathbb{L}_\zeta)$, respectively.

Lemma 8.12 *Assume $\xi \in C(m(\pi), \pi) \cap m(\pi)$, $\xi \in \mathcal{H}$, and $F(\mathbb{L}_0, \mathbb{L}_0, \mathbb{L}_0) \in \Delta_0(\pi)$. If*

$$\mathcal{H} \mathrel{\Big|\!\frac{\alpha}{\rho}} \Gamma, \exists u^\pi \forall x^\pi \exists y^\pi F(u, x, y)$$

then

$$\mathcal{H} \mathrel{\Big|\!\frac{\pi^{1+\alpha}}{\rho}} \Gamma, \exists z^\pi[Ad^\xi(z) \ \wedge \ (\exists u \in z)(\forall x \in z)(\exists y \in z)F(u, x, y)]\,.$$

Note that $\pi^{1+\alpha} = (\omega^\pi)^{1+\alpha} = \omega^{\pi\cdot(1+\alpha)}$.

Proof. We proceed by induction on α. Put $C \equiv \exists u^\pi \forall x^\pi \exists y^\pi F(u, x, y)$. If C is not the principal formula of the last inference, then use the induction hypothesis on the premisses and subsequently apply the same inference.

Assume that C is the principal formula. Then the last inference must be $(\exists)$, and we have

$$\mathcal{H} \mathrel{\Big|\!\frac{\alpha_0}{\rho}} \Gamma, C, \forall x^\pi \exists y^\pi F(s, x, y)$$

for some $\alpha_0 < \alpha$ and $s \in Term(\pi)$. Inductively we get

$$\mathcal{H} \mathrel{\Big|\!\frac{\pi^{1+\alpha_0}}{\rho}} \Gamma, \exists z^\pi[Ad^\xi(z) \ \wedge \ (\exists u \in z)(\forall x \in z)(\exists y \in z)F(u, x, y)], \forall x^\pi \exists y^\pi F(s, x, y)\,.$$

Note that $\pi^{1+\alpha_0} + 1, \pi < \pi^{1+\alpha}$. So, using (Ref^ξ_π), we obtain

$$\mathcal{H} \mathrel{\Big|\!\frac{\pi^{1+\alpha}}{\rho}} \Gamma, \exists z^\pi[Ad^\xi(z) \ \wedge \ (\exists u \in z)(\forall x \in z)(\exists y \in z)F(u, x, y)]\,.$$

$\square$

Lemma 8.13 *Let $\xi \in C(m(\pi),\pi) \cap m(\pi)$ and $\xi > 0$. Assume that $A_1, \dots, A_k$ are subformulae of $\Sigma_3(\pi)$–formelae, $\xi \in \mathcal{H}$, $\pi + \omega \le \rho$, and $\pi < \alpha = \omega^\alpha$. Then,*

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A_1 \wedge \dots \wedge A_k \quad \Longrightarrow \quad \mathcal{H} \overset{\alpha+2}{\underset{\rho}{\vdash}} \Gamma, \exists z^\pi [Ad^\xi(z) \wedge A_1^{(z,\pi)} \wedge \dots \wedge A_n^{(z,\pi)}] \,.$$

Proof. A_i has the form $B_i[\vec{s}]^{\mathbb{L}_\pi}$ with $B_i[\vec{a}]$ being a $\mathcal{L}_{Ad}$–formula. Putting $B[\vec{a}] \equiv B_1[\vec{a}] \wedge \dots \wedge B_n[\vec{a}]$, we have $A_1 \wedge \dots \wedge A_n \equiv B[\vec{s}]^{\mathbb{L}_\pi}$. By going to prenex normal form, coding adjacent quantifiers of the same sort into one quantifier, and, if necessary, inserting dummy quantifiers, we can transform $B[\vec{a}]$ into a Σ_3–formula, say $C[\vec{a}]$. The equivalence of $C[\vec{a}]$ and $B[\vec{a}]$ is provable in GLM[11] since coding tuples of sets just requires Pairing and Extensionality. Therefore, the equivalence of $C[\vec{a}]$ and $B[\vec{a}]$ still holds when we relativize all the quantifiers to a nonempty transitive set which is a model of Pairing; and this can be proved in GLM. So, letting $Pairing := \forall x \forall y \exists u (u = \{x, y\})$, we get

$$GML \vdash \neg B[\vec{a}], C[\vec{a}] \tag{1}$$

and

$$GML \vdash \neg[Tran(b) \wedge b \neq \emptyset \wedge (Pairing)^b], \neg C[\vec{a}]^b, B[\vec{a}]^b. \tag{2}$$

From (1), using Lemma 8.10, we obtain

$$\mathcal{H} \overset{\pi \cdot \omega^m}{\underset{\pi + m}{\vdash}} \neg B[\vec{s}]^{\mathbb{L}_\pi}, C[\vec{s}]^{\mathbb{L}_\pi} \tag{3}$$

for some $0 < m < \omega$. Employing Lemma 8.12, (3) yields

$$\mathcal{H} \overset{\pi^{\pi \cdot \omega^m}}{\underset{\pi + m}{\vdash}} \neg B[\vec{s}]^{\mathbb{L}_\pi}, \exists z^\pi [Ad^\xi(z) \wedge C[\vec{s}]^z] \,. \tag{4}$$

Using (Cut) on (4) and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, B[\vec{s}]^{\mathbb{L}_\pi}$, and noting that $\pi + \omega \le \rho$, one obtains

$$\mathcal{H} \overset{\alpha+1}{\underset{\rho}{\vdash}} \Gamma, \exists z^\pi [Ad^\xi(z) \wedge C[\vec{s}]^z] \,. \tag{5}$$

According to Lemma 8.10, (2) implies

$$\mathcal{H}[\rho] \overset{\pi}{\underset{\pi}{\vdash}} \neg[Tran(\mathbb{L}_\rho) \wedge \mathbb{L}_\rho \neq \emptyset \wedge (Pairing)^{\mathbb{L}_\rho}], \neg C[\vec{s}]^{\mathbb{L}_\rho}, B[\vec{s}]^{\mathbb{L}_\rho} \tag{6}$$

for all $\rho \in M^\xi \cap \pi$ since $\omega^\rho = \rho$ due to $\xi > 0$. But, by Lemma 8.10, we also have, for all $\rho \in M^\xi \cap \pi$,

$$\mathcal{H}[\rho] \overset{\pi}{\underset{\pi}{\vdash}} Tran(\mathbb{L}_\rho) \wedge \mathbb{L}_\rho \neq \emptyset \wedge (Pair)^{\mathbb{L}_\rho} \,,$$

whence (6) implies

$$\mathcal{H}[\rho] \overset{\pi+1}{\underset{\pi}{\vdash}} \neg C[\vec{s}]^{\mathbb{L}_\rho}, B[\vec{s}]^{\mathbb{L}_\rho} \tag{7}$$

[11] This is the only reason why we introduced GML.

for all $\rho \in M^\xi \cap \pi$. From (7) one deduces

$$\mathcal{H}[\rho] \mathrel{\Big|\!\!\frac{\pi+2}{\pi}} \neg C[\vec{s}]^{\mathbb{L}_\rho}, Ad^\xi(\mathbb{L}_\rho) \ \wedge \ B[\vec{s}]^{\mathbb{L}_\rho},$$

whence

$$\mathcal{H}[\rho] \mathrel{\Big|\!\!\frac{\pi+3}{\pi}} \neg C[\vec{s}]^{\mathbb{L}_\rho}, \exists z^\pi (Ad^\xi(z) \ \wedge \ B[\vec{s}]^z) \tag{8}$$

for all $\rho \in M^\xi \cap \pi$. Since, by Corollary 8.4,

$$\mathcal{H}[\rho, t] \mathrel{\Big|\!\!\frac{\pi}{0}} \mathbb{L}_\rho \neq t, \neg C[\vec{s}]^t, C[\vec{s}]^{\mathbb{L}_\rho},$$

(Cut) yields

$$\mathcal{H}[\rho, t] \mathrel{\Big|\!\!\frac{\pi+4}{\pi}} \mathbb{L}_\rho \neq t, \neg C[\vec{s}]^t, \exists z^\pi (Ad^\xi(z) \ \wedge \ B[\vec{s}]^z)$$

for all $\rho \in M^\xi \cap \pi$ und $t \in Term(\pi)$. Whence, via $(\neg Ad^\xi)$,

$$\mathcal{H}[t] \mathrel{\Big|\!\!\frac{\pi+5}{\pi}} \neg Ad^\xi(t), \neg C[\vec{s}]^t, \exists z^\pi (Ad^\xi(z) \ \wedge \ B[\vec{s}]^z)$$

for all $t \in Term(\pi)$. Therefore, employing $(\vee)$ und $(\forall)$,

$$\mathcal{H}[\rho] \mathrel{\Big|\!\!\frac{\pi+8}{\pi}} \forall z^\pi [\neg Ad^\xi(z) \ \vee \ \neg C[\vec{s}]^z], \exists z^\pi (Ad^\xi(z) \ \wedge \ B[\vec{s}]^z). \tag{9}$$

Finally, by linking (5) and (9) via (Cut),

$$\mathcal{H} \mathrel{\Big|\!\!\frac{\alpha+2}{\rho}} \Gamma, \exists z^\pi (Ad^\xi(z) \ \wedge \ B[\vec{s}]^z).$$

$\square$

9 The Operators $\mathcal{H}_\gamma$

In order to be able to remove critical cuts, i.e. cuts which were introduced by $(Ref_\mathcal{K})$ or (Ref^ξ_π) inferences, we have to forgo arbitrary operators. We shall need operators $\mathcal{H}$ such that an $\mathcal{H}$–controlled derivation that satisfies certain extra conditions can be "collapsed" into a derivation with much smaller ordinal labels.

Definition 9.1 The operator $\mathcal{H}_\delta$ is defined by

$$\mathcal{H}_\delta(X) = \bigcap \{C(\alpha, \beta) : \ X \subseteq C(\alpha, \beta) \ \wedge \ \delta < \alpha\}$$

Lemma 9.2 *(i)* $\mathcal{H}_\delta$ *is an operator.*

(ii) $\delta < \delta' \implies \mathcal{H}_\delta(X) \subseteq \mathcal{H}_{\delta'}(X)$.

(iii) $\mathcal{H}_\delta$ *is closed under* φ *and* $(\sigma \mapsto \Omega_\sigma)_{\sigma<\mathcal{K}}$.

(iv) $\xi, \pi, \alpha \in \mathcal{H}_\delta(X) \ \wedge \ \xi \leq \alpha \leq \delta \implies \Psi^\xi_\pi(\alpha) \in \mathcal{H}_\delta(X)$.

(v) $\beta \leq \delta \ \wedge \ \beta \in \mathcal{H}_\delta(X) \implies \Xi(\beta) \in \mathcal{H}_\delta(X)$.

(vi) $\Omega_\sigma \leq \eta \leq \Omega_{\sigma+1} < \mathcal{K} \ \wedge \ \eta \in \mathcal{H}_\delta(X) \implies \sigma, \Omega_\sigma, \Omega_{\sigma+1} \in \mathcal{H}_\delta(X)$.

Proof. (i) follows from Lemma 4.18. (ii) holds by Lemma 4.11(i). (iii) follows from closure of any $C(\alpha, \beta)$ under these functions.

(iv): From $\xi, \pi, \alpha \in \mathcal{H}_\delta(X)$, $X \subseteq C(\alpha', \beta)$ and $\xi \leq \alpha \leq \delta < \alpha'$, it follows $\Psi^\xi_\pi(\alpha) \in C(\alpha', \beta)$; thus $\Psi^\xi_\pi(\alpha) \in \mathcal{H}_\delta(X)$.

The proof of (v) is similar to (iv).

(vi): Suppose $X \subseteq C(\alpha, \beta)$ with $\delta < \alpha$. Then we have to show $\sigma \in C(\alpha, \beta)$. Note that $\eta \in C(\alpha, \beta)$. By induction on n, one verifies

$$(*) \quad \Omega_\sigma \leq \eta \leq \Omega_{\sigma+1} \ \wedge \ \eta \in C_n(\alpha, \beta) \implies \sigma \in C(\alpha, \beta),$$

yielding $\sigma \in C(\alpha, \beta)$. If $\eta = \Omega_\sigma$, then $\sigma \in C(\alpha, \beta)$ by 4.18(iii). Otherwise, there is only one case when $(*)$ is not immediate by the induction hypothesis , namely when $\eta = \Psi^\xi_\pi(\gamma) \in C_n(\alpha, \beta) \backslash C_{n-1}(\alpha, \beta)$ with $\xi, \pi, \gamma \in C_{n-1}(\alpha, \beta)$. According to 4.19,(ii),(iii), we then must have $\xi = 0$ and $\pi = \Omega_{\sigma+1}$; consequently, by Lemma 4.18, $\sigma \in C(\alpha, \beta)$. □

Roughly speaking, the process of collapsing a proof tree, which we will be using in the next Section, involves pruning, grafting, and relabelling the tree with smaller ordinals. The relabelling will be done by applying a variant of Ξ or variants of the functions Ψ^ξ_π to the ordinal labels of the original tree. We are compelled to pass to variants of these functions because Ξ or Ψ^ξ_π may not preserve the order of the ordinals of the given tree, and further $\Psi^\xi_\pi(\alpha) < \pi$ may fail to be the case for some ordinal α of the tree. But that the relabelling be done in an order preserving way, is necessary if this procedure is meant to transform proof trees into proof trees.

To handle the aforementioned difficulties, we will be needing several technical results, the meaning of which will emerge only gradually in the proofs of Theorem 10.1 and Theorem 10.3. I have preferred to ban these "side calculations" from the proofs of the main theorems since the danger is to be feared that they may obscure the central ideas underlying the cut elimination and collapsing procedure.

Definition 9.3 *(i)* $NF(\alpha, \beta)$ means that $\alpha_n \geq \beta_1$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ are the respective Cantor normal forms.

(ii) $\mathfrak{B}(\mathfrak{X}; \gamma) \ :\iff \ \gamma \in \mathcal{H}_\gamma[\mathfrak{X}] \ \wedge \ k(\mathfrak{X}) \subseteq C(\gamma+1, \Xi(\gamma+1))$.

Lemma 9.4 *Assume* $\mathfrak{B}(\mathfrak{X}; \gamma)$, $\pi \in M^{\hat{\alpha}}$, $\alpha \in \mathcal{H}_\gamma[\mathfrak{X}]$, *and* $NF(\gamma, \omega^{\mathcal{K}\cdot\alpha})$, *where* $\hat{\alpha} := \gamma + \omega^{\mathcal{K}\cdot\alpha}$. *For arbitrary* α_0, *let* $\hat{\alpha_0} := \gamma + \omega^{\mathcal{K}\cdot\alpha_0}$.

(i) $\mathcal{H}_\gamma[\mathfrak{X}](\emptyset)\cap\mathcal{K}\subseteq\Xi(\gamma+1)$.

(ii) $\Xi(\hat{\alpha}+\pi)\in\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X},\pi]$.

(iii) $\alpha_0\in\mathcal{H}_\gamma[\mathfrak{X}]\ \wedge\ \alpha_0<\alpha\ \Longrightarrow\ \Xi(\hat{\alpha_0}+\pi)<\Xi(\hat{\alpha}+\pi)$.

(iv) Suppose $t\in Term$*,* $|\,t\,|\leq\alpha_t<\alpha$*, and* $\alpha_t\in\mathcal{H}_\gamma[\mathfrak{X},t]$*. If* $\gamma_t:=\gamma+\omega^{\mathcal{K}\cdot\alpha_t+|t|}$ *and* $\beta_t:=\gamma_t+\omega^{\mathcal{K}\cdot\alpha_t}$*, then*

$$\mathfrak{B}(\mathfrak{X}\cup\{t\};\gamma_t)\quad\textit{and}\quad\beta_t\in\mathcal{H}_{\gamma_t}[\mathfrak{X},t].$$

If in addition $t\in Term(\pi)$*, then also*

$$\Xi(\beta_t+\pi)<\Xi(\hat{\alpha}+\pi)\quad\textit{and}\quad\pi\in M^{\beta_t}.$$

Proof. (i) follows from $k(\mathfrak{X})\subseteq C(\gamma+1,\Xi(\gamma+1))$ in view of the definition of $\mathcal{H}_\gamma[\mathfrak{X}]$.

(ii): Since $\gamma,\alpha,\pi\in\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X},\pi]$, (ii) follows from 9.2(v).

(iii): $\hat{\alpha}+\pi\in C(\hat{\alpha}+\pi,\Xi(\hat{\alpha}+\pi))$ and $NF(\gamma,\omega^{\mathcal{K}\cdot\alpha})$ imply $\gamma\in C(\hat{\alpha}+\pi,\Xi(\hat{\alpha}+\pi))$ by 4.18. Therefore,

$$\alpha_0\in\mathcal{H}_\gamma[\mathfrak{X}]\subseteq C(\gamma+1,\Xi(\gamma+1))\subseteq C(\hat{\alpha}+\pi,\Xi(\hat{\alpha}+\pi)).$$

Thence, $\hat{\alpha_0}+\pi\in C(\hat{\alpha}+\pi,\Xi(\hat{\alpha}+\pi))\cap\hat{\alpha}+\pi$; thus $\Xi(\hat{\alpha_0}+\pi)<\Xi(\hat{\alpha}+\pi)$.

(iv): $\gamma\in\mathcal{H}_\gamma[\mathfrak{X}]$ ensures $\gamma_t,\beta_t\in\mathcal{H}_{\gamma_t}[\mathfrak{X};t]$. $NF(\gamma,\omega^{\mathcal{K}\cdot\alpha})$ and $\alpha_t<\alpha$ yield $NF(\gamma,\omega^{\mathcal{K}\cdot\alpha_t+|t|})$. Hence, from $\gamma_t\in C(\gamma_t,\Xi(\gamma_t))$, we can deduce $\gamma,|\,t\,|\in C(\gamma_t,\Xi(\gamma_t))$ and therefore, $C(\gamma+1,\Xi(\gamma+1))\subseteq C(\gamma_t,\Xi(\gamma_t))$. This shows $\mathfrak{B}(\mathfrak{X}\cup\{t\};\gamma_t)$.

Now suppose $t\in Term(\pi)$. From $NF(\gamma,\omega^{\mathcal{K}\cdot\alpha})$ it follows $\gamma\in C(\hat{\alpha},\Xi(\hat{\alpha}))$ and hence $k(\mathfrak{X}\cup\{t\})\subseteq C(\hat{\alpha},\pi)$ as $\Xi(\hat{\alpha})\leq\pi$ holds because of $\pi\in M^{\hat{\alpha}}$. Whence, $\beta_t\in C(\hat{\alpha},\pi)\cap\hat{\alpha}$. This implies

$$\beta_t+\pi\in C(\hat{\alpha}+\pi,\Xi(\hat{\alpha}+\pi))\cap\hat{\alpha}+\pi;$$

thus

$$\Xi(\beta_t+\pi)<\Xi(\hat{\alpha}+\pi).$$

Finally, from $\beta_t\in C(\hat{\alpha},\pi)\cap\hat{\alpha}$ and $\pi\in M^{\hat{\alpha}}$ we obtain, by 4.11(vi), $\pi\in M^{\beta_t}$. □

Definition 9.5 *(i)* $Card:=\{\mathcal{K}\}\cup\{\Omega_\sigma:0<\sigma<\mathcal{K}\}$.

(ii) For $\mu\in Card$, put

$$\overline{\mu}=\begin{cases}\mu+1 & \text{if } \mu\in Reg\cup\{\mathcal{K}\}\\ \mu & \text{otherwise.}\end{cases}$$

(iii) Let $\mathfrak{A}(\mathfrak{X};\gamma,\pi,\xi,\mu)$ stand for

$$\begin{aligned}&\mathfrak{B}(\mathfrak{X};\gamma)\ \wedge\ \gamma,\pi,\xi,\mu\in\mathcal{H}_\gamma[\mathfrak{X}]\ \wedge\ \xi\in C(m(\pi),\pi)\cap m(\pi)\\ &\wedge\ k(\mathfrak{X})\subseteq C(\gamma+1,\Psi^0_\pi(\gamma+1))\ \wedge\ \pi\in\bigcap\{C(\delta,\Psi^0_\tau(\delta)):\ \delta>\gamma;\ \tau>\pi\}\\ &\wedge\ \xi\leq\gamma\ \wedge\ \mu\in Card\ \wedge\ \pi\leq\mu.\end{aligned}$$

Lemma 9.6 *Assume $\mathfrak{A}(\mathfrak{X};\gamma,\pi,\xi,\mu)$, $NF(\gamma,\omega^{\mu\cdot\alpha})$, and $\alpha\in\mathcal{H}_\gamma[\mathfrak{X}]$. For arbitrary β, let $\hat{\beta}:=\gamma+\omega^{\mu\cdot\beta}$. Then the following properties hold.*

(i) $\Psi^\xi_\pi(\hat{\alpha})\in\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \ \wedge\ \Psi^\xi_\pi(\hat{\alpha})\in M^\xi\cap\pi$.

(ii) $\mathcal{H}_\gamma[\mathfrak{X}](\emptyset)\subseteq C(\gamma+1,\Psi^0_\pi(\gamma+1))$.

(iii) $\alpha_0\in\mathcal{H}_\gamma[\mathfrak{X}]\wedge\alpha_0<\alpha \implies \Psi^\xi_\pi(\hat{\alpha_0})<\Psi^\xi_\pi(\hat{\alpha})$.

(iv) Suppose $\sigma\in\mathcal{H}_\gamma[\mathfrak{X}]$, $\sigma\le\gamma$, $\sigma\in C(m(\pi),\pi)\cap m(\pi)$ and $t\in Term(\pi)$. If $\gamma_t=\gamma+\omega^{\mu\cdot\alpha+|t|}$, then

$$\mathfrak{A}(\mathfrak{X}\cup\{t\};\gamma_t,\pi,\sigma,\mu).$$

(v) If $\alpha_0<\alpha$, $\alpha_0,\tau\in\mathcal{H}_\gamma[\mathfrak{X}]$ and $\pi\le\tau\le\mu$, then

$$\mathfrak{A}(\mathfrak{X};\gamma,\tau,0,\mu)\ \wedge\ \mathfrak{A}(\mathfrak{X};\hat{\alpha_0},\tau,0,\mu).$$

Proof. (i): $\hat{\alpha}\in\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}]$ is obvious. Therefore, $\Psi^\xi_\pi(\hat{\alpha})\in\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}]$ by 9.2(iv). Since $\mathcal{H}_\gamma[\mathfrak{X}](\emptyset)\subseteq C(\gamma+1,\Psi^0_\pi(\gamma+1))\subseteq C(\hat{\alpha},\pi)$, we get $\xi,\pi,\hat{\alpha}\in C(\hat{\alpha},\pi)$. Since also $\xi\in C(m(\pi),\pi)\cap m(\pi)$, we obtain $\Psi^\xi_\pi(\hat{\alpha})\in M^\xi\cap\pi$ using 4.16.

(ii): Immediate as $k(\mathfrak{X})\subseteq C(\gamma+1,\Psi^0_\pi(\gamma+1))$.

(iii): Since $\hat{\alpha},\pi\in C(\hat{\alpha},\Psi^\xi_\pi(\hat{\alpha}))$ by (i), and $NF(\gamma,\omega^{\mu\cdot\alpha})$ involves $\gamma\in C(\hat{\alpha},\Psi^\xi_\pi(\hat{\alpha}))$, it follows $\Psi^0_\pi(\gamma+1)\in C(\hat{\alpha},\Psi^\xi_\pi(\hat{\alpha}))$. From (ii) we get $\Psi^0_\pi(\gamma+1)<\pi$. Therefore, $\Psi^0_\pi(\gamma+1)<\Psi^\xi_\pi(\hat{\alpha})$. In view of (ii), this yields $\mathcal{H}_\gamma[\mathfrak{X}](\emptyset)\subseteq C(\hat{\alpha},\Psi^\xi_\pi(\hat{\alpha}))$ and hence $\hat{\alpha_0}\in C(\hat{\alpha},\Psi^\xi_\pi(\hat{\alpha}))$. $\Psi^\xi_\pi(\hat{\alpha_0})<\pi$ follows by replacing α with α_0 in the proof of (i). Consequently, in view of the above, $\Psi^\xi_\pi(\hat{\alpha_0})<\Psi^\xi_\pi(\hat{\alpha})$.

(iv): $\alpha,\mu,\gamma\in\mathcal{H}_\gamma[\mathfrak{X}]$ guarantees $\mu,\gamma,|t|,\alpha_t\in\mathcal{H}_{\gamma_t}[\mathfrak{X},t]$. Therefore,

$$\gamma_t\in\mathcal{H}_{\gamma_t}[\mathfrak{X},t].$$

We claim that

$$(*)\quad k(\mathfrak{X}\cup\{t\})\subseteq C(\gamma_t+1,\Psi^0_\pi(\gamma_t+1))$$

By (ii), $\alpha,\gamma\in C(\gamma_t+1,\pi)$ and hence $\gamma_t\in C(\gamma_t+1,\pi)$, which implies $\gamma_t\in C(\gamma_t+1,\Psi^0_\pi(\gamma_t+1))$. As $NF(\gamma,\omega^{\mu\cdot\alpha})$, this shows $\gamma\in C(\gamma_t+1,\Psi^0_\pi(\gamma_t+1))$, yielding (note that $\pi\in C(\gamma+1,\pi)$ by (ii)) $\Psi^0_\pi(\gamma+1)<\Psi^0_\pi(\gamma_t+1)$. So we obtain $k(\mathfrak{X})\subseteq C(\gamma_t,\Psi^0_\pi(\gamma_t+1))$ and hence $(*)$.

Finally, from $(*)$ and $\mathcal{H}_\gamma[\mathfrak{X}]\subseteq\mathcal{H}_{\gamma_t}[\mathfrak{X},t]$ and $\gamma_t\in\mathcal{H}_{\gamma_t}[\mathfrak{X},t]$, we get $\mathfrak{A}(\mathfrak{X}\cup\{t\};\gamma_t,\pi,\sigma,\mu)$.

(v): As $\tau\in\mathcal{H}_\gamma[\mathfrak{X}]$, we get $\tau\in C(\gamma+1,\pi)$ due to (ii). If now $\kappa>\tau$ and $\delta>\gamma$, then $\pi\in C(\delta,\Psi^0_\kappa(\delta))$; whence $\tau\in C(\delta,\Psi^0_\kappa(\delta))$. $\mathfrak{A}(\mathfrak{X};\gamma,\tau,0,\mu)$ is now immediate.

To see $\mathfrak{A}(\mathfrak{X};\hat{\alpha_0},\tau,0,\mu)$, it suffices to verify $C(\gamma+1,\Psi^0_\pi(\gamma+1))\subseteq C(\hat{\alpha_0}+1,\Psi^0_\tau(\hat{\alpha_0}+1))$. This is trivial if $\tau>\pi$. In case $\tau=\pi$, we get $\gamma\in C(\hat{\alpha_0}+1,\Psi^0_\pi(\hat{\alpha_0}+1))$ from $NF(\gamma,\omega^{\mu\cdot\alpha})$ and $\hat{\alpha_0}\in C(\hat{\alpha_0}+1,\Psi^0_\pi(\hat{\alpha_0}+1))$. Thus $\Psi^0_\pi(\gamma+1)\in C(\hat{\alpha_0}+1,\Psi^0_\pi(\hat{\alpha_0}+1))$. As $\Psi^0_\pi(\gamma+1)<\pi$, the latter yields the claim. $\square$

10 Impredicative cut elimination and collapsing

In general, the usual cut elimination procedure does not apply when the cut formula has been introduced by a reflection inference. This is, for instance, the case when

$$\mathcal{H} \overset{\alpha}{\underset{\mathcal{K}+1}{\vdash}} \Gamma$$

results from

$$\frac{\mathcal{H} \overset{\xi_0}{\underset{\mathcal{K}}{\vdash}} \Gamma, A}{\mathcal{H} \overset{\xi}{\underset{\mathcal{K}}{\vdash}} \Gamma, \exists z^{\mathcal{K}}[Tran(z) \wedge z \neq \emptyset \wedge A^z]} (Ref_{\mathcal{K}})$$

and

$$\frac{\cdots \mathcal{H}[s] \overset{\xi_s}{\underset{\mathcal{K}}{\vdash}} \Gamma, \neg[Tran(s) \wedge s \neq \emptyset \wedge A^s] \cdots (s \in Term)}{\mathcal{H} \overset{\xi}{\underset{\mathcal{K}}{\vdash}} \Gamma, \forall z^{\mathcal{K}} \neg[Tran(z) \wedge z \neq \emptyset \wedge A^z]} (\forall)$$

using (Cut), where A is a $\Pi_3(\mathcal{K})$–formula. In this situation, the usual procedure of replacing an instance of (Cut) with cuts of lesser rank does not work. In order to overcome this problem, the proof tree has to undergo more radical transformations.

Theorem 10.1 *Suppose $\mathfrak{B}(\mathfrak{X};\gamma)$ and $NF(\gamma, \mathcal{K}^\alpha)$. Let Γ be a set of $RS(\mathcal{K})$–formulae each of which is a subformula of a $\Pi_3(\mathcal{K})$–formula or $\Pi_2(\mathcal{K})$–formula. Furthermore, suppose $\mathcal{H}_\gamma[\mathfrak{X}] \overset{\alpha}{\underset{\mathcal{K}+1}{\vdash}} \Gamma$. Then, for all $\pi \in M^{\hat{\alpha}}$,*

$$\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X}, \pi] \overset{\Xi(\hat{\alpha}+\pi)}{\underset{\Xi(\hat{\alpha}+\pi)}{\vdash}} \Gamma^{(\pi,\mathcal{K})},$$

where $\hat{\alpha} = \gamma + \mathcal{K}^\alpha = \gamma + \omega^{\mathcal{K}\cdot\alpha}$. [12]

Proof by induction on α.

Case 1: The last inference is $(\forall)$ with principal formula $\forall x^{\mathcal{K}} F(x) \in \Gamma$. Then, for all $t \in Term$, there exists α_t satisfying $|t| \leq \alpha_t < \alpha$ and

$$\mathcal{H}_\gamma[\mathfrak{X}, t] \overset{\alpha_t}{\underset{\mathcal{K}+1}{\vdash}} \Gamma, F(t). \tag{10}$$

Define $\gamma_t := \gamma + \omega^{\mathcal{K}\cdot\alpha_t+|t|}$ and $\beta_t := \gamma_t + \mathcal{K}^{\alpha_t} = \gamma_t + \omega^{\mathcal{K}\cdot\alpha_t}$. Then $NF(\gamma_t, \mathcal{K}^{\alpha_t})$. Also $\mathfrak{B}(\mathfrak{X} \cup \{t\}, \gamma_t)$ by 9.4(iv). Therefore, using the induction hypothesis on (10),

$$\mathcal{H}_{\beta_t+\pi}[\mathfrak{X}, t, \pi] \overset{\Xi(\beta_t+\pi)}{\underset{\Xi(\beta_t+\pi)}{\vdash}} \Gamma^{(\pi,\mathcal{K})}, F(t)^{(\pi,\mathcal{K})} \tag{11}$$

[12] An appropiate name for this collapsing technique would be *stationary collapsing* since in order for this procedure to work, a single derivation has to be collapsed into a "stationary" family of derivations.

holds for all $t \in Term$ and $\pi \in M^{\beta_t}$. If $\pi \in M^{\hat{\alpha}}$ and $t \in Term(\pi)$, then, by Lemma 9.4(iv), $\pi \in M^{\beta_t}$ and $\Xi(\beta_t + \pi) < \Xi(\hat{\alpha} + \pi)$. Therefore, from (11), we can conclude

$$\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X}, \pi] \mid\!\frac{\Xi(\hat{\alpha}+\pi)}{\Xi(\hat{\alpha}+\pi)} \Gamma^{(\pi,\mathcal{K})}, \forall x^\pi F(x)^{(\pi,\mathcal{K})}$$

by means of $(\forall)$. Since $\Gamma^{(\pi,\mathcal{K})}, \forall x^\pi F(x)^{(\pi,\mathcal{K})} = \Gamma^{(\pi,\mathcal{K})}$, this provides the desired result.

Case 2: The last inference is $(\bigwedge)$ but does not fall under the previous Case. This implies that the principal formula has a rank $< \mathcal{K}$ or is of the form $A_0 \wedge A_1$. The assertion then follows by simplifying the considerations of the previous Case.

Case 3: The last inference is $(\bigvee)$ with principal formula $C \cong \bigvee(C_\iota)_{\iota \in J} \in \Gamma$. Thus $\mathcal{H}_\gamma[\mathfrak{X}] \mid\!\frac{\alpha_0}{\mathcal{K}+1} \Gamma, C_{\iota_0}$ for some $\iota_0 \in J \restriction \alpha$ satisfying $|\iota_0| < \alpha$ and $k(\iota_0) \subset \mathcal{H}_\gamma[\mathfrak{X}]$. Hence, by the induction hypothesis , for all $\pi \in M^{\hat{\alpha_0}}$,

$$\mathcal{H}_{\hat{\alpha_0}+\pi}[\mathfrak{X}, \pi] \mid\!\frac{\Xi(\hat{\alpha_0}+\pi)}{\Xi(\hat{\alpha_0}+\pi)} \Gamma^{(\pi,\mathcal{K})}, C_{\iota_0}^{(\pi,\mathcal{K})}.$$

The conditions on ι_0 ensure that $|\iota_0| < \Xi(\hat{\alpha}+\pi)$. As $M^{\hat{\alpha}} \subseteq M^{\hat{\alpha_0}}$ is guaranteed by 4.11(vi), and $\Xi(\hat{\alpha_0} + \pi) < \Xi(\hat{\alpha} + \pi)$ holds by 9.4(iii), applying $(\bigvee)$ yields

$$\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X}, \pi] \mid\!\frac{\Xi(\hat{\alpha}+\pi)}{\Xi(\hat{\alpha}+\pi)} \Gamma^{(\pi,\mathcal{K})}, C^{(\pi,\mathcal{K})} \quad (= \Gamma^{(\pi,\mathcal{K})})$$

for $\pi \in M^{\hat{\alpha}}$.

Case 4: The last inference is (Cut). Then

$$\mathcal{H}_\gamma[\mathfrak{X}] \mid\!\frac{\alpha_0}{\mathcal{K}+1} \Gamma, A$$

and

$$\mathcal{H}_\gamma[\mathfrak{X}] \mid\!\frac{\alpha_0}{\mathcal{K}+1} \Gamma, \neg A$$

for some $\alpha_0 < \alpha$ and $RS(\mathcal{K})$–formlae $A, \neg A$ with $rk(A) \le \mathcal{K}$. Since then A as well as $\neg A$ are subformulae of $\Pi_3(\mathcal{K}) \cup \Pi_2(\mathcal{K})$ formulae, we can apply the induction hypothesis to both derivations. Whence, for all $\pi \in M^{\hat{\alpha_0}}$,

$$\mathcal{H}_{\hat{\alpha_0}+\pi}[\mathfrak{X}, \pi] \mid\!\frac{\Xi(\hat{\alpha_0}+\pi)}{\Xi(\hat{\alpha_0}+\pi)} \Gamma^{(\pi,\mathcal{K})}, A^{(\pi,\mathcal{K})}$$

and

$$\mathcal{H}_{\hat{\alpha_0}+\pi}[\mathfrak{X}, \pi] \mid\!\frac{\Xi(\hat{\alpha_0}+\pi)}{\Xi(\hat{\alpha_0}+\pi)} \Gamma^{(\pi,\mathcal{K})}, \neg A^{(\pi,\mathcal{K})}.$$

We also have $M^{\hat{\alpha}} \subseteq M^{\hat{\alpha_0}}$ and

$$rk(A^{(\pi,\mathcal{K})}), \Xi(\hat{\alpha_0} + \pi) < \Xi(\hat{\alpha} + \pi).$$

So the desired derivation is obtained by (Cut).

Case 5: The last inference is $(Ref_{\mathcal{K}})$. Then

$$\mathcal{H}_\gamma[\mathfrak{X}] \mid\!\frac{\alpha_0}{\mathcal{K}+1}\, \Gamma, \forall x^{\mathcal{K}} \exists y^{\mathcal{K}} \forall z^{\mathcal{K}} F(x,y,z)$$

for some $\alpha_0 < \alpha$ and a formula $C \in \Gamma$ of the form

$$C \equiv \exists u^{\mathcal{K}}[Tran(u) \wedge u \neq \emptyset \wedge (\forall x \in u)(\exists y \in u)(\forall z \in u) F(x,y,z)].$$

Set $B \equiv \forall x^{\mathcal{K}} \exists y^{\mathcal{K}} \forall z^{\mathcal{K}} F(x,y,z)$. From the induction hypothesis we then obtain, for all $\tau \in M^{\hat{\alpha_0}}$,

$$\mathcal{H}_{\hat{\alpha_0}+\tau}[\mathfrak{X}, \tau] \mid\!\frac{\Xi(\hat{\alpha_0}+\tau)}{\Xi(\hat{\alpha_0}+\tau)}\, \Gamma^{(\tau,\mathcal{K})}, B^{(\tau,\mathcal{K})}\,. \tag{12}$$

In the sequel, fix $\pi \in M^{\hat{\alpha}}$. If $\tau \in M^{\hat{\alpha_0}}$, then

$$\Vdash Tran(\mathbb{L}_\tau) \wedge \mathbb{L}_\tau \neq \emptyset;$$

therefore, using (12),

$$\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X}, \pi, \tau] \mid\!\frac{\Xi(\hat{\alpha_0}+\tau)+\omega}{\Xi(\hat{\alpha_0}+\tau)}\, \bigvee \Gamma^{(\tau,\mathcal{K})}, \exists u^\pi[Tran(u) \wedge u \neq \emptyset \wedge B^{(u,\mathcal{K})}] \tag{13}$$

for all $\tau \in M^{\hat{\alpha_0}} \cap \pi$.

Now let $s \in Term(\pi)$. In view of Corollary 8.4, we get

$$\Vdash \mathbb{L}_\tau \neq s, \bigwedge \neg\Gamma^{(\tau,\mathcal{K})}, \bigvee \Gamma^{(s,\mathcal{K})}.$$

Using (13) and (Cut),

$$\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X}, \pi, s, \tau] \mid\!\frac{\Xi(\hat{\alpha_0}+|s|)+\omega+1}{\Xi(\hat{\alpha_0}+|s|)}\, \mathbb{L}_\tau \neq s, \bigvee \Gamma^{(s,\mathcal{K})}, C^{(\pi,\mathcal{K})} \tag{14}$$

holds for all $\tau \in M^{\hat{\alpha_0}}$ satisfying $\tau \leq |s|$. Thence, applying $(\neg Ad^{\hat{\alpha_0}})$, we get

$$\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X}, \pi, s] \mid\!\frac{\Xi(\hat{\alpha_0}+|s|)+\omega+2}{\Xi(\hat{\alpha_0}+\pi)}\, \neg Ad^{\hat{\alpha_0}}(s), \bigvee \Gamma^{(s,\mathcal{K})}, C^{(\pi,\mathcal{K})} \tag{15}$$

for $s \in Term(\pi)$. Putting to use $(\vee)$ and subsequently $(\forall)$, we arrive at

$$\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X}, \pi] \mid\!\frac{\Xi(\hat{\alpha_0}+\pi)}{\Xi(\hat{\alpha_0}+\pi)}\, \forall v^\pi[Ad^{\hat{\alpha_0}}(v) \to \bigvee \Gamma^{(v,\mathcal{K})}], C^{(\pi,\mathcal{K})}\,. \tag{16}$$

Furthermore,

$$\Vdash \Gamma^{(\pi,\mathcal{K})}, \bigwedge \neg\Gamma^{(\pi,\mathcal{K})}$$

by 8.2(i). $\bigwedge \neg\Gamma^{(\pi,\mathcal{K})}$ is a conjunction of subformulae of $\Sigma_3(\pi)$–formulae. As a consequence, we can apply 8.13, yielding[13]

$$\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X},\pi] \left|\frac{\Xi(\hat{\alpha_0}+\pi)}{\Xi(\hat{\alpha_0}+\pi)}\right. \Gamma^{(\pi,\mathcal{K})}, \exists v^\pi[Ad^{\hat{\alpha_0}}(v) \wedge \bigwedge \neg\Gamma^{(v,\mathcal{K})}]\,. \tag{17}$$

Since $\Xi(\hat{\alpha_0}+\pi) < \Xi(\hat{\alpha}+\pi)$, (Cut) can be applied on (16) and (17). Hence,

$$\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X},\pi] \left|\frac{\Xi(\hat{\alpha}+\pi)}{\Xi(\hat{\alpha}+\pi)}\right. \Gamma^{(\pi,\mathcal{K})}, C^{(\pi,\mathcal{K})} \quad (= \Gamma^{(\pi,\mathcal{K})}). \tag{18}$$

Case 6: The last inference is (Ref^σ_τ). Thus

$$\mathcal{H}_\gamma[\mathfrak{X}] \left|\frac{\alpha_0}{\mathcal{K}+1}\right. \Gamma, A(s)\,,$$

where $\alpha_0+1, \tau < \alpha$, $A(s) \in \Pi_2(\tau)$, $\sigma \in \mathcal{H}_\gamma$, $\exists z^\tau[Ad^\sigma(z) \wedge (\exists u \in z)A(u)^{(z,\tau)}] \in \Gamma$, and $\sigma \in C(m(\tau),\tau) \cap m(\tau)$.

Here the induction hypothesis provides us with

$$\mathcal{H}_{\hat{\alpha_0}+\pi}[\mathfrak{X},\pi] \left|\frac{\Xi(\hat{\alpha_0}+\pi)}{\Xi(\hat{\alpha_0}+\pi)}\right. \Gamma^{(\pi,\mathcal{K})}, A(s)$$

for all $\pi \in M^{\hat{\alpha}} \subseteq M^{\hat{\alpha_0}}$. Since also $\Xi(\hat{\alpha_0}+\pi)+\tau < \Xi(\hat{\alpha}+\pi)$, because of $\tau < \Xi(\gamma) < \Xi(\hat{\alpha_0}+\pi)$, applying (Ref^σ_τ) gives the assertion. □

Corollary 10.2 *The passage from* $\mathcal{H}_\gamma[\mathfrak{X}] \left|\frac{\alpha}{\mathcal{K}+1}\right. \Gamma$ *to* $\mathcal{H}_{\hat{\alpha}+\pi}[\mathfrak{X},\pi] \left|\frac{\Xi(\hat{\alpha}+\pi)}{\Xi(\hat{\alpha}+\pi)}\right. \Gamma^{(\pi,\mathcal{K})}$ *(for* $\pi \in M^{\hat{\alpha}}$*) only introduces inferences* (Ref^σ_κ) *such that* $\sigma < \hat{\alpha}$.

Proof. New instances of (Ref^σ_κ) were only introduced when we removed an instance of $(Ref_\mathcal{K})$ and those satisfied $\sigma < \hat{\alpha}$. □

Theorem 10.3 *Suppose* $\mathfrak{A}(\mathfrak{X};\gamma,\pi,\xi,\mu)$, $NF(\gamma,\omega^{\mu\cdot\alpha})$, *and* $\Gamma \subset \Sigma_1(\pi) \cup \Delta_o(\pi)$. *Furthermore, assume that*

$$\mathcal{H}_\gamma[\mathfrak{X}] \left|\frac{\alpha}{\mu}\right. \Gamma$$

and that all the inferences of the form (Ref^σ_τ) *that appear in this derivation satisfy* $\sigma \leq \gamma$. *Then, for* $\hat{\alpha} = \gamma + \omega^{\mu\cdot\alpha}$,

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \left|\frac{\Psi^\xi_\pi(\hat{\alpha})}{\Psi^\xi_\pi(\hat{\alpha})}\right. \Gamma\,.$$

[13]This is exactly the place, where the removal of an instance of $(Ref_\mathcal{K})$ forces us to introduce an instance of $(Ad^{\hat{\alpha_0}})$.

Proof. We proceed by main induction on μ and subsidiary induction on α.

Case 1: The last inference is (Ref_π^σ). Then

$$\mathcal{H}_\gamma[\mathfrak{X}] \,\Big|\!\frac{\alpha_0}{\mu}\, \Gamma, A(s)\,,$$

where $\alpha_0+1, \pi < \alpha$, $A(s) \equiv \forall x^\pi \exists y^\pi G(x,y,s) \in \Pi_2(\pi)$, $\sigma, s \in \mathcal{H}_\gamma$, $\sigma \leq \gamma$, and $\exists z^\pi[Ad^\sigma(z) \wedge (\exists u \in z)A(u)^{(z,\pi)}] \in \Gamma$, and $\sigma \in C(m(\pi),\pi) \cap m(\pi)$. Applying Inversion, i.e. 7.1, we have, for all $t \in Term(\pi)$,

$$\mathcal{H}_\gamma[\mathfrak{X}, t] \,\Big|\!\frac{\alpha_0}{\mu}\, \Gamma, \exists y^\pi G(t,y,s) \tag{19}$$

For $t \in Term(\pi)$ and $\gamma_t := \gamma + \omega^{\mu\cdot\alpha_0+|t|}$, by 9.6(iv), it holds $\mathfrak{A}(\mathfrak{X}\cup\{t\}; \gamma_t, \pi, \sigma, \mu)$ and also $\gamma_t \in \mathcal{H}_{\gamma_t}[\mathfrak{X}, t]$. Therefore we can apply the subsidiary induction hypothesis to (19), so that with $\gamma_t' := \gamma_t + \omega^{\mu\cdot\alpha_0}$, for all $t \in Term(\pi)$,

$$\mathcal{H}_{\gamma_t'}[\mathfrak{X}, t] \,\Big|\!\frac{\Psi_\pi^\sigma(\gamma_t+\omega^{\mu\cdot\alpha_0})}{\Psi_\pi^\sigma(\gamma_t+\omega^{\mu\cdot\alpha_0})}\, \Gamma, \exists y^\pi G(t,y,s) \tag{20}$$

Set $\delta_t := \Psi_\pi^\sigma(\gamma_t + \omega^{\mu\cdot\alpha_0})$, $\gamma^* := \gamma + \omega^{\mu\cdot\alpha_0+\pi}$ and let $\eta := \Psi_\pi^\sigma(\gamma + \omega^{\mu\cdot\alpha_0+\pi})$. With the aid of the Bounding Lemma, 7.6, we then obtain from (20),

$$\mathcal{H}_{\gamma^*}[\mathfrak{X}, t] \,\Big|\!\frac{\delta_t}{\delta_t}\, \Gamma, \exists y^\eta G(t,y,s) \tag{21}$$

for $t \in Term(\pi)$ satisfying $\delta_t \leq \eta$. Due to $\mathfrak{A}(\mathfrak{X}; \gamma, \pi, \sigma, \mu)$ and $NF(\gamma, \omega^{\mu\cdot\alpha_0+\pi})$, it follows $\sigma, \pi, \gamma + \omega^{\mu\cdot\alpha_0+\pi} \in C(\gamma + \omega^{\mu\cdot\alpha_0+\pi}, \pi)$. Also $\sigma \in C(m(\pi),\pi) \cap m(\pi)$. Thus M^σ is stationary in π. From this we gather that $\eta = \Psi_\pi^\sigma(\gamma + \omega^{\mu\cdot\alpha_0+\pi}) \in M^\sigma \cap \pi$. Whence,

$$\Vdash Ad^\sigma(\mathbb{L}_\eta). \tag{22}$$

Furthermore, one computes that if $t \in Term(\eta)$, then $\delta_t < \eta$. Therefore

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \,\Big|\!\frac{\eta}{\eta}\, \Gamma, \forall x^\eta \exists y^\eta G(x,y,s) \tag{23}$$

follows from (21). (23) in conjunction with $|s| < \Psi_\pi^0(\gamma) < \eta$ yields

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \,\Big|\!\frac{\eta+1}{\eta}\, \Gamma, s \mathring{\in} \mathbb{L}_\eta \wedge \forall x^\eta \exists y^\eta G(x,y,s)\,. \tag{24}$$

Since $\eta < \pi$,

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \,\Big|\!\frac{\omega^{\eta+\omega}}{\eta}\, \Gamma, \exists z^\pi[Ad^\sigma(z) \wedge (\exists u \in z)A(u)^{(z,\pi)}] \;\; (= \Gamma) \tag{25}$$

by (24) and (22). Finally, it remains to verify $\eta < \Psi_\pi^\xi(\hat{\alpha})$. We have $\gamma + \omega^{\mu\cdot\alpha_0+\pi} < \gamma + \omega^{\mu\cdot\alpha} = \hat{\alpha}$ as $\alpha_0+1, \pi < \alpha$ and $\pi \leq \mu$. From $NF(\gamma, \omega^{\mu\cdot\alpha})$ it follows $\gamma, \mu, \pi, \sigma \in C(\hat{\alpha}, \Psi_\pi^\xi(\hat{\alpha}))$; so $\gamma + \omega^{\mu\cdot\alpha_0+\pi} \in C(\hat{\alpha}, \Psi_\pi^\xi(\hat{\alpha})) \cap \hat{\alpha}$, hence $\eta < \Psi_\pi^\xi(\hat{\alpha})$. Therefore,

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \,\Big|\!\frac{\Psi_\pi^\xi(\hat{\alpha})}{\Psi_\pi^\xi(\hat{\alpha})}\, \Gamma$$

by (25).

Case 2: The last inference is (Ref^{σ}_{κ}) for some $\kappa < \pi$. Then

$$\mathcal{H}_{\gamma}[\mathfrak{X}] \mathrel{\substack{\alpha_0 \\ \vline \overline{} \\ \mu}} \Gamma, A(s)\,,$$

where $\alpha_0 + 1, \kappa < \alpha$, $A(s) \equiv \forall x^{\kappa} \exists y^{\kappa} G(x, y, s) \in \Pi_2(\kappa)$, $\sigma \in \mathcal{H}_{\gamma}$, $\exists z^{\kappa}[Ad^{\sigma}(z) \wedge (\exists u \in z) A(u)^{(z,\kappa)}] \in \Gamma$, and $\sigma \in C(m(\kappa), \kappa) \cap m(\kappa)$. Therefore $A(s) \in \Delta_0(\pi)$ and unlike in the previous Case we can apply the subsidiary induction hypothesis directly, yielding

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \,\big|\!\frac{\Psi^{\xi}_{\pi}(\hat{\alpha_0})}{\Psi^{\xi}_{\pi}(\hat{\alpha_0})}\, \Gamma, A(s)\,.$$

Due to $\Psi^{\xi}_{\pi}(\hat{\alpha_0}) + \kappa < \Psi^{\xi}_{\pi}(\hat{\alpha})$, the same inference (Ref^{σ}_{κ}) leads to

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \,\big|\!\frac{\Psi^{\xi}_{\pi}(\hat{\alpha})}{\Psi^{\xi}_{\pi}(\hat{\alpha})}\, \Gamma\,.$$

Case 3: The last inference is $(\bigvee)$ with principal formula $C \cong \bigvee(C_{\iota})_{\iota \in J} \in \Gamma$. Then

$$\mathcal{H}_{\gamma}[\mathfrak{X}] \,\big|\!\frac{\alpha_0}{\mu}\, \Gamma, C_{\iota_0}$$

for some $\alpha_0 < \alpha$ and $\iota_0 \in J \restriction \alpha$. By subsidiary induction hypothesis , we obtain

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \,\big|\!\frac{\Psi^{\xi}_{\pi}(\hat{\alpha_0})}{\Psi^{\xi}_{\pi}(\hat{\alpha_0})}\, \Gamma, C_{\iota_0}\,,$$

whence,

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \,\big|\!\frac{\Psi^{\xi}_{\pi}(\hat{\alpha})}{\Psi^{\xi}_{\pi}(\hat{\alpha})}\, \Gamma, C \quad (= \Gamma)$$

via $(\bigvee)$.

Case 4: The last inference is $(\bigwedge)$ with principal formula $C \cong \bigwedge(C_{\iota})_{\iota \in J} \in \Gamma$. This means

$$\mathcal{H}_{\gamma}[\mathfrak{X}, \iota] \,\big|\!\frac{\alpha_{\iota}}{\mu}\, \Gamma, C_{\iota}$$

and $|\,\iota\,| \leq \alpha_{\iota} < \alpha$ for $\iota \in J$. The conditions on Γ force $C \in \Delta_0(\pi)$. Due to $k(C) \subset \mathcal{H}_{\gamma}[\mathfrak{X}](\emptyset) \cap \pi \subseteq C(\gamma + 1, \Psi^0_{\pi}(\gamma + 1)) \cap \pi$, we must have $|\,\iota\,| < \Psi^0_{\pi}(\gamma + 1)$ for all $\iota \in J$. Let $\gamma_{\iota} := \gamma + \omega^{\mu \cdot \alpha_{\iota} + |\iota|}$. From $NF(\gamma_{\iota}, \omega^{\mu \cdot \alpha_{\iota}})$ it follows $\mathfrak{A}(\mathfrak{X} \cup \{\gamma_{\iota}\}; \gamma_{\iota}, \pi, \xi, \mu)$ for all $\iota \in J$. The subsidiary induction hypothesis then yields

$$\mathcal{H}_{\delta_{\iota}}[\mathfrak{X}, \iota] \,\big|\!\frac{\Psi^{\xi}_{\pi}(\delta_{\iota})}{\Psi^{\xi}_{\pi}(\delta_{\iota})}\, \Gamma, C_{\iota}$$

for all $\iota \in J$, where $\delta_{\iota} := \gamma_{\iota} + \omega^{\mu \cdot \alpha_{\iota}} \in C(\hat{\alpha}, \Psi^{\xi}_{\pi}(\hat{\alpha}))$. $|\,\iota\,| \leq \alpha_{\iota} < \alpha$ implies $\delta_{\iota} < \hat{\alpha}$; thus $\Psi^{\xi}_{\pi}(\delta_{\iota}) < \Psi^{\xi}_{\pi}(\hat{\alpha})$. So, using $(\bigwedge)$, we conclude

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \,\big|\!\frac{\Psi^{\xi}_{\pi}(\hat{\alpha})}{\Psi^{\xi}_{\pi}(\hat{\alpha})}\, \Gamma\,.$$

Case 5: The last inference is (Cut). Then there exist $\alpha_0 < \alpha$ and an $RS(\mathcal{K})$–formula A with $rk(A) < \overline{\mu}$, so that

$$\mathcal{H}_\gamma[\mathfrak{X}] \overset{\alpha_0}{\underset{\overline{\mu}}{\vdash}} \Gamma, A \tag{26}$$

and

$$\mathcal{H}_\gamma[\mathfrak{X}] \overset{\alpha_0}{\underset{\overline{\mu}}{\vdash}} \Gamma, \neg A. \tag{27}$$

Subcase 5.1: Suppose $\overline{\mu} = \mathcal{K} + 1$. For $\kappa := \Xi(\hat{\alpha_0})$ one obtains, by applying 10.1 to (26) and (27),

$$\mathcal{H}_{\hat{\alpha_0}+\kappa}[\mathfrak{X}] \overset{\Xi(\hat{\alpha_0}+\kappa)}{\underset{\Xi(\hat{\alpha_0}+\kappa)}{\vdash}} \Gamma, A^{(\kappa,\mathcal{K})}$$

and

$$\mathcal{H}_{\hat{\alpha_0}+\kappa}[\mathfrak{X}] \overset{\Xi(\hat{\alpha_0}+\kappa)}{\underset{\Xi(\hat{\alpha_0}+\kappa)}{\vdash}} \Gamma, \neg A^{(\kappa,\mathcal{K})}\,,$$

recalling $\Gamma^{(\kappa,\mathcal{K})} = \Gamma$ (since $\pi < \kappa$) and $\kappa = \Xi(\hat{\alpha_0}) \in \mathcal{H}_{\hat{\alpha_0}+\kappa}[\mathfrak{X}]$. Whence,

$$\mathcal{H}_{\gamma'}[\mathfrak{X}] \overset{\Xi(\hat{\alpha_0}+\kappa)+1}{\underset{\Xi(\hat{\alpha_0}+\kappa)}{\vdash}} \Gamma \tag{28}$$

by means of (Cut), where $\gamma' := \gamma + \omega^{\mathcal{K}\cdot\alpha_0} \cdot 2$.

Since we have lowered the cut rank from $\overline{\mu} = \mathcal{K} + 1$ to $\Xi(\hat{\alpha_0} + \kappa) < \mathcal{K}$, the main induction hypothesis can be applied to (28); hence

$$\mathcal{H}_\eta[\mathfrak{X}] \overset{\Psi^\xi_\pi(\eta)}{\underset{\Psi^\xi_\pi(\eta)}{\vdash}} \Gamma\,,$$

where $\eta := \gamma' + \omega^{\Xi(\hat{\alpha_0}+\kappa)^2+\Xi(\hat{\alpha_0}+\kappa)} = \gamma + \omega^{\mathcal{K}\cdot\alpha_0} + \omega^{\mathcal{K}\cdot\alpha_0} + \omega^{\Xi(\hat{\alpha_0}+\kappa)^2+\Xi(\hat{\alpha_0}+\kappa)}$. Since $\eta < \hat{\alpha}$ und $\Psi^\xi_\pi(\eta) < \Psi^\xi_\pi(\hat{\alpha})$, we deduce

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \overset{\Psi^\xi_\pi(\hat{\alpha})}{\underset{\Psi^\xi_\pi(\hat{\alpha})}{\vdash}} \Gamma\,.$$

In the sequel, we shall assume $\mu < \mathcal{K}$.

Subcase 5.2: $rk(A) < \pi$.

Then $rk(A) < \Psi^\xi_\pi(\hat{\alpha_0})$ and $A \in \Delta_0(\pi)$, hence $\neg A \in \Delta_0(\pi)$. Therefore, applying the subsidiary induction hypothesis to (26) and (27),

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \overset{\Psi^\xi_\pi(\hat{\alpha_0})}{\underset{\Psi^\xi_\pi(\hat{\alpha_0})}{\vdash}} \Gamma, A \quad \text{and} \quad \mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \overset{\Psi^\xi_\pi(\hat{\alpha_0})}{\underset{\Psi^\xi_\pi(\hat{\alpha_0})}{\vdash}} \Gamma, \neg A\,;$$

whence

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \overset{\Psi^\xi_\pi(\hat{\alpha})}{\underset{\Psi^\xi_\pi(\hat{\alpha})}{\vdash}} \Gamma$$

by means of (Cut) since $\Psi^\xi_\pi(\hat{\alpha_0}) < \Psi^\xi_\pi(\hat{\alpha})$.

Subcase 5.3: $rk(A) > \pi$ and $rk(A) \notin Reg$.

We can select $\sigma \in \mathcal{H}_\gamma[\mathfrak{X}]$ so that

$$\Omega_\sigma \leq rk(A) < \Omega_{\sigma+1}.$$

Set $\tau := \Omega_{\sigma+1}$. Then $\tau \leq \mu$, $\mathfrak{A}(\mathfrak{X};\gamma,\tau,0,\mu)$, and $\Gamma \cup \{A, \neg A\} \subset \Delta_0(\tau)$. Using the subsidiary induction hypothesis we get

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \;\vert\!\frac{\Psi^0_\tau(\hat{\alpha_0})}{\Psi^0_\tau(\hat{\alpha_0})}\, \Gamma, A \quad \text{and} \quad \mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \;\vert\!\frac{\Psi^0_\tau(\hat{\alpha_0})}{\Psi^0_\tau(\hat{\alpha_0})}\, \Gamma, \neg A\,,$$

whence,

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \;\vert\!\frac{\Psi^0_\tau(\hat{\alpha_0})+1}{\Psi^0_\tau(\hat{\alpha_0})}\, \Gamma\,, \tag{29}$$

as $rk(A) < \Psi^0_\tau(\hat{\alpha_0})$. Employing predicative cut elimination, 7.4, we obtain

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \;\vert\!\frac{\varphi\eta(\eta+1)}{\nu}\, \Gamma \tag{30}$$

with $\eta := \Psi^0_\tau(\hat{\alpha_0})$ and $\nu := \Omega_\sigma$. Note that $\pi \leq \nu$. Furthermore, $\mathfrak{A}(\mathfrak{X};\hat{\alpha_0},\pi,\xi,\nu)$ and $NF(\hat{\alpha_0}, \omega^{\nu\cdot\varphi\eta(\eta+1)})$. Also $\nu < \mu$. Therefore, letting $\zeta := \hat{\alpha_0} + \omega^{\nu\cdot\varphi\eta(\eta+1)}$, we can use the main induction hypothesis on (30) to conclude

$$\mathcal{H}_\zeta[\mathfrak{X}] \;\vert\!\frac{\Psi^\xi_\pi(\zeta)}{\Psi^\xi_\pi(\zeta)}\, \Gamma\,.$$

Noting that $\zeta < \hat{\alpha}$ and $\Psi^\xi_\pi(\zeta) < \Psi^\xi_\pi(\hat{\alpha})$, this implies

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \;\vert\!\frac{\Psi^\xi_\pi(\hat{\alpha})}{\Psi^\xi_\pi(\hat{\alpha})}\, \Gamma\,.$$

Subcase 5.4: $rk(A) \geq \pi$ and $rk(A) \in Reg$.

Let $\tau := rk(A)$. Then either A or $\neg A$ is of the form $\exists x^\tau F(x)$ with $F(\mathbb{L}_0) \in \Delta_0(\tau)$.

If $\alpha_0 < \tau$, then $\neg A$ never gets used as a principal formula of an inference in $\mathcal{H}_\gamma[\mathfrak{X}] \;\vert\!\frac{\alpha_0}{\mu}\, \Gamma, \neg A$, and therefore, $\mathcal{H}_\gamma[\mathfrak{X}] \;\vert\!\frac{\alpha_0}{\mu}\, \Gamma$. Thus, by subsidiary induction hypothesis , $\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \;\vert\!\frac{\Psi^\xi_\pi(\hat{\alpha_0})}{\Psi^\xi_\pi(\hat{\alpha_0})}\, \Gamma$, whence $\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \;\vert\!\frac{\Psi^\xi_\pi(\hat{\alpha})}{\Psi^\xi_\pi(\hat{\alpha})}\, \Gamma$ since $\Psi^\xi_\pi(\hat{\alpha_0}) < \Psi^\xi_\pi(\hat{\alpha})$.

Now assume $\tau \leq \alpha_0$. Observe that $\mathfrak{A}(\mathfrak{X};\gamma,\tau,0,\mu)$ and $\Gamma, A \subset \Delta_0(\tau) \cup \Sigma_1(\tau)$. Applying the subsidiary induction hypothesis to (26) and using the Bounding Lemma 7.6, we obtain

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \;\vert\!\frac{\Psi^0_\tau(\hat{\alpha_0})}{\Psi^0_\tau(\hat{\alpha_0})}\, \Gamma, A^{(\Psi^0_\tau(\hat{\alpha_0}),\tau)}\,. \tag{31}$$

From (27), by employing 6.15(iii) and $\Psi^0_\tau(\hat{\alpha_0}) \in \mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}]$, we get

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \mathrel{\substack{\alpha_0 \\ \rule{1.5em}{0.4pt} \\ \mu}} \Gamma, \neg A^{(\Psi^0_\tau(\hat{\alpha_0}),\tau)}. \tag{32}$$

Since $\mathfrak{A}(\mathfrak{X}; \hat{\alpha_0}, \tau, 0, \mu)$ and $NF(\hat{\alpha_0}, \omega^{\mu\cdot\alpha_0})$, the subsidiary induction hypothesis can be used on (32), furnishing

$$\mathcal{H}_\delta[\mathfrak{X}] \mathrel{\substack{\Psi^0_\tau(\delta) \\ \rule{3em}{0.4pt} \\ \Psi^0_\tau(\delta)}} \Gamma, \neg A^{(\Psi^0_\tau(\hat{\alpha_0}),\tau)}\,, \tag{33}$$

where $\delta := \hat{\alpha_0} + \omega^{\mu\cdot\alpha_0}$. Using (Cut) on (31) and (32), we obtain

$$\mathcal{H}_\delta[\mathfrak{X}] \mathrel{\substack{\Psi^0_\tau(\delta)+1 \\ \rule{4em}{0.4pt} \\ \Psi^0_\tau(\delta)}} \Gamma\,. \tag{34}$$

If $\tau = \pi$, then (34) implies

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \mathrel{\substack{\Psi^\xi_\pi(\hat{\alpha}) \\ \rule{3em}{0.4pt} \\ \Psi^\xi_\pi(\hat{\alpha})}} \Gamma\,,$$

noting that $\Psi^0_\pi(\delta) < \Psi^\xi_\pi(\hat{\alpha})$.

From now on, let $\pi < \tau$. Again, we can select $\sigma \in \mathcal{H}_\gamma[\mathfrak{X}]$ so that $\Omega_\sigma \leq \Psi^0_\tau(\delta) < \Omega_{\sigma+1} \leq \tau$. Through the use of predicative cut elimination, (34) yields

$$\mathcal{H}_\delta[\mathfrak{X}] \mathrel{\substack{\eta \\ \rule{1em}{0.4pt} \\ \nu}} \Gamma\,, \tag{35}$$

where we put $\eta := \varphi\Psi^0_\tau(\delta)(\Psi^0_\tau(\delta)+1)$ and $\nu := \Omega_\sigma$. Set $\gamma' := \delta + \omega^{\mu\cdot\alpha_0}$. Then $\delta < \gamma'$ and $NF(\gamma', \omega^{\nu\cdot\eta})$ since $\nu < \mu$ as well as $\eta < \nu \leq \alpha_0$. Since $\pi < \tau$ and $\pi \in C(\gamma+1, \Psi^0_\tau(\gamma+1))$, we get $\pi < \Psi^0_\tau(\delta)$; thence $\pi \leq \nu$. Note that $\mathfrak{A}(\mathfrak{X}; \gamma', \pi, \xi, \nu)$. Since $\nu < \mu$, we can use the main induction hypothesis on (35), so that with $\rho := \gamma' + \omega^{\nu\cdot\eta}$,

$$\mathcal{H}_\rho[\mathfrak{X}] \mathrel{\substack{\Psi^\xi_\pi(\rho) \\ \rule{3em}{0.4pt} \\ \Psi^\xi_\pi(\rho)}} \Gamma\,. \tag{36}$$

One readily verifies $\rho < \hat{\alpha}$ and $\rho \in C(\hat{\alpha}, \Psi^\xi_\pi(\hat{\alpha}))$. Therefore, by (36),

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}] \mathrel{\substack{\Psi^\xi_\pi(\hat{\alpha}) \\ \rule{3em}{0.4pt} \\ \Psi^\xi_\pi(\hat{\alpha})}} \Gamma\,.$$

$\square$

Theorem 10.4 *Let $\rho_0 := 1$ and $\rho_{n+1} := \mathcal{K}^{\rho_n}$.*

(i) [14] *If A is a Π_3–sentence of $\mathcal{L}$ and $KP + \Pi_3$–Ref $\vdash A$, then there is an $n < \omega$ such that, for all $\pi \in M^{\rho_n}$,*

$$\mathcal{H}_{\rho_n+\pi} \mathrel{\substack{\Xi(\rho_n+\pi) \\ \rule{4em}{0.4pt} \\ \Xi(\rho_n+\pi)}} A^{\mathbb{L}_\pi}\,.$$

[14]The meaning of (i) can be greatly enhanced by developing the collapsing functions on the basis of a Π_3–reflecting ordinal, say κ_0. It will then be possible, given a proof of a Π_3–sentence in $KP + \Pi_3$–*Ref*, to determine a κ_0–recursively stationary set of reflection points; thereby providing an Herbrand analysis for provable Π_3–sentences of $KP + \Pi_3$–*Ref*.

(ii) *The property of being an admissible set above ω can be expressed by a Δ_0–formula. (For definiteness, let this be the formula displayed in Aczel und Richter [1974].) If B is a Σ_1–sentence and*

$$KP + \Pi_3\text{–}Ref \vdash \forall x[Ad(x) \rightarrow B^x],$$

then there is a $k < \omega$ such that

$$\mathcal{H}_{\rho_k} \,\Big|\!\frac{\Psi^0_{\Omega_1}(\rho_k)}{0}\; B^{\mathbb{L}_{\Psi^0_{\Omega_1}(\rho_k)}}\,.$$

Proof. (i) According to Theorem 8.11, there is an $m < \omega$ satisfying

$$\mathcal{H}_0 \,\Big|\!\frac{\mathcal{K}\cdot\omega^m}{\mathcal{K}+m}\; A^{\mathbb{L}_\mathcal{K}}\,.$$

Applying Corollary 7.5 several times, we get

$$\mathcal{H}_0 \,\Big|\!\frac{\rho_{m+2}}{\mathcal{K}+1}\; A^{\mathbb{L}_\mathcal{K}}\,.$$

Letting $\gamma := \rho_{m+4}$, we have $NF(\gamma, \mathcal{K}^{\rho_{m+2}})$ and $\mathfrak{B}(\emptyset;\gamma)$. So we can apply Theorem 10.1 to get

$$\mathcal{H}_{\rho_n+\pi} \,\Big|\!\frac{\Xi(\rho_n+\pi)}{\Xi(\rho_n+\pi)}\; A^{\mathbb{L}_\pi}$$

for all $\pi \in M^{\rho_n}$, provided that $n > m + 4$.

(ii): By the same procedure as in (i), we obtain an $n < \omega$ satisfying

$$\mathcal{H}_{\rho_n+\pi_0} \,\Big|\!\frac{\Xi(\rho_n+\pi_0)}{\Xi(\rho_n+\pi_0)}\; \neg Ad(\mathbb{L}_{\Omega_1}), B^{\mathbb{L}_{\Omega_1}}\,,$$

where $\pi_0 := \Xi(\rho_n)$. Since

$$\mathcal{H}_0 \,\Big|\!\frac{\Omega_1\cdot\omega}{\Omega_1+\omega}\; Ad(\mathbb{L}_{\Omega_1})\,,$$

it follows

$$\mathcal{H}_{\rho_n+\pi_0} \,\Big|\!\frac{\Xi(\rho_n+\pi_0)+1}{\Xi(\rho_n+\pi_0)}\; B^{\mathbb{L}_{\Omega_1}}\,. \tag{37}$$

Letting $\gamma := \rho_{n+2}$, $\alpha := \Xi(\rho_n + \pi_0) + 1$ and $\mu := \Xi(\rho_n + \pi_0)$, we have $\gamma, \alpha \in \mathcal{H}_\gamma$, $NF(\gamma, \omega^{\mu\cdot\alpha})$, and $\mathfrak{A}(\emptyset; \gamma, \Omega_1, 0, \mu)$. Also, by Corollary 10.2, $\sigma < \gamma$ holds for all inferences (Ref^σ_τ) appearing in (37). Therefore, by Theorem 10.3, we obtain

$$\mathcal{H}_{\hat{\alpha}} \,\Big|\!\frac{\delta}{\delta}\; B^{\mathbb{L}_{\Omega_1}}$$

for $\hat{\alpha} := \gamma + \omega^{\mu\cdot\alpha}$ and $\delta := \Psi^0_{\Omega_1}(\hat{\alpha})$. Using predicative cut elimination, Theorem 7.4, this leads to

$$\mathcal{H}_{\hat{\alpha}} \,\Big|\!\frac{\varphi\delta\delta}{0}\; B^{\mathbb{L}_{\Omega_1}}\,.$$

For $k := n+3$, one easily verifies $\hat{\alpha} < \rho_k$ and $\varphi\delta\delta < \Psi^0_{\Omega_1}(\rho_k)$. Hence,

$$\mathcal{H}_{\rho_k} \mid\frac{\Psi^0_{\Omega_1}(\rho_k)}{0} B^{\mathbb{L}_{\Omega_1}} .$$

□

Corollary 10.5

$$\mid KP + \Pi_3\text{–}\mathit{Ref} \mid \leq \Psi^0_{\Omega_1}(\varepsilon_{\mathcal{K}+1}).$$

($\mid KP + \Pi_3$–Ref $\mid$ denotes the proof–theoretic ordinal of $KP + \Pi_3$–Ref.)

□

Remark 10.6 The bound given in 10.5 is indeed sharp. But we will not give a proof for that in this paper.

11 Conclusions

A notation system which is suitable for an ordinal analysis of *KP*+Π_{n+2}–*reflection* ($n > 1$) can be derived from collapsing functions based on Π^1_m indescribable cardinals, where $0 < m \leq n$. Here one employs the thinning–operation

$$M_{k+1}(X) = \{\pi{\in}X : \pi \mathit{is}\ \Pi^1_k\ \mathit{indescribable\ on}\ X\},$$

where π is Π^1_k *indescribable on* X if for all $U_1, \ldots, U_i \subseteq V_\pi$ and every Π^1_k sentence F, whenever $\langle V_\pi, \in, U_1, \ldots, U_i\rangle \models F$, then there exists a $\rho{\in}X \cap \pi$ such that

$$\langle V_\rho, \in, U_1 \cap V_\rho, \ldots, U_i \cap V_\rho\rangle \models F.$$

As a matter of fact, if κ is Π^1_{k+1} indescribable and $X \subseteq \kappa$ is stationary in κ then $M_k(X)$ is also stationary in κ. So, analogously to Definition 4.8, given a Π_{n+1} indescribable cardinal $\mathfrak{R}$, one defines a hierarchy of subsets $M^{\mathfrak{R},\alpha}_n$ of $\mathfrak{R}$ (using M_n in place of M) which induces a collapsing function $\Xi^{\mathfrak{R}}_{n+1}$ by letting

$$\Xi^{\mathfrak{R}}_{n+1}(\alpha) = \mathrm{least}\nu[\nu{\in}M^{\mathfrak{R},\alpha}_n].$$

We have already pointed out that the use of large cardinals in the development of collapsing functions is merely an exaggeration that simplifies proofs, but could be avoided by employing their recursively large analogoues (see Rathjen [1993c]). However, regarding a consistency proof for $KP + \Pi_3$–*Ref* (or, more generally, $KP + \Pi_{n+2}$–*reflection*) we would like to have some kind of constructive justification for the well–foundedness of $\langle \mathcal{T}(\mathcal{K}), <\rangle$. First, let us delimit in which metatheory such a consistency proof can be accomplished. A rough estimate would be first order arithmetic augmented by the scheme of transfinite induction along the ordering of $\mathcal{T}(\mathcal{K})$. To see this, note that $\langle \mathcal{T}(\mathcal{K}), < \rangle$ is primitive recursive (after some coding) and that recursive $RS(\mathcal{K})$ derivations suffice for the results of Sections 6 through 10. Now, recursive $RS(\mathcal{K})$ derivations can be formalized in first order arithmetic (see Schwichtenberg [1977]). But we can do even better. For a particular arithmetic theorem of $KP + \Pi_3$–*Ref*, say A, an n can be determined (depending on the proof of A) such that there is a cut free controlled recursive derivation of A that utilizes solely ordinals from $\mathcal{T}_n(\mathcal{K}) = C(\rho_n, 0)$, where $\rho_0 = 1$ and $\rho_{k+1} = \mathcal{K}^{\rho_n}$. So the upshot is that any arithmetic theorem of $KP + \Pi_3$–*Ref* is provable in first order arithmetic augmented by the schemes of transfinite induction for all the orderings $<_n$ arising by restricting $<$ to $\mathcal{T}_n(\mathcal{K})$. Finally, by results of Friedman and Sheard [1993], Theorem 4.5, the consistency (even the 1–consistency) of the latter theory is provable in primitive recursive arithmetic plus a scheme expressing that there is no infinite primitive recursive[15] descending sequence in the notation system determined by $C(\varepsilon_{\mathcal{K}+1}, 0) \subseteq \mathcal{T}(\mathcal{K})$.

By now we have managed to reduce the consistency of $KP + \Pi_3$–*Ref* to the principle (say $FT(<)$) that every concrete strictly decreasing sequence of members of $C(\varepsilon_{\mathcal{K}+1}, 0)$ terminates in a finite number of steps. How can we assure ourselves of the validity of $FT(<)$? Takeuti (see [1985],[1987]) refers to such proofs as *accessibility proofs*. In his work he has given accessibility proofs for the ordinal diagrams that he used for his consistency proof of Π^1_1 comprehension. As to the methods allowed for such proofs, Takeuti delimits a kind of concrete constructivity. In the words of Takeuti [1987, p.96]: "We believe that our standpoint is a natural extension of Hilbert's finitist standpoint, similar to that introduced by Gentzen, and we call it the Hilbert–Gentzen finitist standpoint."

However, Takeuti does not formally lay bare what he counts as acceptable from his stance, this especially applies to what he calls (using Hilbert's jargon) "performing a Gedankenexperiment". Of course, ultimately, justification can only come about by halting at some intuitively convincing grounds, and no explanation can substitute for each individuals understanding. Incidentally, the author convinced himself of the accessibility of $\mathcal{T}(\mathcal{K})$ along the lines delineated by Takeuti.

Nonetheless, it might be desirable to obtain different accessibility proofs based on different styles of constructivity. There are prospects that extensions of Martin–Löf's intuitionistic type theory with higher universes can provide a uniform setting for consistency proofs. Palmgren (in [1990]) has outlined an intuitionistic theory of types with transfinite universes that provides a means of understanding constructive Mahlo numbers.

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